STABLE BASIS FOR $T^*(G/B)$ AND ITS APPLICATIONS

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The main reference for the first two parts are [5]. We refer the interested readers to Chapter 3 in [4] for a general definition of stable envelope in a much more general setting.

1. Definition of stalbe basis

Let G be a semisimple linear algebraic group. Let $A \subset B \subset G$ be a maximal torus and a Borel subgroup respectively. Let $\mathcal{B}$ be the flag variety G/B. Let us first define the stable basis in $T^*\mathcal{B}$.

1.1. **Fixed point set.** The A-fixed points of $T^*\mathcal{B}$ is in one-to-one correspondence with the Weyl group W. The fixed point corresponds to $w \in W$ is denoted by wB. For any cohomology class $\alpha \in H_T^*(T^*\mathcal{B})$, let $\alpha|_w$ denote the restriction of α to the fixed point wB.

1.2. **Chamber decomposition.** The cocharacters

$$\sigma : \mathbb{C}^* \to A$$

form a lattice. Let

$$\mathfrak{a}_\mathbb{R} = \text{cochar}(A) \otimes_\mathbb{Z} \mathbb{R}.$$

Define the torus roots to be the A-weights occurring in the normal bundle to $(T^*\mathcal{B})^A$. Then the root hyperplanes partition $\mathfrak{a}_\mathbb{R}$ into finitely many chambers

$$\mathfrak{a}_\mathbb{R} \setminus \bigcup \alpha_i^\perp = \coprod \mathfrak{C}_i.$$

It is easy to see in this case that the torus roots are just the roots for G. Let $+$ denote the chamber such that all roots in R^+ are positive on it, and $-$ the opposite chamber.

1.3. **Stable leaves.** Let $\mathfrak{C}$ be a chamber. For any fixed point yB, define the stable leaf of yB by

$$\text{Leaf}_\mathfrak{C}(yB) = \left\{ x \in T^*\mathcal{B} \,\middle|\, \lim_{z \to 0} \sigma(z) \cdot x = yB \right\}$$

where σ is any cocharacter in $\mathfrak{C}$; the limit is independent of the choice of $\sigma \in \mathfrak{C}$. In the $T^*\mathcal{B}$ case, $\text{Leaf}_+(yB) = T^*_{ByB/B}\mathcal{B}$, and $\text{Leaf}_-(yB) = T^*_{B^-yB/B}\mathcal{B}$, where B^- is the opposite Borel subgroup.

Define a partial order on the fixed points as follows:

$$wB \preceq_\mathfrak{C} yB \quad \text{if} \quad \overline{\text{Leaf}_\mathfrak{C}(yB)} \cap wB \neq \emptyset.$$

By the description of $\text{Leaf}_+(yB)$, the order $\preceq_+$ is the same as the Bruhat order $\leq$, and $\preceq_-$ is the opposite order. Define the slope of a fixed point yB by

$$\text{Slope}_\mathfrak{C}(yB) = \bigcup_{wB \preceq_\mathfrak{C} yB} \text{Leaf}_\mathfrak{C}(wB).$$

1.4. **Stable basis.** For each $y \in W$, let $T_y^*\mathcal{B}$ and $T_y(T^*\mathcal{B})$ denote $T^*_{yB}\mathcal{B}$ and $T_{yB}(T^*\mathcal{B})$ respectively, and define $\epsilon_y = e^A(T_y^*\mathcal{B})$. Here, e^A denotes the A-equivariant Euler class. Let N_y denote the normal bundle of $T^*\mathcal{B}$ at the fixed point yB. The chamber $\mathfrak{C}$ gives a decomposition of the normal bundle

$$N_y = N_{y,+} \oplus N_{y,-}$$

into A-weights which are positive and negative on $\mathfrak{C}$ respectively. The sign in $\pm e(N_{y,-})$ is determined by the condition

$$\pm e(N_{y,-})|_{H_A^*(\text{pt})} = \epsilon_y.$$

Theorem 1.1. *There exists a unique map of $H_T^*(pt)$-modules*

$$\mathrm{stab}_{\mathfrak{C}} : H_T^*((T^*\mathcal{B})^A) \to H_T^*(T^*\mathcal{B})$$

such that for any $y \in W$, $\Gamma = \mathrm{stab}_{\mathfrak{C}}(y)$ satisfies:

(1) $\operatorname{supp}\Gamma \subset \mathrm{Slope}_{\mathfrak{C}}(yB)$,

(2) $\Gamma|_y = \pm e(N_{-,y})$, *with sign according to ϵ_y,*

(3) $\Gamma|_w$ *is divisible by $\hbar$, for any $wB \prec_{\mathfrak{C}} yB$,*

where y in $\mathrm{stab}_{\mathfrak{C}}(y)$ is the unit in $H_T^*(yB)$.

Remark 1.2.

(1) The map is defined by a Lagrangian correspondence between $(T^*\mathcal{B})^A \times T^*\mathcal{B}$, hence maps middle degree to middle degree.

(2) From the characterization, the transition matrix from $\{\mathrm{stab}_{\mathfrak{C}}(y)|y \in W\}$ to the fixed point basis is a triangular matrix with nontrivial diagonal terms. Hence, after localization, $\{\mathrm{stab}_{\mathfrak{C}}(y)|y \in W\}$ form a basis for the cohomology, which we call the **stable basis**.

(3) Maulik and Okounkov prove that $\{\mathrm{stab}_{\mathfrak{C}}(y)|y \in W\}$ and $\{(-1)^n \mathrm{stab}_{-\mathfrak{C}}(y)|y \in W\}$ are dual bases, i.e.,

$$(\mathrm{stab}_{\mathfrak{C}}(y), (-1)^n \mathrm{stab}_{-\mathfrak{C}}(w)) = \delta_{y,w}.$$

Here $n = \dim_{\mathbb{C}} \mathcal{B}$.

2. Restriction formulas

Let $\pm$ denote the positive/negative chamber. Then the formula I proved is:

Theorem 2.1. *Let $y = \sigma_1\sigma_2\cdots\sigma_l$ be a reduced expression for $y \in W$. Then*

$$\mathrm{stab}_-(w)|_y = (-1)^{l(y)} \prod_{\alpha\in R^+\setminus R(y)} (\alpha - \hbar) \sum_{\substack{1\leq i_1<i_2<\cdots<i_k\leq l\\ w=\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}}} \hbar^{l-k} \prod_{j=1}^{k} \beta_{i_j}, \tag{1}$$

where σ_i is the simple reflection associated to a simple root α_i, $\beta_i = \sigma_1\cdots\sigma_{i-1}\alpha_i$, $R(y) = \{\beta_i|1 \leq i \leq l\}$, and $\mathrm{stab}_-(w)|_y$ denotes the restriction of $\mathrm{stab}_-(w)$ to the fixed point yB.

For the positive chamber, we have

Theorem 2.2. *Let $y = \sigma_1\sigma_2\cdots\sigma_l$ be a reduced expression for $y \in W$, and $w \leq y$. Then*

$$\mathrm{stab}_+(y)|_w = \sum_{\substack{1\leq i_1<i_2<\cdots<i_k\leq l\\ w=\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}}} (-1)^l \prod_{j=1}^{k} \frac{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j} - \hbar}{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j}} \frac{\hbar^{l-k}}{\prod_{j=0}^{k}\prod_{i_j<r<i_{j+1}} \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_r} \prod_{\alpha\in R^+} \alpha.$$

The proof is very similar to the proof of the restriction formula of Schubert variety. The basic idea is the following. I learned this from [3], which is a very good reference if you want to learn equivariant cohomology and Schubert calculus.

Let Q be the quotient field of $H_T^*(\mathrm{pt})$, and $F(W,Q)$ be the functions from W to Q. Restriction to fixed points gives a map

$$H_T^*(T^*\mathcal{B}) \to H_T^*((T^*\mathcal{B})^T) = \bigoplus_{w\in W} H_T^*(wB)$$

and embeds $H_T^*(T^*\mathcal{B})$ into $F(W,Q)$.

For each simple root $\alpha \in \Delta$, let Y_α be the orbit corresponding to the reflection σ_α. Then

$$\overline{Y_\alpha} = \mathcal{B} \times_{\mathcal{P}_\alpha} \mathcal{B}$$

where $\mathcal{P}_\alpha = G/P_\alpha$ and P_α is the minimal parabolic subgroup corresponding to the simple root α. Let $T^*_{\overline{Y_\alpha}}(\mathcal{B}\times\mathcal{B})$ be the conormal bundle to $\overline{Y_\alpha}$. This is a Lagrangian correspondence in $T^*\mathcal{B}\times T^*\mathcal{B}$, and defines a map

$$D_\alpha : H_T^*(T^*\mathcal{B}) \to H_T^*(T^*\mathcal{B}).$$

Define an operator $A_0 : F(W,Q) \to F(W,Q)$ by the formula

$$(A_0\psi)(w) = \frac{\psi(w\sigma_\alpha) - \psi(w)}{w\alpha}(w\alpha - \hbar).$$

Then we have the following important commutative diagram.

Proposition 2.3. *The diagram*

$$\begin{array}{ccc} H_T^*(T^*\mathcal{B}) & \hookrightarrow & F(W,Q) \\ \Big\downarrow{\scriptstyle D_\alpha} & & \Big\downarrow{\scriptstyle A_0} \\ H_T^*(T^*\mathcal{B}) & \hookrightarrow & F(W,Q) \end{array}$$

commutes.

Apply this diagram to stable basis, we get some recursive formulas for the restriction, which finally lead to the proof of Theorems 2.1 and 2.2.

3. Applications

This is joint work with Leonardo C Mihalcea.

3.1. **First relation with CSM classes.** Let $c_* : L_{\mathbb{C}^*}(T^*(G/B)) \to H_*(G/B)$ be the map define by Ginzburg in the appendix of [2] between the Lagrangian cycles in the cotangent bundle of the flag manifold G/B and the homology of G/B. We found

$$c_*(\mathrm{stab}_+(w)) = \pm CSM(X(w)^\circ).$$

This is essentially due to Ginzburg.

3.2. **Second relation with CSM classes.** With the formula proved in [1], we proved the following Theorem:

Theorem 3.1. *Let i be the inclusion of X into T^*X, then*

$$(-1)^{\dim X} i^*(\mathrm{stab}_+(y))|_{\hbar=1} = CSM(X(y)^\circ). \tag{2}$$

In particular, let $y = \sigma_1\sigma_2\cdots\sigma_l$ be a reduced expression for $y \in W$, and $w \le y$. Then

$$CSM(X(y)^\circ)|_w = (-1)^{\dim X + \ell(y)} \sum_{\substack{1\le i_1 < i_2 < \cdots < i_k \le l \\ w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}}} \prod_{j=1}^{k} \frac{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j} - 1}{\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_{i_j}}$$

$$\frac{1}{\prod_{j=0}^{k} \prod_{i_j < r < i_{j+1}} \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_j}\alpha_r} \prod_{\alpha\in R^+} \alpha.$$

With this formula, we can check in some simple cases the conjecture in [1].

References

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