Algebraic construction of oriented cohomology of flag varieties

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Notations

k: field

h: an oriented cohomology theory in the sense of Levine-Morel

$R = h(Spec(k))$: the coefficient ring

G: a split semisimple linear algebraic group

$B \supset T$: Borel subgroup and maximal torus

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$R = h(Spec(k))$: the coefficient ring
G: a split semisimple linear algebraic group
$B \supset T$: Borel subgroup and maximal torus
$\Lambda = T^*$: the group of characters of T
$\Sigma, \Pi = \{\alpha_1, ..., \alpha_n\}$: the sets of roots and simple roots
W: the Weyl group, generated by $s_i = s_{\alpha_i}, i = 1, ..., n$.
$J \subset \Pi$
$W_J < W$: the subgroup generated by J
$P_J \supset B$: the parabolic subgroup

Goal

To construct the following diagram:

$$\begin{array}{ccccccc}
h_T(pt) & \longrightarrow & h_T(G/B) & \xrightarrow{p_{J*}} & h_T(G/P_J) & \xrightarrow{p_J^*} & h_T(G/B) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
S & \xrightarrow{c_S} & \mathbf{D}_F^* & \longrightarrow & (\mathbf{D}_F^*)^{W_J} & \hookrightarrow & \mathbf{D}_F^*
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The work follows Demazure (Chow group), Bernstein–Gelfand–Gelfand (singular cohomology), Arabia (equivariant cohomology), Kostant–Kumar (equivariant cohomology and equivariant Grothendieck group), and many others.

Formal group laws

Definition

A formal group law (FGL) F over a ring R is a power series $F(x,y) \in R[[x,y]]$ satisfying:

$$F(x,0) = x, \quad F(x,y) = F(y,x), \quad F(F(x,y),z) = F(x,F(y,z)).$$

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Example

- The additive FGL: $F_a = x + y$.
- The multiplicative FGL: $F_m = x + y - xy$
- The universal FGL: F_U over the Lazard ring $\mathbb{L}$.

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Examples of oriented cohomology are Chow group, Grothendieck group and the algebraic cobordism.
The push-forward can be used to define characteristic classes.
Moreover, it defines a FGL F over $R := h(k)$

$$c_1^h(L_1 \otimes L_2) = F(c_1^h(L_1), c_1^h(L_2)),$$

L_1, L_2 are line bundles.

Example

$$CH \rightsquigarrow F_a, \quad K_0 \rightsquigarrow F_m, \quad \text{algebraic cobordism } \rightsquigarrow F_u.$$

The formal group algebra: $h_T(Spec(k))$

[Calmès–Petrov–Zainoulline]
Let $R[[x_\Lambda]] = R[[x_\lambda | \lambda \in \Lambda]]$, and define

$$S := R[[\Lambda]]_F = R[[x_\Lambda]]/(x_{\lambda+\mu} - F(x_\lambda, x_\mu), x_0).$$

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Example

$$R[[\Lambda]]_{F_a} = S_R^*(\Lambda)^\wedge, \quad R[[\Lambda]]_{F_m} = R[\Lambda]^\wedge.$$

$$R[[\Lambda]]_F \cong R[[t_1, ..., t_n]].$$

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Difficulty for generalization

- S is not (Laurent) polynomial ring but power series ring.
- The Bott-Samelson classes depend on the choices of reduced sequences.

Formal affine Demazure algebra

For each root α, define Demazure operators (divided difference operators, or BGG operators)

$$\Delta_\alpha(z) = \frac{z - s_\alpha(z)}{x_\alpha} \in S, \quad z \in S.$$

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Object to study

The subring of $End(S)$ generated by S and $\Delta_\alpha, \alpha \in \Sigma$.

Define

$$Q = S[\frac{1}{x_\alpha} | \alpha \in \Sigma], \quad Q_W = Q \rtimes R[W], \quad S_W = S \rtimes R[W]$$

Q_W has Q-basis $\{\delta_w\}_{w \in W}$ and the product is

$$q\delta_w \cdot q'\delta_{w'} = qw(q')\delta_{ww'}, \quad q, q' \in Q, w, w' \in W.$$

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We define the Demazure element

$$X_\alpha = \frac{1}{x_\alpha}(1 - \delta_{s_\alpha}).$$

Then

$$X_\alpha \cdot z = \Delta_\alpha(z).$$

Denote $X_i = X_{\alpha_i}$.

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2. $X_i q = s_i(q) X_i + \Delta_i(q), q \in Q$.
3. $\underbrace{X_i X_j X_i \cdots}_{m_{ij}} - \underbrace{X_j X_i X_j \cdots}_{m_{ij}} =$ extra terms, where $(s_i s_j)^{m_{ij}} = 1$.

The formal affine Demazure algebra

Definition (Hoffnung–MalagónLópez–Savage–Zainoulline)

We define the *formal affine Demazure algebra*

$$\mathbf{D}_F = R < S, X_\alpha | \alpha \in \Sigma > \subset Q_W.$$

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For each $w = s_{i_1} \cdots s_{i_k}$, let $I_w = (i_1, ..., i_k)$ and

$$X_{I_w} = X_{i_1} \cdots X_{I_k}.$$

It depends on the choice of I_w unless F is F_a or F_m.

Theorem (Calmès-Zainoulline-Z.)

$\mathbf{D}_F$ *is a free* S*-module with basis* $\{X_{I_w}\}_{w \in W}$

1. $\mathbf{D}_{F_a}$ = affine nil-Hecke algebra
2. $\mathbf{D}_{F_m}$ = affine 0-Hecke algebra

Both were constructed by Kostant-Kumar

$h_T(G/B)$

Recall that $S_W = S \rtimes R[W]$.

Definition

$\mathbf{D}_F^* = Hom_S(\mathbf{D}_F, S)$ and $S_W^* = Hom_S(S_W, S) \cong Hom(W, S)$.

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Definition

$\mathbf{D}_F^* = Hom_S(\mathbf{D}_F, S)$ and $S_W^* = Hom_S(S_W, S) \cong Hom(W, S)$.

Let $\{f_w\}_{w \in W}$ be the standard basis of S_W^* with product

$$f_v f_w = \delta_{v,w} f_v, \quad v, w \in W.$$

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Theorem (Calmes-Zainoulline-Z.)

$h_T(G/B) \cong \mathbf{D}_F^*$, and the embedding $\mathbf{D}_F^* \hookrightarrow S_W^*$, corresponds to $h_T(G/B) = h_T(G/T) \to h_T((G/T)^T) = h_T(W) \cong S_W^*$. Moreover,

$$\mathbf{D}_F^* = \{\sum_{w\in W} q_w f_w \in S_W^* | \frac{q_w - q_{s_\alpha w}}{x_\alpha} \text{ for all } \alpha \in \Sigma\}.$$

$h_T(G/B)$

Definition

For $J \subset \Pi$, define

$$x_J = \prod_{\alpha \in \Sigma_J^-} x_\alpha \in S,$$

$$Y_J = \sum_{w \in W_J} \delta_w \frac{1}{x_J} \in \mathbf{D}_F$$

$$[pt] = x_\Pi f_e \in \mathbf{D}_F^* \subset S_W^*,$$

$$\mathbf{1} = \sum_{w \in W} f_w \in \mathbf{D}_F^* \subset S_W^*$$

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Via $\mathbf{D}_F^* \cong h_T(G/B)$,

$$\begin{aligned} [pt] &= \text{the class of the identity point} \\ \mathbf{1} &= [G/B]. \end{aligned}$$

$h_T(G/B)$

$\mathbf{D}_F$ acts on $\mathbf{D}_F^*$ by

$$(z \bullet f)(z') = f(z'z), \quad z, z' \in \mathbf{D}_F.$$

$$(p\delta_v) \bullet (qf_w) = qwv^{-1}(p)f_{wv^{-1}}, \quad w, v \in W, p, q \in S.$$

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Theorem (Calmes-Zainoulline-Z.)

1. $(\mathbf{D}_F^*)^{W_J} \cong h_T(G/P_J)$
2. $Y_J \bullet \_ : \mathbf{D}_F^* \to (\mathbf{D}_F^*)^{W_J} \subset \mathbf{D}_F^*$ *gives* $h_T(G/B) \to h_T(G/P_J)$.
3. $\mathbf{D}_F^*$ *is a free* $\mathbf{D}_F$*-module via the* $\bullet$*-action, with basis* $[pt]$.

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The Bott–Samelson class is

$$X_{I_w^{-1}} \bullet [pt] \in \mathbf{D}_F^* \cong h_T(G/B).$$

Theorem (Calmes-Zainoulline-Z.)

The equivariant characteristic map $h_T(Spec(k)) \to h_T(G/B)$ *is*

$$c_S : S \to \mathbf{D}_F^*, \quad q \mapsto q \bullet \mathbf{1}.$$

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We define

$$\rho : S \otimes_{S^W} S \to \mathbf{D}_F^*, q_1 \otimes q_2 \mapsto q_1 c_S(q_2) \in \mathbf{D}_F^*.$$

Theorem (Calmes-Zainoulline-Z.)

If $R \supset \mathbb{Q}$, or for type A, C, or if $F = F_m$, ρ is an isomorphism.

We define another action of $\mathbf{D}_F$ on $\mathbf{D}_F^*$ by

$$q\delta_w \odot (pf_v) = qw(p)f_{wv}, \quad q, p \in S, w, v \in W.$$

For $z \in \mathbf{D}_F$,

$$\begin{array}{ccc} S \otimes_{S^W} S & \xrightarrow{\rho} & \mathbf{D}_F^* \\ {\scriptstyle (z\cdot -)\otimes 1}\downarrow & & \downarrow{\scriptstyle z\odot -} \\ S \otimes_{S^W} S & \xrightarrow{\rho} & \mathbf{D}_F^* \end{array} \qquad \begin{array}{ccc} S \otimes_{S^W} S & \xrightarrow{\rho} & \mathbf{D}_F^* \\ {\scriptstyle 1\otimes(z\cdot -)}\downarrow & & \downarrow{\scriptstyle z\bullet -} \\ S \otimes_{S^W} S & \xrightarrow{\rho} & \mathbf{D}_F^*. \end{array}$$

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$\bullet$ commutes with $\odot$.

- $\bullet$: the right Hecke action
- $\odot$: the left Hecke action.

The $\odot$-action for singular cohomology was studied by Brion, Knutson, Peterson, Tymoczko.

$h_T(G/P_J)$

Theorem (Lenart–Zainoulline–Z.)

$(\mathbf{D}_F^*)^{W_J}$ is a $\mathbf{D}_F$-module via the $\odot$ action, generated by $Y_J \bullet pt \in (\mathbf{D}_F^*)^{W_J}$.

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The Bott–Samelson class of $h_T(G/P_J)$ is given by

$$X_{I_w} \odot (Y_J \bullet [pt]) = Y_J X_{I_w^{-1}} \bullet [pt], \quad w \in W^J.$$

Remark

- *There is an isomorphism* $(h_T(G/B), \circ) \cong \mathbf{D}_F$, *which gives some relation between some integral representation category of* $\mathbf{D}_F$ *and the category of Chow motives (Neshitov-Petrov-Semenov-Zainoulline).*
- *We are trying to generalize the above construction to the Kac-Moody setting (Calmes-Zainoulline-Z.)*
- *There is a parallel construction of formal affine Hecke algebra* $\mathbf{H}_F$ *for* F, *which generalizes the affine degenerate Hecke algebra (for* F_a*) and the affine Hecke algebra (for* F_m*). It is isomorphic to* $h_{G\times G_m}(Z)$ *where* Z *is the Steinberg variety. (G. Zhao–Z.)*
- *For elliptic formal group law,* H_F *is isomorphic to the stalk of Ginzburg-Kapranov-Vasserot's elliptic Hecke algebra. (G. Zhao–Z.)*

[Deodhar(1987), Lenart-Zainoulline-Z (to appear soon).]
There is chain complex of $\mathbf{D}_F$-modules

$$0 \to h_T(G/B) \to^{d_0} \bigoplus_{|J|=1} h_T(G/P_J) \to^{d_1} \bigoplus_{|J|=2} h_T(G/P_J) \to^{d_2} \cdots$$

d_i is alternating sum of

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It is exact except at $h_T(G/B)$, whose cohomology is a free S-module of rank 1 in "some" cases, generated by $X_{w_0} \bullet [pt]$.