COMINUSCULE RICHARDSON VARIETIES

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Abstract. A Richardson variety in a cominuscule flag variety is defined by a skew shape of boxes. We prove that any cominuscule Richardson variety has a unique decomposition as a product of indecomposable cominuscule Richardson varieties, which are given by the connected components of the skew shape. We also prove that the first Chow group and the Picard group of a cominuscule Richardson variety are dual to each other; both are free abelian groups of rank equal to the number of indecomposable factors. Along the way, we construct an explicit desingularization of any cominuscule Richardson variety of classical Lie type as a tower of (isotropic) Grassmannian bundles. Our results imply that the Hilbert polynomial of the minimal ample line bundle is a well-defined geometric invariant of any cominuscule Richardson variety. We give a positive formula for this Hilbert polynomial.

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1. Introduction

Let $X = G/P_X$ be a flag variety defined by a reductive linear algebraic group G and a parabolic subgroup P_X. Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P_X \subset G$. Each Weyl group element $u \in W$ defines the Schubert varieties $X_u = \overline{Bu.P_X}$ and $X^u = \overline{B^- u.P_X}$. A non-empty intersection $X_u^v = X_u \cap X^v$ of opposite Schubert varieties is called a *Richardson variety*. In this paper we study these Richardson varieties when X is a cominuscule flag variety. We will show that any cominuscule Richardson variety X_u^v has a unique factorization as a product of *indecomposable* cominuscule Richardson varieties. In addition, we prove that the first Chow group $A_1(X_u^v)$ and the Picard group $\mathrm{Pic}(X_u^v)$ are dual to each other; both are free abelian groups of rank equal to the number of indecomposable factors of X_u^v. The family of cominuscule flag varieties consists of Grassmannians of type

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A, Lagrangian Grassmannians, maximal orthogonal Grassmannians, quadric hypersurfaces, and two exceptional varieties of Lie type E called the Cayley Plane E_6/P_6 and the Freudenthal variety E_7/P_7 (see Table 1). More generally, any product of these varieties will be considered a cominuscule flag variety. Our computation of the Picard groups of cominuscule Richardson varieties of type E uses a new result of Anderson and Payne [AP26].

Some properties of Richardson varieties can be proved from the corresponding properties of Schubert varieties. In particular, classical results show that Schubert varieties are normal, Cohen-Macaulay, and have rational singularities [MR85, RR85, Ram85]. These properties are inherited by Richardson varieties, thanks to a generalization of Kleiman's transversality theorem [Kle74, Bri02, KLS14]. Both Schubert varieties and Richardson varieties are rational varieties. Moreover, any Schubert variety X_u is stratified by the B-orbits it contains, each of which is isomorphic to an affine space. Each Chow group $A_d(X_u)$ is therefore a free abelian group generated by the classes of all d-dimensional Schubert varieties contained in X_u. The rationality of Richardson varieties follows from Deodhar's decomposition [Deo77], which expresses any Richardson variety as a disjoint union of locally closed cells of the form $\mathbb{A}^n \times (\mathbb{A}^1 \smallsetminus \{0\})^m$. However, Deodhar's decomposition is not a stratification, and the Chow groups of Richardson varieties are only known in special cases (see, e.g., [GLL26] for the case of toric Richardson varieties in Grassmannians).

The Picard group $\mathrm{Pic}(X_u)$ of any Schubert variety was obtained by Mathieu [Mat88]. The usual strategy for computing the Picard group of a normal variety Y is first to compute the divisor class group $\mathrm{C}\ell(Y) = A_{\dim(Y)-1}(Y)$ and then identify $\mathrm{Pic}(Y)$ as a subgroup of $\mathrm{C}\ell(Y)$. However, aside from toric Richardson varieties and Schubert varieties, the divisor class group $\mathrm{C}\ell(X_u^v)$ appears to be unknown. The fact that X_u^v is rational implies that $\mathrm{C}\ell(X_u^v)$ is finitely generated. The only other result in this direction that we are aware of is that, when $X = \mathrm{Gr}(m,n)$ is a Grassmannian of type A, the class group $\mathrm{C}\ell(\mathring{X}_u^v)$ of any open Richardson variety $\mathring{X}_u^v = Bu.P_X \cap B^- v.P_X$ is zero [LP22].

Given any complete variety Y, the natural pairing $\mathrm{Pic}(Y) \otimes A_1(Y) \to \mathbb{Z}$ given by $L \otimes [C] \mapsto \deg(L|_C)$ defines a group homomorphism $\mathrm{Pic}(Y) \to A_1(Y)^*$ that we call the *degree map* of Y. Here $A_1(Y)^* = \mathrm{Hom}_{\mathbb{Z}}(A_1(Y), \mathbb{Z})$ denotes the dual of the first Chow group. Rather than relying on the class group, we show that the degree map $\mathrm{Pic}(X_u^v) \to A_1(X_u^v)^*$ is an isomorphism for any cominuscule Richardson variety X_u^v. We note that Mathieu obtained the Picard group $\mathrm{Pic}(X_u)$ of a Schubert variety as a subgroup of the class group $\mathrm{C}\ell(X_u)$, but he also proved [Mat88, Prop. 6] that the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_u)$ is surjective and has the same kernel as the degree map $\mathrm{Pic}(X) \to A_1(X_u)^*$; this implies $\mathrm{Pic}(X_u) \cong A_1(X_u)^*$, as the degree map of a Schubert variety is clearly surjective. While this isomorphism generalizes to cominuscule Richardson varieties, the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_u^v)$ generally fails to be surjective for Richardson varieties.

The flag variety $X = G/P_X$ is called *cominuscule* if P_X is a maximal parabolic subgroup defined by a cominuscule simple root γ, that is, the coefficient of γ in the highest root is 1. We assume this in the following. The combinatorics of X is then centered around the partially ordered set $\mathcal{P}_X = \{\alpha \in \Phi \mid \alpha \geq \gamma\}$, where Φ is the root system, and we write $\alpha \geq \beta$ if $\alpha - \beta$ is a sum of simple roots (see Table 1). In particular, a result of Proctor [Pro84] shows that the subset $W^X \subset W$

of minimal length representatives is in bijective correspondence with the set of lower order ideals in $\mathcal{P}_X$; this bijection sends any element $u \in W^X$ to its inversion set $I(u) = \{\alpha \in \Phi^+ \mid u.\alpha \in \Phi^-\}$. The lower order ideals in $\mathcal{P}_X$ generalize the (shifted) Young diagrams used for classical Grassmannians. For this reason, the order ideals will be called *straight shapes* and the roots in $\mathcal{P}_X$ will be called *boxes*. Given a straight shape $\lambda \subset \mathcal{P}_X$, we let $w_\lambda \in W^X$ denote the element with $I(w_\lambda) = \lambda$, and write $X_\lambda = \overline{Bw_\lambda.P_X}$ and $X^\lambda = \overline{B^- w_\lambda.P_X}$ for the associated Schubert varieties. We then have $\dim(X_\lambda) = \operatorname{codim}(X^\lambda, X) = |\lambda|$, and the Bruhat order on X is given by

$$\mu \subset \lambda \;\Leftrightarrow\; X_\mu \subset X_\lambda \;\Leftrightarrow\; X^\mu \supset X^\lambda \;\Leftrightarrow\; X_\lambda \cap X^\mu \neq \emptyset\,.$$

Given straight shapes $\mu \subset \lambda \subset \mathcal{P}_X$, the associated *cominuscule Richardson variety* is denoted by $X_\lambda^\mu = X_\lambda \cap X^\mu$. This variety depends only on the skew shape $\theta = \lambda/\mu := \lambda \smallsetminus \mu$ up to translation; we let X_θ denote the isomorphism class of X_λ^μ as an algebraic variety.

An irreducible variety of positive dimension will be called *indecomposable* if it is not a product of varieties of strictly smaller dimension. Our main results are the following theorems.

Theorem 1.1. *Let X be a cominuscule flag variety and $\theta \subset \mathcal{P}_X$ a skew shape. Then the following are equivalent.* (a) *θ is connected (as a poset).* (b) *X_θ is indecomposable.* (c) *$A_1(X_\theta) \cong \mathbb{Z}$.* (d) *$\operatorname{Pic}(X_\theta) \cong \mathbb{Z}$.*

Theorem 1.2. *Let X be a cominuscule flag variety, let $\theta \subset \mathcal{P}_X$ be a skew shape, and let $\theta_1, \dots, \theta_r$ be the connected components of θ.*

(a) *We have $X_\theta \cong X_{\theta_1} \times \cdots \times X_{\theta_r}$.*

(b) *The map $A_1(X_\theta) \to A_1(X_{\theta_1}) \oplus \cdots \oplus A_1(X_{\theta_r})$ defined by proper pushforward to the factors is an isomorphism of groups.*

(c) *The map $\operatorname{Pic}(X_{\theta_1}) \oplus \cdots \oplus \operatorname{Pic}(X_{\theta_r}) \to \operatorname{Pic}(X_\theta)$ defined by pullback from the factors is an isomorphism of groups.*

(d) *The degree map $\operatorname{Pic}(X_\theta) \to A_1(X_\theta)^*$ is an isomorphism of groups.*

In particular, both $A_1(X_\theta)$ and $\operatorname{Pic}(X_\theta)$ are free abelian groups of rank equal to the number of indecomposable factors of X_θ. Part (a) of Theorem 1.2 was proved for cominuscule Grassmannians of classical Lie types in [BR12]. We obtain it as a special case of a factorization result for Richardson varieties in arbitrary flag varieties (see Proposition 3.11).

In a paper in preparation with Ayah Almousa and Daoji Huang [ABH], we have generalized the above results to all positroid varieties and used this to determine which positroid varieties are Gorenstein.

Given a cominuscule Richardson variety $X_\theta \subset X$, we define the *minimal embedding* of X_θ to be $L.X_\theta$, where $L \subset G$ is the minimal Levi subgroup such that X_θ is contained in a single L-orbit. We show that this minimal embedding is itself a Richardson variety in X, and also a cominuscule flag variety of L; if X_θ has several indecomposable factors, then $L.X_\theta$ will be a product of several indecomposable cominuscule flag varieties.

Theorem 1.3. *Let $i : X_\theta \subset X$ be a cominuscule Richardson variety such that X is the minimal embedding of X_θ. Then the pushforward map $i_* : A_1(X_\theta) \to A_1(X)$ and the pullback map $i^* : \operatorname{Pic}(X) \to \operatorname{Pic}(X_\theta)$ are isomorphisms.*

When $\theta \subset \mathcal{P}_X$ is any connected skew shape, the pushforward $A_1(X_\theta) \to A_1(X)$ and pullback $\mathrm{Pic}(X) \to \mathrm{Pic}(X_\theta)$ are always injective. Both maps are isomorphisms if and only if θ contains a long box (or the root system Φ is simply-laced); otherwise the images of the maps are $2A_1(X)$ and $2\,\mathrm{Pic}(X_\theta)$, respectively.

A result of Hirschowitz [Hir73] implies that the first Chow group $A_1(X_\theta)$ is generated by the T-stable curves contained in X_θ. To compute the group $A_1(X_\theta)$, we must account for the relations. We will say that two T-fixed points in X are *adjacent* if they are joined by a T-stable curve; in this case the T-stable curve is unique, and the points are comparable in the Bruhat order of X^T. Our computation of $A_1(X_\theta)$ is based on the following result. Part (a) holds for all flag varieties of type different from G_2, whereas part (b) is true only if X is cominuscule or has type G_2 and Picard rank 1.

Proposition 1.4. *Let X be a cominuscule flag variety.*

(a) *Assume that $x, y, z \in X^T$ are pairwise adjacent T-fixed points. Then there exists a unique surface $\Omega \subset X$ with fixed locus $\Omega^T = \{x, y, z\}$. Moreover, Ω is a translate of a Schubert variety.*

(b) *Let $\tau \subset \sigma \subset \mathcal{P}_X$ be straight shapes such that $w_\tau.P_X$ and $w_\sigma.P_X$ are adjacent T-fixed points, and assume that $|\sigma/\tau| \geq 2$. Then there exists a straight shape κ such that $\tau \subsetneq \kappa \subsetneq \sigma$ and $w_\kappa.P_X$ is adjacent to both $w_\tau.P_X$ and $w_\sigma.P_X$.*

We also use the fact that any Richardson variety $X_\theta \subset X$ is *T-convex* in X [BCP25], that is, if $\Omega \subset X$ is any T-stable closed subvariety such that $\Omega^T \subset X_\theta$, then $\Omega \subset X_\theta$. Part (a) of the proposition therefore implies that, if three T-stable curves contained in X_θ form a triangle, then they define the same class in $A_1(X_\theta)$, up to a constant. Indeed, the surface Ω defined by the vertices is contained in X_θ since X_θ is T-convex, and the three curves are contained in Ω since Ω is itself T-convex. Using this, part (b) implies that $A_1(X_\theta)$ is generated by the classes of all Richardson curves in X_θ. When θ is connected, it follows from similar arguments that these Richardson curves generate a subgroup of rank 1 in $A_1(X_\theta)$. Then the pushforward map $A_1(X_\theta) \to A_1(X)$ is injective, and in fact an isomorphism when θ contains a long box. The surjectivity of the degree map $\mathrm{Pic}(X_\theta) \to A_1(X_\theta)^*$ also follows, using that the degree map of X is surjective.

To prove that the degree map $\mathrm{Pic}(X_\theta) \to A_1(X_\theta)^*$ is injective, we use the existence of a desingularization $\pi : \widetilde{X_\theta} \to X_\theta$ such that the degree map of $\widetilde{X_\theta}$ is injective. In fact, these degree maps fit in a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(X_\theta) & \xrightarrow{\pi^*} & \mathrm{Pic}(\widetilde{X_\theta}) \\ \downarrow & & \downarrow \\ A_1(X_\theta)^* & \xrightarrow{(\pi_*)^*} & A_1(\widetilde{X_\theta})^* \,, \end{array}$$

and the normality of X_θ implies that π^* is injective. For cominuscule Richardson varieties of classical types, we explicitly construct the desingularization $\widetilde{X_\theta}$ as a tower of (isotropic) Grassmannian bundles, in the style of Zelevinsky's desingularization of Schubert varieties in Grassmannians of type A [Zel83]. This desingularization is of independent interest, as it is simpler and more explicit than the usual 'brick' resolution studied in [Bri02, Esc16]. In response to our questions, David Anderson and Sam Payne have proved that the brick resolution indeed satisfies the required

injectivity property (and much more!). We rely on this result to complete the computation of the Picard groups of cominuscule Richardson varieties of type E. We also use this result for Richardson varieties in the orthogonal Grassmannian $\mathrm{OG}(n, 2n)$ when the ground field has characteristic 2.

One source of motivation for this paper is the problem of classifying cominuscule Richardson varieties up to isomorphism as algebraic varieties. Richmond and Slofstra have previously obtained such a classification of the Schubert varieties in full flag varieties G/B [RS22]. The classification of the Schubert varieties in Grassmannians of type A was obtained in [TX23] and generalized to all cominuscule Schubert varieties in [RTX24].

Define a *long/short poset* to be a partially ordered set $\mathcal{P}$ together with a partition $\mathcal{P} = \mathcal{P}_{\mathrm{long}} \coprod \mathcal{P}_{\mathrm{short}}$ into disjoint subsets of long and short boxes, such that every connected component of $\mathcal{P}$ has non-empty intersection with $\mathcal{P}_{\mathrm{long}}$. It was proved in [RTX24] that two cominuscule Schubert varieties X_λ and $X'_{\lambda'}$ are isomorphic if and only if the associated straight shapes $\lambda \subset \mathcal{P}_X$ and $\lambda' \subset \mathcal{P}_{X'}$ are isomorphic as long/short posets. Knowledge of the Picard group $\mathrm{Pic}(X_\lambda)$ was important in this work; the Schubert classes in the total Chow group $A_*(X_\lambda)$ can be identified as the generators of the cone of effective classes, after which the action of the Chern class $c_1(\mathcal{L})$ of the ample generator of $\mathrm{Pic}(X_\lambda)$ reveals the Bruhat order, and λ is recovered as the subset of join-irreducible elements in the Bruhat order.

It follows from Theorem 1.2 that two cominuscule Richardson varieties are isomorphic if and only if their indecomposable factors are isomorphic. This reduces the classification problem to that of classifying the indecomposable cominuscule Richardson varieties. The following conjecture was posed in [TX23] for Richardson varieties in Grassmannians of type A.

Conjecture 1.5. *Let X and X' be cominuscule flag varieties and let $\theta \subset \mathcal{P}_X$ and $\theta' \subset \mathcal{P}_{X'}$ be connected skew shapes. Then the Richardson varieties X_θ and $X'_{\theta'}$ are isomorphic as varieties if and only if θ and θ' are isomorphic or anti-isomorphic as long/short posets.*

An *anti-isomorphism* of long/short posets reverses the order but preserves the lengths of boxes. In a sequel [Buc] to this paper we will prove one direction: if θ and θ' are isomorphic or anti-isomorphic as long/short posets, then X_θ and $X'_{\theta'}$ are isomorphic as varieties. We have also verified by computer that Conjecture 1.5 is true when θ and θ' have at most 15 boxes.

The determination of the Picard group of a cominuscule Richardson variety X_θ makes it possible to define related geometric invariants that may be helpful for the classification problem. One example is the Hilbert polynomial defined by the minimal ample line bundle in $\mathrm{Pic}(X_\theta)$; this line bundle exists and is unique by Theorem 1.2. In Section 7 we give a simple recursive formula for this Hilbert polynomial, which shows that it has a positive expansion in the binomial basis.

This paper is organized as follows. In Section 2 we fix our notation for flag varieties and recall some standard definitions. We also recall the construction of flag varieties of classical Lie types as varieties of (isotropic) flags in a fixed vector space. In Section 3 we define flag subvarieties and minimal embeddings of Richardson varieties in arbitrary flag varieties, and we prove a decomposition theorem that expresses certain Richardson varieties as products of smaller Richardson varieties. Section 4 interprets these results for cominuscule flag varieties. We also prove that the minimal embedding of a cominuscule Richardson variety is a cominuscule

flag variety. Section 5 determines the first Chow group and the Picard group of a cominuscule Richardson variety. Section 6 constructs a desingularization of any cominuscule Richardson variety of classical Lie type as a tower of (isotropic) Grassmannian bundles. Finally, Section 7 computes the Hilbert polynomial defined by the minimal ample line bundle on any cominuscule Richardson variety.

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2. Flag varieties

In this section we fix our notation for flag varieties and recall some results.

2.1. **General notation.** Let G be a connected reductive linear algebraic group over an algebraically closed field $\mathbb{K}$, and choose a maximal torus T and a Borel subgroup B such that $T \subset B \subset G$. The opposite Borel subgroup $B^- \subset G$ is defined by $B \cap B^- = T$. Let Φ be the root system of (G, T), with positive roots Φ^+ and simple roots $\Delta \subset \Phi^+$ given by B. We will not assume that Φ is irreducible. We identify Δ with the set of nodes in the Dynkin diagram of Φ. Let $W = N_G(T)/T$ be the Weyl group of G. The reflection along a root $\alpha \in \Phi$ is denoted s_α. The inversion set of $u \in W$ is $I(u) = \{\alpha \in \Phi^+ \mid u.\alpha < 0\}$. We have $\ell(u) = \#I(u)$.

A complete homogeneous G-variety X will be called a *flag variety* of G if the morphism $G \to X$, $g \mapsto g.x$ defined by any point $x \in X$ is separable. Any such flag variety X contains a unique B-fixed point. We denote the parabolic subgroup stabilizing this point by $P_X \subset G$ and the point itself by $1.P_X$. Each element $g \in G$ defines a point $g.P_X = g.(1.P_X)$ in X. We identify X with the quotient $G/P_X = \{g.P_X \mid g \in G\}$. Let $W_X = N_{P_X}(T)/T$ be the Weyl group of P_X, and let $W^X \subset W$ be the subset of minimal representatives of the cosets in W/W_X. Each element $u \in W$ defines a T-fixed point $u.P_X \in X^T$ and the *Schubert varieties* $X_u = \overline{Bu.P_X}$ and $X^u = \overline{B^- u.P_X}$. These Schubert varieties depend only on the coset uW_X in W/W_X, and for $u \in W^X$ we have $\dim(X_u) = \operatorname{codim}(X^u, X) = \ell(u)$. Let $\leq$ denote the Bruhat order on W. For $u, v \in W^X$ we have

$$v \leq u \quad \Leftrightarrow \quad X_v \subset X_u \quad \Leftrightarrow \quad X^v \supset X^u \quad \Leftrightarrow \quad X_u \cap X^v \neq \emptyset\,.$$

In this case the intersection $X_u^v = X_u \cap X^v$ is called a *Richardson variety*; this variety is reduced, irreducible, rational, and $\dim(X_u^v) = \ell(u) - \ell(v)$; see [Kle74, Deo77] and [BK05, §2]. In addition, X_u^v is normal and has rational singularities [MR85, RR85, Ram85, Bri02, KLS14]. It was proved in [BCP25, Thm. 6.3] that X_u^v is T-*convex* in X, that is, if $Z \subset X$ is any T-stable closed subvariety such that the fixed point set Z^T is contained in X_u^v, then $Z \subset X_u^v$. Unless stated otherwise, a Richardson variety in X will always be defined relative to the chosen Borel subgroups B and B^-.

Each element $u \in W$ has a unique factorization $u = u^X u_X$ for which $u^X \in W^X$ and $u_X \in W_X$, called the *parabolic factorization* of u with respect to W_X. This factorization is *reduced* in the sense that $\ell(u) = \ell(u^X) + \ell(u_X)$. The parabolic factorization of the longest element $w_0 \in W$ is denoted $w_0 = w_0^X w_{0,X}$, where w_0^X is the longest element in W^X and $w_{0,X}$ is the longest element in W_X. Notice that $u \in W^X$ if and only if $u < us_\beta$ holds for all $\beta \in \Delta$ with $s_\beta \in W_X$.

2.2. **Flag varieties of classical Lie types.** In the rest of this section we fix our notation for flag varieties of the classical Lie types A_n, B_n, C_n, and D_n. This notation will be used in examples as well as in some type-dependent proofs.

The root system Φ is identified with a subset of $\mathbb{R}^{n'}$, where $n' = n+1$ in type A_n, and $n' = n$ in the other classical types. The basis of simple roots will be denoted $\Delta = \{\beta_1, \ldots, \beta_n\}$ with the standard numbering (see [Hum78, §11.4] or Table 1). Let $\{\epsilon_1, \ldots, \epsilon_{n'}\}$ be the coordinate basis of $\mathbb{R}^{n'}$. We set $\beta_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i \le n-1$. The last simple root β_n depends on the type and is given in the following table together with Φ. The root systems B_2 and C_2 are isomorphic, as are A_3 and D_3.

Type	Rank	Root system Φ	Last simple root
A_n	$n \ge 1$	$\{\epsilon_i - \epsilon_j \mid i \ne j \in [1, n+1]\}$	$\beta_n = \epsilon_n - \epsilon_{n+1}$
B_n	$n \ge 2$	$\{\epsilon_i - \epsilon_j,\ \epsilon_i + \epsilon_j,\ \epsilon_i \mid i \ne j \in [1, n]\}$	$\beta_n = \epsilon_n$
C_n	$n \ge 2$	$\{\epsilon_i - \epsilon_j,\ \epsilon_i + \epsilon_j,\ 2\epsilon_i \mid i \ne j \in [1, n]\}$	$\beta_n = 2\epsilon_n$
D_n	$n \ge 3$	$\{\epsilon_i - \epsilon_j,\ \epsilon_i + \epsilon_j \mid i \ne j \in [1, n]\}$	$\beta_n = \epsilon_{n-1} + \epsilon_n$

The Weyl group W is identified with a subgroup of the permutation group S_N, where N depends on the Lie type and is given by the table below. When Φ has type A_n, we have $W = S_N$. When Φ has type B_n, C_n, or D_n, it is convenient to write $\bar{i} = N + 1 - i$ for $i \in [1, N]$. In type A_n we set $\bar{i} = i$ for $i \in [1, N]$. Then $W \subset S_N$ is the subgroup of permutations satisfying the condition in the following table.

Type	N	W is the set of permutations $w \in S_N$ satisfying
A_n	$n+1$	$w \in S_N$
B_n	$2n+1$	$w(\bar{i}) = \overline{w(i)}$ for $i \in [1, N]$
C_n	$2n$	$w(\bar{i}) = \overline{w(i)}$ for $i \in [1, N]$
D_n	$2n$	$w(\bar{i}) = \overline{w(i)}$ for $i \in [1, N]$ and $\#\{i \in [1, n] : w(i) > n\}$ is even

For $i, j \in [1, N]$, we let $\tau_{i,j} \in S_N$ be the permutation defined by $\tau_{i,j}(i) = j$, $\tau_{i,j}(j) = i$, $\tau_{i,j}(\bar{i}) = \bar{j}$, $\tau_{i,j}(\bar{j}) = \bar{i}$, and $\tau_{i,j}(k) = k$ for $k \in [1, N] \smallsetminus \{i, j, \bar{i}, \bar{j}\}$. The reflection $s_\alpha \in W$ defined by each root $\alpha \in \Phi$ is given by the following table. The simple reflections are denoted $s_i = s_{\beta_i}$ for $i \in [1, n]$.

Root $\alpha \in \Phi$	$\epsilon_i - \epsilon_j$	$\epsilon_i + \epsilon_j$	$\epsilon_i,\ 2\epsilon_i$
Reflection $s_\alpha \in W$	$\tau_{i,j}$	$\tau_{i,\bar{j}}$	$\tau_{i,\bar{i}}$

For $w \in W$ and $1 \le i \le n$, we have $w < ws_i$ in the Bruhat order if and only if $w(i) < w(i+1)$, with one exception: If Φ has type D_n and $i = n$, then $w < ws_n$ holds if and only if $w(n-1) < w(n+1)$.

Any element $w \in W$ can be represented in *one-line notation* by the sequence of numbers $w(1)\, w(2) \cdots w(n')$, where 'bar' notation is used for integers larger than n'. For example, $w = 4\,\bar{1}\,2\,\bar{5}\,\bar{3}$ means $w(1) = 4$, $w(2) = \bar{1}$, $w(3) = 2$, $w(4) = \bar{5}$, and $w(5) = \bar{3}$. Notice that since the number of bars is odd, this element belongs to W only when Φ has type B_5 or C_5. Notice also that $\bar{3} = 11 + 1 - 3 = 9$ when Φ has type B_5, whereas $\bar{3} = 10 + 1 - 3 = 8$ when Φ has type C_5.

Flag varieties of the classical Lie types will be constructed as spaces of isotropic flags in $\mathbb{K}^N$. The standard construction used here requires that the field $\mathbb{K}$ does not

have characteristic 2 when Φ has type B_n or D_n. We first define a bilinear form ω on $\mathbb{K}^N$ which is zero in type A_n, is non-degenerate and symmetric in types B_n and D_n, and is non-degenerate and anti-symmetric in type C_n. Let $\{e_1, \ldots, e_N\}$ be the coordinate basis of $\mathbb{K}^N$. When ω is not zero, it is determined by $\omega(e_i, e_j) = \delta_{i+j,N+1}$ for $1 \leq i \leq j \leq N$. Let $G \subset \mathrm{SL}(N)$ be the subgroup preserving ω, that is

$$G = \{g \in \mathrm{SL}(N) \mid \omega(g.x, g.y) = \omega(x, y) \ \forall\, x, y \in \mathbb{K}^N\}\,.$$

This group is $\mathrm{SL}(N)$ in type A_n, is the special orthogonal group $\mathrm{SO}(N)$ in types B_n and D_n, and is the symplectic group $\mathrm{Sp}(N)$ in type C_n. Let $T \subset G$ be the maximal torus of all diagonal matrices in G, let $B \subset G$ be the Borel subgroup of upper triangular matrices, and let $B^- \subset G$ be the lower triangular matrices.

A vector subspace $V \subset \mathbb{K}^N$ is called *isotropic* if the bilinear form ω vanishes on V, that is $\omega(V, V) = 0$. This condition implies $\dim(V) \leq n'$. All subspaces $V \subset \mathbb{K}^N$ are isotropic when Φ has type A_n. Given a subset $a = \{a_1 < a_2 < \cdots < a_r\}$ of $[0, n'] = \{0, 1, \ldots, n'\}$, an a-dimensional isotropic *flag* $V_\bullet \subset \mathbb{K}^N$ is an increasing sequence $V_\bullet = (V_{a_1} \subset V_{a_2} \subset \cdots \subset V_{a_r})$ of isotropic subspaces of $\mathbb{K}^N$ with $\dim(V_{a_i}) = a_i$ for $1 \leq i \leq r$. Note that the steps of a flag are indexed by their dimensions. Let $F_\omega(a; N) = F_\omega(a_1, \ldots, a_r; N)$ denote (one connected component of) the variety of all a-dimensional isotropic flags in $\mathbb{K}^N$. Then $F_\omega(a; N)$ is a flag variety of G. Given vectors $v_1, \ldots, v_\ell \in \mathbb{K}^N$, we will use the notation $\langle v_1, \ldots, v_\ell\rangle = \mathrm{Span}_{\mathbb{K}}\{v_1, \ldots, v_\ell\}$. The B-stable point of $X = F_\omega(a; N)$ is then given by

$$1.P_X = (\langle e_1, \ldots, e_{a_1}\rangle \subset \langle e_1, \ldots, e_{a_2}\rangle \subset \cdots \subset \langle e_1, \ldots, e_{a_r}\rangle)\,.$$

More generally, $V_\bullet \in F_\omega(a; N)$ is a T-fixed point if and only if each subspace V_{a_i} is spanned by a subset of the basis $\{e_1, \ldots, e_N\}$. The variety of all isotropic flags of dimension a is connected unless Φ has type D_n and $a_r = n$; in this case there are two components, and we let $F_\omega(a; N)$ be the component containing $1.P_X$. In type D_n we also have $F_\omega(a; N) \cong F_\omega(a \cup \{n\}; N)$ whenever $n - 1 \in a$, so we will only consider dimension sets $a = \{a_1, \ldots, a_r\}$ satisfying $n - 1 \notin a$ or $n \in a$. When this convention is followed, the parabolic subgroup P_X defining $X = F_\omega(a; N)$ is determined in all Lie types by $s_i \in W^X$ if and only if $i \in \{a_1, \ldots, a_r\}$.

Let each permutation $w \in S_N$ act on $\mathbb{K}^N$ by $w(e_i) = e_{w(i)}$ for $i \in [1, N]$; this identifies S_N with the group of permutation matrices in $\mathrm{GL}(N)$. The action of W on the fixed point locus $F_\omega(a; N)^T$ is given by $w.V_\bullet = (w.V_{a_1} \subset \cdots \subset w.V_{a_r})$. The permutation matrix corresponding to $w \in W$ belongs to G if and only if $\det(w) = 1$. When $\det(w) = -1$, we obtain an element $w' \in G$ by switching the sign of any entry of the permutation matrix defined by w, in which case w is represented by w' as an element of $N_G(T)/T$.

Let $F_\omega(N) = F_\omega(1, 2, \ldots, n; N)$ denote the variety of complete isotropic flags in $\mathbb{K}^N$. The space $F_\omega(m; N)$, for $m \in [1, n]$, is the Grassmannian of m-dimensional isotropic subspaces $V \subset \mathbb{K}^N$. The following notation will be used to refer to flag varieties of all classical types.

Type	N	ω	$F_\omega(a; N)$	$F_\omega(N)$	$F_\omega(m; N)$
A_n	$n+1$	zero	$\mathrm{Fl}(a; N)$	$\mathrm{Fl}(N)$	$\mathrm{Gr}(m, N)$
B_n	$2n+1$	orthogonal	$\mathrm{OF}(a; N)$	$\mathrm{OF}(N)$	$\mathrm{OG}(m, N)$
C_n	$2n$	symplectic	$\mathrm{SF}(a; N)$	$\mathrm{SF}(N)$	$\mathrm{SG}(m, N)$
D_n	$2n$	orthogonal	$\mathrm{OF}(a; N)$	$\mathrm{OF}(N)$	$\mathrm{OG}(m, N)$

The symplectic Grassmannian $\mathrm{SG}(n,2n)$ of maximal isotropic subspaces in $\mathbb{K}^{2n}$ is called the *Lagrangian Grassmannian* and is also denoted $\mathrm{LG}(n,2n)$. The *quadric hypersurface* of dimension $d \geq 3$ is denoted $Q^d = \mathrm{OG}(1,d+2)$. We have $Q^3 \cong \mathrm{LG}(2,4)$ and $Q^4 \cong \mathrm{Gr}(2,4)$. In addition, the maximal orthogonal Grassmannians $\mathrm{OG}(n,2n)$ and $\mathrm{OG}(n-1,2n-1)$, of types D_n and B_{n-1}, are isomorphic by the map $f : \mathrm{OG}(n,2n) \cong \mathrm{OG}(n-1,2n-1)$ defined by $f(V) = \iota^{-1}(V)$, where $\iota : \mathbb{K}^{2n-1} \to \mathbb{K}^{2n}$ is the linear map given by $\iota(e_i) = e_i$ for $0 < i < n$, $\iota(e_i) = e_{i+1}$ for $n < i < 2n$, and $\iota(e_n) = \sqrt{1/2}\,(e_n + e_{n+1})$.

3. Product decomposition of Richardson varieties

In this section we let $X = G/P_X$ be an arbitrary flag variety. We will show that certain Richardson varieties in X are products of smaller Richardson varieties, generalizing the decomposition result in [BR12] for cominuscule Richardson varieties of classical types.

3.1. **The poset of a flag variety.** The root lattice $\mathrm{Span}_{\mathbb{Z}}(\Delta)$ has a partial order defined by $\alpha' \leq \alpha$ if and only if $\alpha - \alpha'$ is a sum of positive roots. Define the partially ordered set

$$\mathcal{P}_X \;=\; I(w_0^X) \;=\; \Phi^+ \smallsetminus I(w_{0,X}) \;=\; \{\alpha \in \Phi^+ \mid w_0^X.\alpha < 0\}\,,$$

where the partial order is inherited from $\mathrm{Span}_{\mathbb{Z}}(\Delta)$. Equivalently, $\mathcal{P}_X$ is the set of roots $\alpha \in \Phi$ for which $-\alpha$ is a T-weight of the tangent space $T_X(1.P_X)$. The cardinality of $\mathcal{P}_X$ is $\#\mathcal{P}_X = \dim(X)$, and the minimal elements of $\mathcal{P}_X$ are the excluded simple roots of P_X,

$$\min(\mathcal{P}_X) \;=\; \mathcal{P}_X \cap \Delta \;=\; \{\beta \in \Delta \mid s_\beta \in W^X\}\,.$$

The corresponding Schubert divisor classes $\{[X^{s_\beta}] \mid \beta \in \min(\mathcal{P}_X)\}$ form a $\mathbb{Z}$-basis of the Picard group $\mathrm{Pic}(X)$. The identity $\mathcal{P}_X = \Phi^+ \smallsetminus I(w_{0,X})$ implies that

$$\mathcal{P}_X \;=\; \{\alpha \in \Phi \mid \exists\, \beta \in \min(\mathcal{P}_X) : \beta \leq \alpha\}\,. \tag{1}$$

Remark 3.1. One can check that $\alpha_1 \leq \alpha_2$ is a covering relation in $\mathcal{P}_X$ if and only if $\alpha_2 - \alpha_1 \in \Delta$ is a simple root. We will not need this fact.

3.2. **Flag subvarieties.** Given a subset $S \subset \Delta$ of the simple roots, let $G_S \subset G$ be the Levi subgroup containing T with root system $\Phi_S = \mathrm{Span}_{\mathbb{Z}}(S) \cap \Phi$, see [Spr98, (8.4.1)]. Our chosen Borel subgroups of G_S are $B_S = G_S \cap B$ and $B_S^- = G_S \cap B^-$, and the positive roots defined by B_S are $\Phi_S^+ = \Phi_S \cap \Phi^+$. The Weyl group of G_S is the subgroup $W_S \subset W$ generated by the simple reflections $\{s_\beta \mid \beta \in S\}$.

Definition 3.2. Let $\kappa.P_X \in X$ be a T-fixed point and let $S \subset \Delta$. The G_S-orbit $G_S\kappa.P_X \subset X$ will be called a *flag subvariety* of X.

This definition of a flag subvariety depends on the group G acting on X, as well as the chosen maximal torus T and Borel subgroup B. Our definition implies a close relationship between Richardson varieties in X and Richardson varieties in flag subvarieties of X, which suits the goals of this paper, see Proposition 3.7.

Any flag subvariety $Y = G_S\kappa.P_X \subset X$ is a flag variety of G_S by Proposition 3.3 below. The B_S-fixed point in Y is denoted $1.P_Y$ and has stabilizer $P_Y \subset G_S$. Let $(W_S)^Y \subset W_S$ be the subset of minimal representatives modulo $(W_S)_Y = N_{P_Y}(T)/T$. Let $w_0^Y \in (W_S)^Y$ denote the longest element, and let $\mathcal{P}_Y = I(w_0^Y) \subset \Phi_S^+$ be the associated partially ordered set. For $\kappa \in W^X$, let

$\kappa_S \in W^X$ denote the unique minimal element of the set $\{w\kappa \mid w \in W_S\}$. Equivalently, κ_S^{-1} is the left factor of the parabolic factorization of κ^{-1} with respect to W_S. Part (a) of the following result implies that the inclusion $Y \to X$ is given by $g.P_Y \mapsto g\kappa_S.P_X$ for $g \in G_S$.

Proposition 3.3. *Let $X = G/P_X$ be a flag variety and let $\kappa \in W^X$. Let $Y = G_S\kappa.P_X \subset X$ be the flag subvariety defined by $S \subset \Delta$ and $\kappa.P_X$.*

(a) *Y is a flag variety of G_S with B_S-fixed point $1.P_Y = \kappa_S.P_X$.*

(b) *$Y = X^{\kappa_S}_{w_0^Y\kappa_S}$ is a Richardson variety in X.*

(c) *The map $w \mapsto w\kappa_S$ is an order isomorphism from the Bruhat order of $(W_S)^Y$ to the interval $[\kappa_S, w_0^Y\kappa_S]$ in the Bruhat order of W^X.*

(d) *The map $\alpha \mapsto \kappa_S^{-1}.\alpha$ is an order-preserving bijection $\mathcal{P}_Y \to I(w_0^Y\kappa_S) \smallsetminus I(\kappa_S)$.*

Proof. Since $\kappa.P_X$ is stabilized by the Borel subgroup $G_S \cap \kappa B\kappa^{-1}$ of G_S [Spr98, (6.4.7)], Y is a flag variety of G_S. The T-fixed points are $Y^T = \{w\kappa.P_X \mid w \in W_S\}$. Since $\overline{B_S\kappa_S.P_X}^T \subset X_{\kappa_S} \cap Y^T = \{\kappa_S.P_X\}$, it follows that $\kappa_S.P_X$ is a B_S-fixed point of Y, which proves part (a). The definition of κ_S implies that the product $w\kappa_S$ is reduced for each $w \in W_S$, so the map $w \mapsto w\kappa_S$ is an order isomorphism $W_S \to W_S\kappa_S$. The element $w \in W_S$ belongs to $(W_S)^Y$ if and only if w is the minimal representative of the point $w.P_Y = w\kappa_S.P_X$. Using that $\kappa_S \in W^X$, we deduce that $w \in (W_S)^Y$ if and only if $w\kappa_S \in W^X$. This proves part (c), which in turn implies $Y^T \subset X^{\kappa_S}_{w_0^Y\kappa_S}$ and $\dim(Y) = \dim(X^{\kappa_S}_{w_0^Y\kappa_S})$. Part (b) follows from this, using that Richardson varieties are T-convex [BCP25]. Finally, part (d) holds because $\kappa_S^{-1}.\alpha > 0$ for all $\alpha \in \Phi_S^+$ by the definition of κ_S. □

Remark 3.4. The set $-\mathcal{P}_Y = \{-\alpha \mid \alpha \in \mathcal{P}_Y\}$ consists of all T-weights of the tangent space $T_Y(1.P_Y) \subset T_X(\kappa_S.P_X)$, while $-\kappa_S^{-1}.\mathcal{P}_Y$ is the set of T-weights of the translate $\kappa_S^{-1}.T_Y(1.P_Y) \subset T_X(1.P_X)$. This explains the geometric relevance of Proposition 3.3(d).

We will show in Proposition 4.9 that when X is a cominuscule flag variety, all flag subvarieties of X are also cominuscule. The following example shows that the general situation is more complicated.

Example 3.5. Let $X = \mathrm{SG}(2,6)$ be the symplectic Grassmannian of 2-dimensional isotropic subspaces of $\mathbb{K}^6$, see Section 2.2. Let $S = \{\beta_1, \beta_2\} \subset \Delta$ be the two short simple roots, so that G_S has type A_2, and set $\kappa = \kappa_S = s_3s_2$ and $Y = G_S\kappa.P_X$. Then $\kappa_S.P_X = s_3s_2.\langle e_1, e_2\rangle = \langle e_1, e_4\rangle$, and the map $Y \to \mathrm{Fl}(3)$ defined by $V \mapsto (V \cap \langle e_1, e_2, e_3\rangle, V^\perp \cap \langle e_1, e_2, e_3\rangle)$ is an isomorphism. The list of flag subvarieties of X includes $\mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, $\mathbb{P}^3$, $\mathrm{Fl}(3)$, $\mathrm{LG}(2,4)$, $\mathrm{SG}(2,6)$, and the single point $\mathrm{Spec}(\mathbb{K})$. Some of these varieties have several inclusions as flag subvarieties of X.

When X is a cominuscule flag variety, it follows from [BCMP22, Lemma 4.2(a)] that the map of Proposition 3.3(d) is an order isomorphism, in the sense that the inverse map $\kappa_S : I(w_0^Y\kappa_S) \smallsetminus I(\kappa_S) \to \mathcal{P}_Y$ is also order-preserving. The following example shows that this is not the case for arbitrary flag varieties.

Example 3.6. Let $X = \mathrm{Fl}(1,2;4)$ be the variety of two-step flags in $\mathbb{K}^4$ with dimension $(1,2)$, and set $S = \{\beta_1, \beta_3\}$, $\kappa = \kappa_S = s_2s_1$, and $Y = G_S\kappa.P_X$. We obtain $\kappa_S.P_X = (\langle e_3\rangle, \langle e_1, e_3\rangle)$, $Y = \mathbb{P}^1 \times \mathbb{P}^1$, $w_0^Y = s_1s_3$, $\mathcal{P}_Y = S$, $\kappa_S^{-1}.\beta_1 = \beta_2$,

and $\kappa_S^{-1}.\beta_3 = \beta_1 + \beta_2 + \beta_3$. The two roots in $\kappa_S^{-1}.\mathcal{P}_Y = I(w_0^Y \kappa_S) \smallsetminus I(\kappa_S)$ are comparable, while the roots in $\mathcal{P}_Y$ are not.

3.3. **Minimal embeddings.** We next show that each Richardson variety X_u^v is contained in a unique minimal flag subvariety of X. The Schubert varieties $Y_{u'} = \overline{B_S u'.P_Y}$ and $Y^{v'} = \overline{B_S^- v'.P_Y}$ in the flag subvariety $Y = G_S\kappa.P_X \subset X$, defined for $v', u' \in W_S$, may not be Schubert varieties in X, but any Richardson variety $Y_{u'}^{v'} = Y_{u'} \cap Y^{v'}$ is also a Richardson variety in X by the following result.

Proposition 3.7. *Let $X = G/P_X$ be a flag variety, let $u, v, \kappa \in W^X$ satisfy $v \leq u$, and let $Y = G_S\kappa.P_X$ be the flag subvariety of X defined by $\kappa.P_X$ and $S \subset \Delta$.*

(a) *The intersection $X_u^v \cap Y$ is either empty or a Richardson variety in Y.*

(b) *We have $X_u^v \subset Y$ if and only if $u\kappa^{-1} \in W_S$ and $v\kappa^{-1} \in W_S$.*

(c) *Given $v' \leq u'$ in $(W_S)^Y$, we have $Y_{u'}^{v'} = X_{u'\kappa_S}^{v'\kappa_S}$.*

Proof. Assume $X^v \cap Y \neq \emptyset$. Using that all Richardson varieties in X are compatibly Frobenius split [BK05, §2], it follows from Proposition 3.3(b) that $X^v \cap Y$ is reduced. Since $X^v \cap Y$ is also B_S^--stable, it is a union of B_S^--stable Schubert varieties in Y. The T-fixed points are $(X^v \cap Y)^T = \{w\kappa_S.P_X \mid w \in W_S \text{ and } w\kappa_S \geq v\}$. Since the set $\{w \in W_S \mid w\kappa_S \geq v\}$ has a unique minimal element by [KLS14, Prop. 2.1], it follows that $X^v \cap Y$ is a single B_S^--stable Schubert variety in Y. Similarly, $Y \cap X_u$ is either empty or a B_S-stable Schubert variety. Part (a) follows from this. Part (b) follows from Proposition 3.3(c) because $u\kappa^{-1} \in W_S$ is equivalent to $\kappa_S \leq u \leq w_0^Y \kappa_S$. Proposition 3.3(c) also implies that $Y_{u'}^{v'}$ and $X_{u'\kappa_S}^{v'\kappa_S}$ have the same T-fixed points, which proves part (c), using that $X_{u'\kappa_S}^{v'\kappa_S}$ is T-convex in X [BCP25]. □

Corollary 3.8. *Let $X = G/P_X$ be a flag variety, and let $Y = G_S\kappa.P_X$ be a flag subvariety of X.*

(a) *Any flag subvariety of Y is also a flag subvariety of X.*

(b) *Any flag subvariety of X contained in Y is also a flag subvariety of Y.*

(c) *Any non-empty intersection of flag subvarieties of X is a flag subvariety.*

(d) *Given $S_1, S_2 \subset \Delta$ and $\kappa \in W$, we have $G_{S_1}\kappa.P_X \cap G_{S_2}\kappa.P_X = G_{S_1 \cap S_2}\kappa.P_X$.*

Proof. Part (a) is clear from the definitions, and parts (b) and (c) follow from part (d). Let S_1, S_2, and κ be as in (d). Since both $G_{S_1}\kappa.P_X \cap G_{S_2}\kappa.P_X$ and $G_{S_1 \cap S_2}\kappa.P_X$ are Richardson varieties in X by Proposition 3.3(b) and Proposition 3.7(a,c), it is enough to show that these varieties have the same T-fixed points. Equivalently, we must show that $W_{S_1}\kappa W_X \cap W_{S_2}\kappa W_X = W_{S_1 \cap S_2}\kappa W_X$ holds as subsets of W. This is equivalent to $W_X\kappa^{-1}W_{S_1} \cap W_X\kappa^{-1}W_{S_2} = W_X\kappa^{-1}W_{S_1 \cap S_2}$. Since $W_X\kappa^{-1}$ is an interval in the Bruhat order of W, this follows from [BB05, Thm. 2.6.1]. □

The *support* of $w \in W$ is the subset $S \subset \Delta$ of simple roots β for which s_β occurs in any reduced expression of w.

Definition 3.9. Let $X_u^v \subset X$ be the Richardson variety defined by $v \leq u \in W^X$. The *minimal embedding* of X_u^v is the flag subvariety $Y = G_S u.P_X$, where $S \subset \Delta$ is the support of uv^{-1}.

It follows from Proposition 3.7(b) that the minimal embedding of X_u^v is the unique minimal flag subvariety of X that contains X_u^v. The minimal embedding depends on the chosen Borel subgroup $B \subset G$, as shown by the following example.

Example 3.10. Let $X = \mathrm{Fl}(4)$ be the variety of complete flags in $\mathbb{K}^4$. The minimal embedding of the Richardson variety X_u^v defined by $u = s_1 s_2 s_3$ and $v = s_1 s_3$ is $G_{\{\beta_1,\beta_2\}} u.P_X \cong \mathrm{Fl}(3)$. The translate $s_2.X_u^v = X_{s_2 u}^{s_2 v} = G_{\{\beta_1\}} s_2 u.P_X \cong \mathbb{P}^1$ is its own minimal embedding.

3.4. **Product decompositions.** It was shown in [BR12] that certain Richardson varieties in classical cominuscule flag varieties are products of smaller Richardson varieties. We proceed to generalize this result. Two subsets $S_1, S_2 \subset \Phi$ will be called *orthogonal* if every root in S_1 is orthogonal to every root in S_2. If $S_1, S_2 \subset \Delta$ consist of simple roots, then this holds if and only if $S_1 \cap S_2 = \emptyset$ and no edge in the Dynkin diagram of Φ connects a root in S_1 to a root in S_2.

Proposition 3.11. *Let X be a flag variety of G, and let $v \le \kappa \le u \in W$. Assume that $\Delta = S_1 \coprod S_2$ is a disjoint union of orthogonal subsets. Set $Y_i = G_{S_i}\kappa.P_X$, $P_{\overline{Y}_i} = G_{S_{3-i}} P_X$, and $\overline{Y}_i = G/P_{\overline{Y}_i}$, for $i \in \{1,2\}$.*

(a) *The map $g.P_X \mapsto g.P_{\overline{Y}_i}$, for $g \in G_{S_i}$, is an isomorphism $Y_i \cong \overline{Y}_i$.*

(b) *The map $g.P_X \mapsto (g.P_{\overline{Y}_1}, g.P_{\overline{Y}_2})$, for $g \in G$, is an isomorphism $X \cong \overline{Y}_1 \times \overline{Y}_2$.*

(c) *The map of part* (a) *restricts to an isomorphism $Y_i \cap X_u^v \cong (\overline{Y}_i)_u^v$, and the map of part* (b) *restricts to an isomorphism $X_u^v \cong (\overline{Y}_1)_u^v \times (\overline{Y}_2)_u^v$.*

(d) *The map $w \mapsto (w^{\overline{Y}_1}, w^{\overline{Y}_2})$ is an order isomorphism $W^X \cong W^{\overline{Y}_1} \times W^{\overline{Y}_2}$ of Bruhat orders, with inverse $(w_1, w_2) \mapsto w_1 w_2$, and $(W_{S_i})^{Y_i} = W^{\overline{Y}_i} = W_{S_i} \cap W^X$.*

(e) *We have $\mathcal{P}_{Y_i} = \mathcal{P}_{\overline{Y}_i} = \Phi_{S_i} \cap \mathcal{P}_X$ and $\mathcal{P}_X = \mathcal{P}_{Y_1} \coprod \mathcal{P}_{Y_2}$.*

Proof. The assumptions imply that $\Phi = \Phi_{S_1} \coprod \Phi_{S_2}$, where Φ_{S_1} and Φ_{S_2} are orthogonal subsets of Φ. It follows that the set of products $P_{\overline{Y}_i} = G_{S_{3-i}} P_X$ is a parabolic subgroup of G, see e.g. [Spr98, (8.2.3)]. It also follows that the product map $W_{S_1} \times W_{S_2} \to W$ is an isomorphism of groups, which restricts to an isomorphism $(W_{S_1} \cap W_X) \times (W_{S_2} \cap W_X) \cong W_X$. Write $\kappa = \kappa_1 \kappa_2$ with $\kappa_i \in W_{S_i}$. Then the B_{S_i}-stable point of Y_i is $1.P_{Y_i} = \kappa_{3-i}.P_X$ by Proposition 3.3(a). For $\beta \in S_i$ we have $s_\beta \kappa_{3-i}.P_X = \kappa_{3-i} s_\beta.P_X$, hence $s_\beta.P_{Y_i} = 1.P_{Y_i}$ holds if and only if $s_\beta.P_X = 1.P_X$. We obtain $(W_{S_i})_{Y_i} = W_{S_i} \cap W_X = W_{S_i} \cap W_{\overline{Y}_i}$. Part (d) follows from this. In particular, we obtain $w_0^X = w_0^{\overline{Y}_1} w_0^{\overline{Y}_2} = w_0^{Y_1} w_0^{Y_2}$, which proves part (e). Part (a) follows because $G_{S_i} P_{\overline{Y}_i} = G$ and $G_{S_i} \cap P_{\overline{Y}_i} = G_{S_i} \cap P_X = P_{Y_i}$, see [Spr98, (8.4.3)]. Similarly, the identity $P_{\overline{Y}_1} \cap P_{\overline{Y}_2} = P_X$ shows that the map of part (b) is injective, and surjectivity holds because $\dim(X) = \dim(\overline{Y}_1 \times \overline{Y}_2)$ by part (e). Finally, part (d) implies that $\dim(Y_i \cap X_u^v) = \dim((\overline{Y}_i)_u^v)$ and $\dim(X_u^v) = \dim((\overline{Y}_1)_u^v \times (\overline{Y}_2)_u^v)$, which proves part (c). □

Corollary 3.12. *Let $X = G/P_X$ be a flag variety, let $v \le \kappa \le u$ in W^X, and let $S_1, S_2 \subset \Delta$ be orthogonal subsets such that $uv^{-1} \in W_{S_1 \cup S_2}$. Set $Y_i = G_{S_i}\kappa.P_X$ for $i \in \{1,2\}$. Then $X_u^v \cong (Y_1 \cap X_u^v) \times (Y_2 \cap X_u^v)$ as T-varieties.*

Proof. Set $S = S_1 \cup S_2$. Since $v_S \le \kappa_S \le u_S = v_S$, it follows from Proposition 3.7 that X_u^v is a Richardson variety in $Y = G_S \kappa.P_X$. The result therefore follows from Proposition 3.11(c). □

3.5. **Indecomposable varieties.** Cominuscule Richardson varieties that satisfy the following definition play an important role in this paper.

Definition 3.13. An irreducible algebraic variety of positive dimension will be called *indecomposable* if it is not isomorphic to a product of varieties of strictly smaller dimension.

Lemma 3.14. *Let Z be an irreducible projective variety. If* $\mathrm{Pic}(Z) = \mathbb{Z}$ *or* $A_1(Z) = \mathbb{Z}$*, then Z is indecomposable.*

Proof. Assume that $Z = Z_1 \times Z_2$, where Z_1 and Z_2 have positive dimension. Choose a point $p_i \in Z_i$, a curve $C_i \subset Z_i$, and an embedding $\phi_i : Z_i \subset \mathbb{P}^N$, for $i \in \{1, 2\}$. Then pushforward along $\phi_1 \times \phi_2 : Z \to \mathbb{P}^N \times \mathbb{P}^N$ maps $[C_1 \times p_2]$ and $[p_1 \times C_2]$ to linearly independent classes in $A_1(\mathbb{P}^N \times \mathbb{P}^N)$, and pullback along $\phi_1 \times \phi_2$ maps the generators of $\mathrm{Pic}(\mathbb{P}^N \times \mathbb{P}^N)$ to linearly independent classes in $\mathrm{Pic}(Z)$. □

Definition 3.15. Let $\mathcal{P}$ be a partially ordered set. Two elements $\alpha_1, \alpha_2 \in \mathcal{P}$ are *comparable* if $\alpha_1 \leq \alpha_2$ or $\alpha_2 \leq \alpha_1$; otherwise α_1 and α_2 are *incomparable*. Two subsets $A_1, A_2 \subset \mathcal{P}$ are *incomparable* if no element from A_1 is comparable to an element from A_2. A *connected component* of $\mathcal{P}$ is a maximal non-empty subset that is not a union of two non-empty incomparable subsets. The partially ordered set $\mathcal{P}$ is *connected* if it is a connected component of itself.

Notice that Definition 3.15 implies that the empty set is not connected and has no connected components.

Lemma 3.16. *Let $X = G/P_X$, and assume that $\mathcal{P}_X = \mathcal{P}_1 \coprod \mathcal{P}_2$ is a union of two incomparable subsets. Let $S_i \subset \Delta$ be the subset of all simple roots that are connected to* $\min(\mathcal{P}_i)$ *by a path in the Dynkin diagram of Φ, for $i \in \{1, 2\}$. Then $S_1 \cap S_2 = \emptyset$.*

Proof. If $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cap S_2$ contains a connected component $\Delta' \subset \Delta$ of the Dynkin diagram, hence both $\mathcal{P}_1$ and $\mathcal{P}_2$ have non-empty intersection with Δ'. This contradicts the assumption that $\mathcal{P}_1$ and $\mathcal{P}_2$ are incomparable, as (1) implies that the highest root of $\mathrm{Span}_{\mathbb{Z}}(\Delta') \cap \Phi$ belongs to either $\mathcal{P}_1$ or $\mathcal{P}_2$. □

Corollary 3.17. *Let $X = G/P_X$ be a flag variety.*

(a) *If* $\#\min(\mathcal{P}_X) = 1$*, then X is indecomposable.*

(b) *If X is indecomposable, then $\mathcal{P}_X$ is connected.*

Proof. This follows from Lemma 3.14, Lemma 3.16, and Proposition 3.11. □

Conjecture 3.18. *The flag variety $X = G/P_X$ is indecomposable if and only if $\mathcal{P}_X$ is connected.*

Remark 3.19. Let $X = G/P_X$ be a flag variety, and let X_u^v be the Richardson variety defined by $v \leq u \in W^X$. If X_u^v is indecomposable, then the support of uv^{-1} is a connected subset of the Dynkin diagram of Φ by Corollary 3.12. The converse holds when X is cominuscule, but is not true in general. For example, for $X = \mathrm{Fl}(4)$ we have $X_{4132}^{1423} = \{V_\bullet \in X \mid V_2 = \langle e_1, e_4 \rangle\} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

4. Cominuscule Richardson varieties

In this section we interpret the results of Section 3 for cominuscule flag varieties. We prove that flag subvarieties of cominuscule flag varieties are again cominuscule, and show that any cominuscule Richardson variety is a product of indecomposable cominuscule Richardson varieties.

4.1. **Cominuscule flag varieties.** Given a simple root $\beta \in \Delta$, we let $\omega_\beta^\vee$ denote the corresponding fundamental coweight. We have $\alpha = \sum_{\beta\in\Delta}(\alpha, \omega_\beta^\vee)\beta$ for $\alpha \in \Phi$.

Definition 4.1. The flag variety $X = G/P_X$ is called *cominuscule* if, for each root $\alpha \in \Phi$, we have

$$\sum_{\gamma\in\min(\mathcal{P}_X)} (\alpha, \omega_\gamma^\vee) \ \in \ \{\pm 1, 0\}\,.$$

An indecomposable cominuscule flag variety is the same as what is called a cominuscule flag variety in many other sources; see Corollary 4.3 below. A representative collection of indecomposable cominuscule flag varieties is shown in Table 1. Notice that Definition 4.1 depends on the group G acting on X. For example, the maximal orthogonal Grassmannian $\mathrm{OG}(m, 2m) \cong \mathrm{OG}(m-1, 2m-1)$ is cominuscule for the action of $\mathrm{SO}(2m)$, but not for the action of $\mathrm{SO}(2m-1)$.

Lemma 4.2. *Let $X = G/P_X$ be a cominuscule flag variety. The following are equivalent:* (a) *X is indecomposable.* (b) *$\mathcal{P}_X$ is connected.* (c) $\#\min(\mathcal{P}_X) = 1$.

Proof. In view of Corollary 3.17, it suffices to show that (b) implies (c). If $\mathcal{P}_X$ is connected, then $\min(\mathcal{P}_X)$ is contained in a single connected component $\Delta' \subset \Delta$ of the Dynkin diagram. By letting α be the highest root of $\mathrm{Span}_{\mathbb{Z}}(\Delta') \cap \Phi$ in Definition 4.1, we obtain $\#\min(\mathcal{P}_X) = 1$, as required. □

A simple root $\gamma \in \Delta$ is called *cominuscule* if $(\alpha, \omega_\gamma^\vee) \in \{\pm 1, 0\}$ for all $\alpha \in \Phi$. A non-trivial linear algebraic group will be called *simple* if it is semi-simple and its root system is irreducible.

Corollary 4.3. *The flag variety $X = G/P_X$ is an indecomposable cominuscule flag variety if and only if P_X is a maximal parabolic subgroup of G defined by a cominuscule simple root. In this case there exists a simple subgroup $G' \subset G$ such that G' acts transitively on X, and X is cominuscule for this action.*

Proof. The first claim follows from Lemma 4.2. For the second claim, let $S_1 \subset \Delta$ be the connected component in the Dynkin diagram of the unique cominuscule simple root in $\min(\mathcal{P}_X)$, and set $S_2 = \Delta \smallsetminus S_1$. Then X is a flag variety of G_{S_1} by Proposition 3.11. Now let $G' = (G_{S_1}, G_{S_1})$ be the commutator subgroup. □

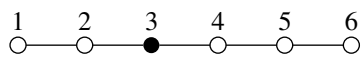
Corollary 4.4. *A flag variety is cominuscule if and only if it is a product of indecomposable cominuscule flag varieties.*

Proof. It follows from Proposition 3.11 and Lemma 4.2 that any cominuscule flag variety is a product of indecomposable cominuscule factors. On the other hand, if $X = X_1 \times \cdots \times X_m$ is any product of cominuscule flag varieties $X_i = G_i/P_{X_i}$, then X is a cominuscule flag variety of $G_1 \times \cdots \times G_m$. □

4.2. **Shapes of boxes.** Let X be a cominuscule flag variety and let $\mathcal{P}_X \subset \Phi^+$ be the associated partially ordered set. A lower order ideal $\lambda \subset \mathcal{P}_X$ will be called a *straight shape* in $\mathcal{P}_X$. Straight shapes generalize the Young diagrams used for Grassmannians of type A; for this reason, the roots in $\mathcal{P}_X$ will be called *boxes*. The boxes in $\mathcal{P}_X$ will be designated as *long* or *short* according to their lengths as roots in Φ. By convention, all roots in simply-laced components of Φ are considered long. When X is indecomposable, all boxes of $\mathcal{P}_X$ are long unless X is a Lagrangian Grassmannian or an odd quadric. A box in $\mathcal{P}_{\mathrm{LG}(n,2n)}$ is short if and only if it is not on the southwest diagonal, while the only short box in $\mathcal{P}_{Q^{2n+1}}$ is the middle

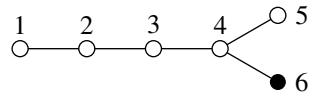
TABLE 1. Partially ordered sets $\mathcal{P}_X$ of indecomposable cominuscule flag varieties. In each case the partial order is given by $\alpha' \leq \alpha$ if and only if α' is weakly northwest of α. The numbers in the boxes indicate the labeling $\delta_X : \mathcal{P}_X \to \Delta$ by simple roots.

Grassmannian $\mathrm{Gr}(3,7)$ of type A



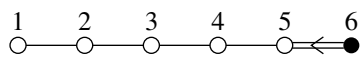
3	4	5	6
2	3	4	5
1	2	3	4

Max. orthog. Grassmannian $\mathrm{OG}(6,12)$



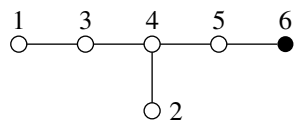
6	4	3	2	1
	5	4	3	2
		6	4	3
			5	4
				6

Lagrangian Grassmannian $\mathrm{LG}(6,12)$



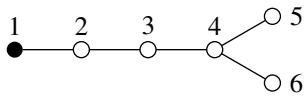
6	5	4	3	2	1
	6	5	4	3	2
		6	5	4	3
			6	5	4
				6	5
					6

Cayley Plane E_6/P_6



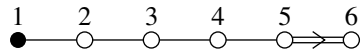
6	5	4	2				
		3	4	5	6		
		1	3	4	5		
				2	4	3	1

Even quadric $Q^{10} \subset \mathbb{P}^{11}$



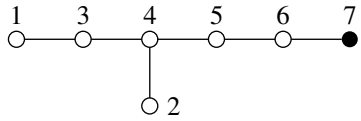
1	2	3	4	5				
			6	4	3	2	1	

Odd quadric $Q^{11} \subset \mathbb{P}^{12}$



1	2	3	4	5	6	5	4	3	2	1

Freudenthal variety E_7/P_7



7	6	5	4	3	1					
			2	4	3					
				5	4	2				
				6	5	4	3	1		
				7	6	5	4	3		
							2	4		
								5		
								6		
								7		

box (see Table 1). Given straight shapes $\mu \subset \lambda$ in $\mathcal{P}_X$, we let λ/μ denote the set of boxes in λ that are not in μ. A subset $\theta \subset \mathcal{P}_X$ of the form $\theta = \lambda/\mu$ is called a *skew shape*. Equivalently, θ is a convex subset of $\mathcal{P}_X$, that is, $\alpha \in \theta$ holds whenever $\alpha' < \alpha < \alpha''$ and $\alpha', \alpha'' \in \theta$.

Parts (a) and (b) of the following result were proved by Proctor [Pro84], and part (c) was proved by Stembridge [Ste96]. Notice that the general statement follows easily from the indecomposable case via Proposition 3.11.

Theorem 4.5. *Let X be a cominuscule flag variety.*

(a) *The Bruhat order on W^X is a distributive lattice that agrees with the weak left Bruhat order.*

(b) *The map $u \mapsto I(u)$ is a bijection from W^X to the set of straight shapes in $\mathcal{P}_X$. We have $v \leq u$ in W^X if and only if $I(v) \subset I(u)$.*

(c) *All elements of W^X are fully commutative.*

Given a straight shape $\lambda \subset \mathcal{P}_X$, we let $w_\lambda \in W^X$ denote the unique element with inversion set $I(w_\lambda) = \lambda$. The Schubert and Richardson varieties in X will frequently be indexed by straight shapes using the notation $X_\lambda = X_{w_\lambda}$, $X^\mu = X^{w_\mu}$, and $X^\mu_\lambda = X_\lambda \cap X^\mu$, where $\mu \subset \lambda \subset \mathcal{P}_X$ are straight shapes. Notice that $\dim(X^\mu_\lambda) = |\lambda/\mu|$ is the number of boxes in the skew shape λ/μ.

An ordering $\mu = \{\alpha_1, \alpha_2, \ldots, \alpha_{|\mu|}\}$ of the boxes of the straight shape μ is called *increasing* if $\alpha_i < \alpha_j$ implies $i < j$. In this case we have [BS16, Thm. 2.4]

$$w_\mu = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{|\mu|}} \,. \tag{2}$$

For any root $\alpha \in \mathcal{P}_X$, consider the shape $\lambda(\alpha) = \{\alpha' \in \mathcal{P}_X \mid \alpha' < \alpha\}$, and set $\delta_X(\alpha) = w_{\lambda(\alpha)}.\alpha$. Since $s_{\delta_X(\alpha)} = w_{\lambda(\alpha)} s_\alpha w^{-1}_{\lambda(\alpha)} = w_{\lambda(\alpha)\cup\{\alpha\}} w^{-1}_{\lambda(\alpha)}$ has length 1 by Theorem 4.5, it follows that $\delta_X : \mathcal{P}_X \to \Delta$ is a labeling of the boxes in $\mathcal{P}_X$ by simple roots, such that

$$w_\mu = s_{\delta_X(\alpha_{|\mu|})} \cdots s_{\delta_X(\alpha_2)} s_{\delta_X(\alpha_1)} \tag{3}$$

is a reduced expression of w_μ; see [BS16, Cor. 2.10]. Examples of this labeling are provided in Table 1. More generally, if $\theta \subset \mathcal{P}_X$ is any skew shape, and $\theta = \{\alpha'_1, \alpha'_2, \ldots, \alpha'_{|\theta|}\}$ is an increasing ordering, we define

$$w_\theta = s_{\delta_X(\alpha'_{|\theta|})} \cdots s_{\delta_X(\alpha'_2)} s_{\delta_X(\alpha'_1)} \,.$$

This is a reduced expression of w_θ, and we have $w_\theta = w_\lambda w_\mu^{-1}$ whenever $\theta = \lambda/\mu$ is written as a difference between straight shapes $\mu \subset \lambda \subset \mathcal{P}_X$.

Lemma 4.6. *Let $\alpha_1, \alpha_2 \in \mathcal{P}_X$ be boxes and let $\theta \subset \mathcal{P}_X$ be a skew shape.*

(a) *If α_2 covers α_1, then $\delta_X(\alpha_1)$ and $\delta_X(\alpha_2)$ are distinct and not orthogonal.*

(b) *If α_1 and α_2 are incomparable, then $\delta_X(\alpha_1)$ and $\delta_X(\alpha_2)$ are orthogonal.*

(c) *θ is connected if and only if $\delta_X(\theta)$ is a connected subset of the Dynkin diagram.*

Proof. This follows from Theorem 4.5 and (3). □

Given a straight shape $\lambda \subset \mathcal{P}_X$ and $u \in W$, we set $u \bullet \lambda = I((uw_\lambda)^X)$, that is, $u \bullet \lambda \subset \mathcal{P}_X$ is the unique straight shape such that $u.(w_\lambda.P_X) = w_{u\bullet\lambda}.P_X$. The action of W on X^T is expressed in terms of shapes as follows.

Lemma 4.7. *Let $\lambda \subset \mathcal{P}_X$ be a straight shape. For $\beta \in \Delta$ we have*

$$s_\beta \bullet \lambda = \begin{cases} \lambda \cup \{\alpha\} & \text{if } \mathcal{P}_X \smallsetminus \lambda \text{ contains a minimal box } \alpha \text{ satisfying } \delta_X(\alpha) = \beta; \\ \lambda \smallsetminus \{\alpha\} & \text{if } \lambda \text{ contains a maximal box } \alpha \text{ satisfying } \delta_X(\alpha) = \beta; \\ \lambda & \text{otherwise.} \end{cases}$$

Proof. We may assume $\lambda \subset s_\beta \bullet \lambda$. If $\lambda \neq s_\beta \bullet \lambda$, then $s_\beta \bullet \lambda = \lambda \cup \{\alpha\}$ for some box $\alpha \in \mathcal{P}_X$, and (3) implies that $\delta_X(\alpha) = \beta$. □

Identify $\mathbb{Z}^2$ with a grid of boxes in the plane. The point $(r, c) \in \mathbb{Z}^2$ represents a box in row r and column c, where the row number increases from north to south, and the column number increases from east to west. The *northwest-to-southeast order* on $\mathbb{Z}^2$ is defined by $(r', c') \leq (r, c)$ if and only if $r' \leq r$ and $c' \leq c$. This is a distributive lattice with meet $(r_1, c_1) \wedge (r_2, c_2) = (\min(r_1, r_2), \min(c_1, c_2))$ and join $(r_1, c_1) \vee (r_2, c_2) = (\max(r_1, r_2), \max(c_1, c_2))$. Given simple roots $\beta_1, \beta_2 \in \Delta$ in the same connected component of the Dynkin diagram of Φ, we let $[\beta_1, \beta_2] \subset \Delta$ denote the subset of simple roots occurring on the shortest path from β_1 to β_2 in the Dynkin diagram. The following result implies that $\mathcal{P}_X$ is a distributive lattice when X is indecomposable.

Lemma 4.8. *Let X be an indecomposable cominuscule flag variety.*

(a) *The poset $\mathcal{P}_X$ is isomorphic to a subset of $\mathbb{Z}^2$ that is closed under the join and meet operations. Moreover, every covering $\alpha_1 \lessdot \alpha_2$ in $\mathcal{P}_X$ is also a covering in $\mathbb{Z}^2$.*

(b) *Let $\alpha_1, \alpha_2 \in \mathcal{P}_X$ be incomparable boxes. For all $\alpha_1 \wedge \alpha_2 \leq \alpha \leq \alpha_1 \vee \alpha_2$ we have $\delta_X(\alpha) \in [\delta_X(\alpha_1), \delta_X(\alpha_2)] \subset \Delta$.*

Proof. This follows by inspection of Table 1. □

4.3. **Cominuscule flag subvarieties.** We next show that any flag subvariety of the cominuscule flag variety $X = G/P_X$ is also cominuscule. Given a straight shape $\lambda \subset \mathcal{P}_X$ and a subset $S \subset \Delta$, let $\lambda_S \subset \lambda$ be the minimal straight shape for which $\delta_X(\lambda/\lambda_S) \subset S$, and let $\lambda^S \supset \lambda$ be the maximal straight shape for which $\delta_X(\lambda^S/\lambda) \subset S$. With the notation of Section 3.2 we have $w_{\lambda_S} = (w_\lambda)_S$ and $w_{\lambda^S} = w_0^Y (w_\lambda)_S$, with $Y = G_S w_\lambda.P_X$; see Proposition 3.3(c).

Proposition 4.9. *Let $X = G/P_X$ be a cominuscule flag variety and let $Y = G_S w_\lambda.P_X \subset X$ be the flag subvariety defined by $S \subset \Delta$ and $\lambda \subset \mathcal{P}_X$.*

(a) *$Y = X_{\lambda^S}^{\lambda_S}$ is a cominuscule flag variety of G_S with B_S-stable point $w_{\lambda_S}.P_X$.*

(b) *The map $\alpha \mapsto w_{\lambda_S}^{-1}.\alpha$ is an order isomorphism $\mathcal{P}_Y \cong \lambda^S/\lambda_S$.*

(c) *The labeling $\delta_Y : \mathcal{P}_Y \to S$ is given by $\delta_Y(\alpha) = \delta_X(w_{\lambda_S}^{-1}.\alpha)$.*

(d) *Given straight shapes $\tau \subset \sigma \subset \mathcal{P}_Y$, we have $Y_\sigma^\tau = X_{\overline{\sigma}}^{\overline{\tau}}$, where $\overline{\tau} = \lambda_S \cup w_{\lambda_S}^{-1}.\tau$ and $\overline{\sigma} = \lambda_S \cup w_{\lambda_S}^{-1}.\sigma$ are straight shapes in $\mathcal{P}_X$.*

Proof. It follows from Proposition 3.3 that $Y = X_{\lambda^S}^{\lambda_S}$, and $w_{\lambda_S}.P_X$ is the B_S-stable point in Y. To see that Y is cominuscule, we may assume by Proposition 3.11 and Corollary 4.4 that the Dynkin diagram Δ is connected, and that S is a connected subset of Δ. We may also assume that Y is not a point, so that $\lambda_S \neq \lambda^S$. By Corollary 4.3 we must show there exists $\gamma \in S$ such that γ is a cominuscule simple root of $\Phi' = \mathrm{Span}_{\mathbb{Z}}(S) \cap \Phi$ and $s_\beta \bullet \lambda_S = \lambda_S$ for $\beta \in S \smallsetminus \{\gamma\}$. By construction of λ_S, no maximal box α' of λ_S satisfies $\delta_X(\alpha') \in S$. Assume that two minimal boxes $\alpha_1 \neq \alpha_2$ of $\mathcal{P}_X \smallsetminus \lambda_S$ satisfy $\delta_X(\alpha_1), \delta_X(\alpha_2) \in S$. Then $\alpha_1 \wedge \alpha_2 \in \lambda_S$. Let $\alpha' \in \lambda_S$ be maximal with $\alpha' \geq \alpha_1 \wedge \alpha_2$. Since $\alpha_1 \not\leq \alpha'$ and $\alpha_2 \not\leq \alpha'$, it follows from Lemma 4.8(a) that $\alpha' \leq \alpha_1 \vee \alpha_2$, after which Lemma 4.8(b) implies $\delta_X(\alpha') \in S$, a contradiction. We deduce that $\mathcal{P}_X \smallsetminus \lambda_S$ contains a unique minimal box α with $\delta_X(\alpha) \in S$. Set $\gamma = \delta_X(\alpha)$. By Lemma 4.7 we have $s_\beta \bullet \lambda_S = \lambda_S$ for $\beta \in S \smallsetminus \{\gamma\}$; we must show that γ is a cominuscule simple root of Φ'. If Φ'

has type A, then any simple root γ is cominuscule, so assume that Φ' is not of type A. Then S contains either two simple roots of distinct lengths, or a simple root connected to three other simple roots in the Dynkin diagram of Φ'. If X is a Lagrangian Grassmannian or a maximal orthogonal Grassmannian, it follows that $\delta_X(\alpha') \in S$ for all boxes α' on the southwest diagonal of $\mathcal{P}_X$. Since some box α' on the southwest diagonal is either maximal in λ_S or minimal in $\mathcal{P}_X \smallsetminus \lambda_S$, we deduce that $\alpha = \alpha'$ and γ is cominuscule. If X is a quadric, then γ is the simple root in S of lowest index using the numbering of Table 1, hence γ is cominuscule in Φ'. Finally, it has been checked using a computer that γ is cominuscule in the finitely many cases for E_6/P_6 and E_7/P_7. This proves part (a). Part (b) is a special case of [BCMP22, Lemma 4.2(a)]; see also Proposition 3.3(d). Given any straight shape $\sigma \subset \mathcal{P}_Y$, it follows from part (b) that $\overline{\sigma} = \lambda_S \cup w_{\lambda_S}^{-1}.\sigma$ is a straight shape in $\mathcal{P}_X$, and (2) implies that $w_{\overline{\sigma}} = w_\sigma w_{\lambda_S}$. Part (d) therefore follows from Proposition 3.7(c). Let $\alpha \in \mathcal{P}_Y$ and set $\sigma = \{\alpha' \in \mathcal{P}_Y \mid \alpha' < \alpha\}$, so that $\delta_Y(\alpha) = w_\sigma.\alpha$. Since $w_{\lambda_S}^{-1}.\alpha$ is a minimal box of $\mathcal{P}_X \smallsetminus \overline{\sigma}$, it follows from [BCMP22, Lemma 4.2(b)] that $\delta_X(w_{\lambda_S}^{-1}.\alpha) = w_{\overline{\sigma}}.(w_{\lambda_S}^{-1}.\alpha) = \delta_Y(\alpha)$, which proves part (c). □

4.4. **Richardson varieties defined by skew shapes.** The next result shows that any skew shape $\theta \subset \mathcal{P}_X$ defines a Richardson variety X_θ in X up to translation.

Proposition 4.10. *Let $X = G/P_X$ be a cominuscule flag variety, and let X_λ^μ and $X_{\lambda'}^{\mu'}$ be Richardson varieties in X such that $\lambda/\mu = \lambda'/\mu' \subset \mathcal{P}_X$. Let $S \subset \Delta$ be the subset of simple roots that are orthogonal to $\delta_X(\lambda/\mu)$.*

(a) *For $w \in W_S$ we have $w \bullet \mu \subset w \bullet \lambda$, $(w \bullet \lambda)/(w \bullet \mu) = \lambda/\mu$, and $X_{w\bullet\lambda}^{w\bullet\mu} = w.X_\lambda^\mu$.*

(b) *There exists $w \in W_S$ such that $w.X_\lambda^\mu = X_{\lambda'}^{\mu'}$.*

(c) *Assume that X_λ^μ and $X_{\lambda'}^{\mu'}$ are both contained in a larger Richardson variety X_a^b. Then $[X_\lambda^\mu] = [X_{\lambda'}^{\mu'}]$ holds in the Chow group $A_{|\lambda/\mu|}(X_a^b)$.*

Proof. We first prove part (a) when $w = s_\beta$ is a simple generator of W_S, given by $\beta \in S$, such that $s_\beta \bullet \mu \neq \mu$. By Lemma 4.7 we have either $s_\beta \bullet \mu = \mu \cup \{\alpha\}$ or $s_\beta \bullet \mu = \mu \smallsetminus \{\alpha\}$, with $\delta_X(\alpha) = \beta$, and the definition of S gives $\alpha \notin \lambda/\mu$. If $s_\beta \bullet \mu = \mu \cup \{\alpha\}$, then α is a minimal box of both $\mathcal{P}_X \smallsetminus \mu$ and $\mathcal{P}_X \smallsetminus \lambda$, hence $s_\beta \bullet \lambda = \lambda \cup \{\alpha\}$. Otherwise we have $s_\beta \bullet \mu = \mu \smallsetminus \{\alpha\}$, where α is a maximal box of μ. If $\alpha < \alpha'$ is a covering in $\mathcal{P}_X$, then Lemma 4.6(a) and the definition of S show that $\alpha' \notin \lambda/\mu$, hence α is also a maximal box of λ, and we have $s_\beta \bullet \lambda = \lambda \smallsetminus \{\alpha\}$. In either case we obtain $s_\beta \bullet \mu \subset s_\beta \bullet \lambda$ and $(s_\beta \bullet \lambda)/(s_\beta \bullet \mu) = \lambda/\mu$. We also have $s_\beta.X_\lambda^\mu = X_{s_\beta\bullet\lambda}^{s_\beta\bullet\mu}$, as both varieties are T-convex in X [BCP25] and contain the same T-fixed points by Lemma 4.7. A symmetric argument proves part (a) when $w = s_\beta$ and $s_\beta \bullet \lambda \neq \lambda$, and the general case of (a) follows from these cases. For parts (b) and (c), we may assume that $\lambda' = \lambda \cup \{\alpha\}$ and $\mu' = \mu \cup \{\alpha\}$ for some box $\alpha \in \mathcal{P}_X \smallsetminus \lambda$. Set $\beta = \delta_X(\alpha)$. Since α is incomparable to λ/μ, we obtain $\beta \in S$ by Lemma 4.6(b), so part (a) shows that $s_\beta.X_\lambda^\mu = X_{\lambda'}^{\mu'}$. This proves part (b). The group $G_{\{\beta\}}$ stabilizes $X_{\lambda'}^\mu$, as it stabilizes both $X_{\lambda'}$ and X^μ. We deduce that $[X_\lambda^\mu] = [X_{\lambda'}^{\mu'}]$ holds in $A_*(X_{\lambda'}^\mu)$. Part (c) follows from this since $X_{\lambda'}^\mu \subset X_a^b$. □

Definition 4.11. Given any skew shape $\theta \subset \mathcal{P}_X$, we let X_θ denote the isomorphism class of the Richardson variety $X_\lambda^\mu \subset X$, for any choice of straight shapes $\mu \subset \lambda$ in $\mathcal{P}_X$ such that $\lambda/\mu = \theta$. Moreover, if $\phi \subset \theta$ is any smaller skew shape, then we set

$[X_\phi] = [X^{\mu'}_{\lambda'}] \in A_{|\phi|}(X_\theta)$, for any choice of straight shapes $\mu', \lambda' \subset \mathcal{P}_X$ such that $\mu \subset \mu' \subset \lambda' \subset \lambda$ and $\phi = \lambda'/\mu'$.

It follows from Proposition 4.10 that the variety X_θ and class $[X_\phi] \in A_*(X_\theta)$ are well-defined. One way to choose the shapes $\mu', \lambda' \subset \mathcal{P}_X$ in Definition 4.11 is to let λ' be the order ideal generated by $\mu \cup \phi$ and set $\mu' = \lambda' \smallsetminus \phi$.

Remark 4.12. More generally, if $\phi \subset \theta \subset \mathcal{P}_X$ are skew shapes, then there exists a well-defined group homomorphism $A(X_\phi) \to A(X_\theta)$ that agrees with the map $A(X^{\mu'}_{\lambda'}) \to A(X^{\mu}_{\lambda})$ whenever $\mu \subset \mu' \subset \lambda' \subset \lambda$ are straight shapes such that $\phi = \lambda'/\mu'$ and $\theta = \lambda/\mu$. We will not need this fact.

Remark 4.13. Let $X^{\mu'}_{\lambda'}$ and $X^{\mu''}_{\lambda''}$ be Richardson varieties in X, and set $\lambda = \lambda' \cap \lambda''$ and $\mu = \mu' \cup \mu''$. Using that $X_{\lambda'} \cap X_{\lambda''} = X_\lambda$ and $X^{\mu'} \cap X^{\mu''} = X^\mu$, we obtain

$$X^{\mu'}_{\lambda'} \cap X^{\mu''}_{\lambda''} = X_\lambda \cap X^\mu = \begin{cases} X^\mu_\lambda & \text{if } \mu \subset \lambda; \\ \emptyset & \text{if } \mu \not\subset \lambda. \end{cases}$$

When this intersection is not empty, the isomorphism class of X^μ_λ is given by the skew shape $\lambda/\mu = (\lambda'/\mu') \cap (\lambda''/\mu'')$.

Lemma 4.14. *Let $\theta \subset \mathcal{P}_X$ be a skew shape and set $S = \delta_X(\theta)$. There exists a unique maximal skew shape $\widehat{\theta} \subset \mathcal{P}_X$ containing θ such that $\delta_X(\widehat{\theta}) = S$. Moreover, if $\mu \subset \lambda$ are straight shapes in $\mathcal{P}_X$ with $\theta = \lambda/\mu$, then $\widehat{\theta} = \lambda^S/\lambda_S$.*

Proof. Let $\mu \subset \lambda$ be straight shapes in $\mathcal{P}_X$ such that $\theta = \lambda/\mu$, and let $\xi \subset \mathcal{P}_X$ be a skew shape such that $\theta \subset \xi$ and $\delta_X(\xi) = S$. We must show that $\xi \subset \lambda^S/\lambda_S$. Choose straight shapes $\tau' \subset \mu' \subset \lambda' \subset \sigma'$ such that $\xi = \sigma'/\tau'$ and $\theta = \lambda'/\mu'$. Let $S' \subset \Delta$ be the simple roots orthogonal to S. By Proposition 4.10 we can choose $w \in W_{S'}$ such that $w \bullet \mu' = \mu$ and $w \bullet \lambda' = \lambda$. Set $\tau = w \bullet \tau'$ and $\sigma = w \bullet \sigma'$. We obtain $\tau \subset \mu \subset \lambda \subset \sigma$ and $\xi = \sigma/\tau$. In particular, we have $\delta_X(\lambda/\tau) \subset S$ and $\delta_X(\sigma/\lambda) \subset S$. This implies $\lambda_S \subset \tau$ and $\sigma \subset \lambda^S$, and therefore $\xi \subset \lambda^S/\lambda_S$. □

The skew shape $\widehat{\theta}$ of Lemma 4.14 will be called the *minimal embedding* of θ, and $X_{\widehat{\theta}}$ will be called the *minimal embedding* of X_θ. Indeed, $X_{\widehat{\theta}}$ is isomorphic to the minimal embedding of X^μ_λ whenever $\lambda/\mu = \theta$. Notice that $X_{\widehat{\theta}}$ is a cominuscule flag variety by Proposition 4.9.

Corollary 4.15. *Let $\theta \subset \mathcal{P}_X$ be a skew shape. Assume that $\theta = \theta_1 \cup \theta_2$, where θ_1 and θ_2 are incomparable subsets of $\mathcal{P}_X$. Then $X_\theta \cong X_{\theta_1} \times X_{\theta_2}$.*

Proof. Choose straight shapes $\mu \subset \lambda \subset \mathcal{P}_X$ such that $\theta = \lambda/\mu$, and set $S_i = \delta_X(\theta_i)$ and $Y_i = G_{S_i} w_\lambda . P_X$ for $i \in \{1, 2\}$. Since $w_\lambda w_\mu^{-1} = w_{\lambda/\mu} \in W_{S_1 \cup S_2}$, and S_1 and S_2 are orthogonal subsets of Δ by Lemma 4.6(b), it follows from Corollary 3.12 that $X_\theta \cong (Y_1 \cap X^\mu_\lambda) \times (Y_2 \cap X^\mu_\lambda)$. Proposition 4.9(a) gives $Y_i = X^{\lambda_{S_i}}_{\lambda^{S_i}}$, so it suffices to show that $\theta_i = \theta \cap (\lambda^{S_i}/\lambda_{S_i})$ by Remark 4.13. Using that θ_1 and θ_2 are incomparable, it follows that $\lambda \smallsetminus \theta_i$ is a straight shape in $\mathcal{P}_X$, so we obtain $\widehat{\theta}_i = \lambda^{S_i}/\lambda_{S_i}$ by Lemma 4.14. The result follows from this since $\widehat{\theta}_i \cap \theta_{3-i} = \emptyset$. □

Corollary 4.15 will be generalized in Theorem 5.14, where we show that any cominuscule Richardson variety has a unique factorization as a product of indecomposable cominuscule Richardson varieties.

Remark 4.16. Let $X = G/P_X$ be a cominuscule flag variety and let $\theta \subset \mathcal{P}_X$ be a skew shape. Then Lemma 4.14 shows that X is the minimal embedding of X_θ if and only if $\delta_X(\theta) = \delta_X(\mathcal{P}_X)$. When studying the Richardson variety X_θ, it is frequently convenient to replace X with the minimal embedding, so that the identity $\delta_X(\theta) = \delta_X(\mathcal{P}_X)$ can be assumed. By Corollary 4.3 and Corollary 4.4 we may additionally replace G with a smaller group, so that $\delta_X(\theta) = \Delta$. In this case it follows from Lemma 4.6(c) that θ, $\mathcal{P}_X$, and the Dynkin diagram of Φ have the same number of connected components.

5. The first Chow group and the Picard group

In this section we compute the first Chow group $A_1(X_\theta)$ and the Picard group $\mathrm{Pic}(X_\theta)$ of any cominuscule Richardson variety X_θ.

5.1. **T-stable curves in flag varieties.** We start by recalling some facts about T-stable curves in an arbitrary flag variety $X = G/P_X$. The proofs given in [FW04, §3] are valid for flag varieties defined over any algebraically closed field. A T-stable curve in X is assumed to be irreducible. Given a root $\alpha \in \mathcal{P}_X$, there exists a unique T-stable curve $C_\alpha \subset X$ that contains the points $1.P_X$ and $s_\alpha.P_X$. Any T-stable curve in X has the form $w.C_\alpha$ for some $\alpha \in \mathcal{P}_X$ and $w \in W$.

Any W_X-invariant character $\lambda : T \to \mathbb{G}_m$ extends to a character of P_X. Let $\mathbb{K}_\lambda$ be the 1-dimensional representation of P_X defined by $p.z = \lambda(p)z$, and let $L_\lambda = G \times^{P_X} \mathbb{K}_{-\lambda}$ be the associated line bundle on X. The total space of this bundle consists of all pairs $[g, z]$ with $g \in G$ and $z \in \mathbb{K}$, subject to the relation $[gp, \lambda(p)z] = [g, z]$ for all $p \in P_X$. The degree of L_λ restricted to C_α is given by

$$\int_{C_\alpha} c_1(L_\lambda) \,=\, (\alpha^\vee, \lambda)\,, \tag{4}$$

where $\alpha^\vee$ is the coroot of α.

We will say that two distinct T-fixed points $x, y \in X^T$ are *adjacent* if both points are contained in some T-stable curve in X. Equivalently, we have $x \neq y = s_\alpha.x$ for some positive root $\alpha \in \Phi^+$. This root $\alpha \in \Phi^+$ is unique by [FW04, Lemma 4.1(i)]. For convenience we will also say that two elements $u, v \in W^X$ are adjacent if the corresponding T-fixed points $x = u.P_X$ and $y = v.P_X$ are adjacent. In this case we denote the unique T-stable curve through these points by either $C_{x,y}$ or $C_{u,v}$. Recall from Definition 4.11 that, if X is a cominuscule flag variety, then any skew shape $\phi \subset \mathcal{P}_X$ defines a Chow class $[X_\phi] \in A_{|\phi|}(X)$.

Lemma 5.1. *Let X be an indecomposable cominuscule flag variety, and let $\gamma \in \Delta$ be the cominuscule simple root defining X. For $\alpha \in \mathcal{P}_X$ we have in $A_1(X)$ that*

$$[X_{\{\alpha\}}] \,=\, [C_\alpha] \,=\, \begin{cases} [X_{s_\gamma}] & \text{if } \alpha \text{ is long,} \\ 2[X_{s_\gamma}] & \text{if } \alpha \text{ is short.} \end{cases}$$

Proof. It follows from equation (2) that the class $[X_{\{\alpha\}}] \in A_1(X)$ is represented by the curve $X^{\lambda(\alpha)}_{\lambda(\alpha)\cup\{\alpha\}} = w_{\lambda(\alpha)}.C_\alpha$, so it is enough to prove the second equality. Using equation (4) and the assumption that γ is cominuscule, we obtain

$$\int_{C_\alpha} c_1(L_{\omega_\gamma}) \,=\, (\alpha^\vee, \omega_\gamma) \,=\, \frac{(\gamma,\gamma)}{(\alpha,\alpha)}(\alpha, \omega_\gamma^\vee) \,=\, \frac{(\gamma,\gamma)}{(\alpha,\alpha)}\,.$$

The result follows from this since the root system Φ does not have type G_2, γ is long, and $A_1(X) \cong \mathbb{Z}$ is generated by the Schubert curve class $[C_\gamma] = [X_{s_\gamma}]$ by Lemma 4.2. $\square$

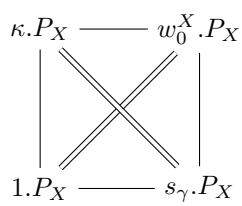
Given a cominuscule flag variety X, we will say that two boxes $\alpha', \alpha \in \mathcal{P}_X$ are *neighbors* if one of the boxes covers the other.

Lemma 5.2. *Let X be a cominuscule flag variety and let $\mu \subset \lambda \subset \mathcal{P}_X$ be straight shapes. Let $\alpha', \alpha \in \lambda/\mu$ be neighbor boxes such that α is at least as long as α'. Then we have in $A_1(X_\lambda^\mu)$ that*

$$[X_{\{\alpha'\}}] = \begin{cases} [X_{\{\alpha\}}] & \text{if } \alpha' \text{ and } \alpha \text{ have the same length,} \\ 2[X_{\{\alpha\}}] & \text{if } \alpha' \text{ is shorter than } \alpha. \end{cases} \tag{5}$$

Proof. Choose straight shapes $\tau, \sigma \subset \mathcal{P}_X$ with $\sigma/\tau = \{\alpha', \alpha\}$ and $\mu \subset \tau \subset \sigma \subset \lambda$, and set $S = \delta_X(\sigma/\tau)$ and $Y = G_S w_\sigma.P_X$. Since G_S has type A_2 or B_2, and Y is a cominuscule flag variety of G_S by Proposition 4.9, we must have $Y = \mathbb{P}^2$ or $Y = Q^3$. Equation (5) therefore holds in $A_1(Y)$ by Lemma 5.1. Since X_σ^τ is a Schubert variety in Y by Proposition 3.7, we obtain $A_1(X_\sigma^\tau) \cong A_1(Y)$ by [Mat88], which shows that (5) holds in $A_1(X_\sigma^\tau)$. Finally, since $X_\sigma^\tau \subset X_\lambda^\mu$, we deduce that (5) holds in $A_1(X_\lambda^\mu)$. $\square$

Example 5.3. Consider the quadric threefold $X = \mathrm{LG}(2,4) \cong Q^3$ of type C_2, and let $\gamma \in \Delta$ be the cominuscule simple root defining X. Since $\mathcal{P}_X$ contains two long roots and one short root, it follows that each T-fixed point in X is contained in three T-stable curves, two of which have class $[X_{s_\gamma}]$ and generate $A_1(X)$, while the third curve has class $2[X_{s_\gamma}]$. Since X contains four T-fixed points, it follows that all distinct T-fixed points in X are adjacent. We have $W^X = \{1, s_\gamma, \kappa, w_0^X\}$, where $\ell(\kappa) = 2$. Using that $C_{s_\gamma,\kappa}$ is the Richardson variety defined by the short root in $\mathcal{P}_X$, we deduce from Lemma 5.1 that the T-stable curves in X have the degrees in the following picture, where single lines indicate the class $[X_{s_\gamma}]$ and double lines indicate twice this class.



5.2. **Triangles of T-stable curves in Richardson varieties.** We next prove some facts about triples of adjacent T-fixed points in arbitrary flag varieties of Lie type different from G_2. Recall that a subgroup of a Weyl group is *parabolic* if it is generated by simple reflections.

Lemma 5.4. *Let Φ be a root system of rank 3 that does not contain a subsystem of type G_2. Let W be the Weyl group of Φ, let $P \subset W$ be a parabolic subgroup, and let $\alpha_1, \alpha_2, \alpha_3 \in \Phi^+$ be distinct positive roots such that $s_{\alpha_1} \notin P$, $s_{\alpha_3} \notin P$, and $s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} \in P$. Then the roots $\alpha_1, \alpha_2, \alpha_3$ are linearly dependent.*

Proof. This has been checked using a computer for the root systems A_3, B_3, $A_2 \times A_1$, $B_2 \times A_1$, and $A_1 \times A_1 \times A_1$. $\square$

Proposition 5.5. *Let $X = G/P_X$ be a flag variety, and assume that the root system of G does not contain a subsystem of type G_2. Let $x, y, z \in X^T$ be pairwise adjacent T-fixed points. There exists a unique irreducible closed T-stable subvariety $\Omega \subset X$ such that $\Omega^T = \{x, y, z\}$. Furthermore, there exists $w \in W$ such that $w.\Omega$ is a Richardson variety in X whose minimal embedding is $\mathbb{P}^2$, $\mathrm{SG}(1,4)$, or Q^3. In particular, Ω is isomorphic to either $\mathbb{P}^2$ or the Schubert divisor in Q^3.*

Proof. It suffices to show that Ω exists with the stated properties, as uniqueness then follows because Ω is T-convex. Choose $\alpha_1, \alpha_2, \alpha_3 \in \Phi^+$ such that $s_{\alpha_1}.x = y$, $s_{\alpha_2}.y = z$, and $s_{\alpha_3}.z = x$, and set $\Phi' = \mathrm{Span}_{\mathbb{R}}\{\alpha_1, \alpha_2, \alpha_3\} \cap \Phi$. Since any basis of simple roots in Φ' can be extended to a basis of Φ, there exists $w \in W$ such that $w.\Phi' = \Phi_S$ for some subset $S \subset \Delta$. By replacing x, y, z with $w.x$, $w.y$, $w.z$, we may assume $\Phi' = \Phi_S$. By replacing X with the flag subvariety $G_S.x$, we may further assume that Φ has rank at most 3. By translating x, y, z by a second element from W, we may assume $x = 1.P_X$. Then Lemma 5.4 implies that Φ has rank 2. Since our assumptions imply that $P_X \neq B$ and Φ does not have type $A_1 \times A_1$ or G_2, we deduce that X is one of the spaces $\mathbb{P}^2$, $\mathrm{SG}(1,4) \cong \mathbb{P}^3$, or $\mathrm{LG}(2,4) \cong Q^3$. After a final translation of x, y, z by an element from W, we obtain $\{x, y, z\} = \Omega^T$, where $\Omega \subset X$ is a 2-dimensional Schubert variety in X. This completes the proof. □

Corollary 5.6. *Let $X = G/P_X$ be a flag variety, and assume that the root system of G does not contain a subsystem of type G_2. Let $X_u^v \subset X$ be a Richardson variety, and let $x, y, z \in (X_u^v)^T$ be pairwise adjacent T-fixed points. Then the curve classes $[C_{x,y}]$, $[C_{y,z}]$, $[C_{z,x}]$ generate a free abelian subgroup of rank 1 in $A_1(X_u^v)$, and at least two of the three curve classes individually generate this subgroup.*

Proof. Let $\Omega \subset X$ be as in Proposition 5.5. We have $A_1(\Omega) = \mathbb{Z}$. Since $\Omega^T \subset X_u^v$ and X_u^v is T-convex [BCP25], we have $\Omega \subset X_u^v$. Since Ω is itself T-convex, the three curves $C_{x,y}$, $C_{y,z}$, $C_{z,x}$ are contained in Ω. If $\Omega \cong \mathbb{P}^2$, then $[C_{x,y}] = [C_{y,z}] = [C_{z,x}]$ holds in $A_1(\Omega)$; notice that $\mathrm{SG}(1,4)$ is minuscule, so all T-stable curves in $\mathrm{SG}(1,4)$ have the same degree. If Ω is isomorphic to the Schubert divisor in Q^3, then Example 5.3 shows that $[C_{x,y}] = 2[C_{y,z}] = 2[C_{z,x}]$ holds, up to permuting the three points. The result follows from this since $A_1(\Omega)$ is a subgroup of $A_1(X_u^v)$. □

Corollary 5.6 is particularly useful for cominuscule Richardson varieties, as these contain many triangles of T-stable curves by the following result.

Proposition 5.7. *Let X be a cominuscule flag variety, and let $v, w \in W^X$ be adjacent elements such that $\ell(w) - \ell(v) \geq 2$. Then there exists $u \in W^X$ such that $v < u < w$ and u is adjacent to both v and w.*

Proof. We may assume that X is indecomposable. The claim has been verified using a computer when X is one of the exceptional spaces E_6/P_6 and E_7/P_7, so we may assume that $X = F_\omega(m; N)$ is a Grassmannian of classical Lie type. In particular, the Weyl group W is identified with a subgroup of the symmetric group S_N, see Section 2.2. Let $\widetilde{W^X}$ be the set of all m-element subsets $J \subset [1, N]$ for which $i \in J$ implies $\bar{i} = N + 1 - i \notin J$, and endow this set with the partial order defined by

$$\{i_1 < \cdots < i_m\} \leq \{j_1 < \cdots < j_m\} \iff i_p \leq j_p \text{ for all } p \in [1, m].$$

Then the map $\iota : W \to \widetilde{W^X}$ defined by $\iota(u) = \{u(1), \ldots, u(m)\}$ is order-preserving, satisfies $\iota(u'u) = u'.\iota(u)$ for $u', u \in W$, and $\iota(u) = \iota(u')$ holds if and only if

$u.P_X = u'.P_X$. We will say that $I, J \in \widetilde{W^X}$ are adjacent if $s_\alpha.I = J$ holds for some root $\alpha \in \Phi$. Then $u.P_X$ and $u'.P_X$ are adjacent T-fixed points in X if and only if $\iota(u)$ and $\iota(u')$ are adjacent in $\widetilde{W^X}$.

Assume that $X = Q^{N-2} = \mathrm{OG}(1, N)$ is a quadric hypersurface. Then we have $\widetilde{W^X} = \{\{j\} \mid 1 \leq j \leq N \text{ and } 2j \neq N+1\}$. Since $\tau_{i,k}(\{i\}) = \{k\}$, it follows that $\{i\}$ and $\{k\}$ are adjacent if and only if $\tau_{i,k} \in W$. If N is odd, then all distinct points in X^T are adjacent and the result is clear, so assume that $N = 2n$ is even. Then $\{i\}$ and $\{k\}$ are adjacent if and only if $i \neq k$ and $i + k \neq 2n + 1$. Equivalently, $u, u' \in W^X$ are adjacent if and only if $u \neq u'$ and $\ell(u) + \ell(u') \neq 2n - 2$. Notice also that $u' < u$ holds in W^X if and only if $\ell(u') < \ell(u)$. If $\ell(v) < n - 1 < \ell(w)$, then any element $u \in W^X$ with $\ell(u) = n - 1$ will satisfy the proposition, and otherwise we can choose any $u \in W^X$ such that $\ell(v) < \ell(u) < \ell(w)$.

Assume that $X = \mathrm{LG}(n, 2n)$ is a Lagrangian Grassmannian. Then $\widetilde{W^X}$ is the set of all n-element subsets $J \subset [1, 2n]$ such that $j \in J$ or $\bar{j} \in J$ for all $j \in [1, n]$. Set $I = \iota(v)$ and $K = \iota(w)$. It suffices to show that there exists $J \in \widetilde{W^X}$ such that $I < J < K$ and J is adjacent to both I and K. Indeed, if $J = s_\alpha.I$, then $u = (s_\alpha v)^X \in W^X$ satisfies the proposition. Choose $i, k \in [1, 2n]$ such that $i < k \leq \bar{i}$ and $\tau_{i,k}(I) = K$. We have $I = (I \cap K) \cup \{i, \bar{k}\}$ and $K = (I \cap K) \cup \{k, \bar{i}\}$ since $I < K$, and $k - i \geq 2$ holds because K does not cover I. Set $j = i + 1$ and $A = (I \cap K) \smallsetminus \{j, \bar{j}\}$. If $I \cap K \cap \{j, \bar{j}\} = \emptyset$, then we must have $j \in \{i, \bar{k}, k, \bar{i}\}$, but only $j = \bar{k}$ is possible. We can therefore use one of the following three cases to define the required element $J \in \widetilde{W^X}$:

Case	I	J	K
$j \in I \cap K$	$A \cup \{i, j, \bar{k}\}$	$\tau_{j,k}(I) = A \cup \{i, k, \bar{j}\} = \tau_{i,j}(K)$	$A \cup \{j, k, \bar{i}\}$
$\bar{j} \in I \cap K$	$A \cup \{i, \bar{k}, \bar{j}\}$	$\tau_{i,j}(I) = A \cup \{j, \bar{k}, \bar{i}\} = \tau_{j,k}(K)$	$A \cup \{k, \bar{j}, \bar{i}\}$
$j = \bar{k}$	$A \cup \{i, \bar{k}\}$	$\tau_{i,\bar{i}}(I) = A \cup \{\bar{k}, \bar{i}\} = \tau_{\bar{k},k}(K)$	$A \cup \{k, \bar{i}\}$

If $X = \mathrm{Gr}(m, n)$ is a Grassmannian of type A_{n-1}, then X is a flag subvariety of $\mathrm{LG}(n, 2n)$, so the result for X follows from the case just established. Finally, assume that X is a maximal orthogonal Grassmannian. Then X is isomorphic to the space $\mathrm{OG}(n, 2n+1)$ of type B_n. Since the Weyl groups of types B_n and C_n are isomorphic as Coxeter groups, and the isomorphism identifies the parabolic subgroups corresponding to $\mathrm{OG}(n, 2n + 1)$ and $\mathrm{LG}(n, 2n)$, the proposition for $\mathrm{OG}(n, 2n + 1)$ also follows from the case of $\mathrm{LG}(n, 2n)$. This completes the proof. □

Example 5.8. Let $Y = G_{\{\beta_1,\beta_2\}} s_3 s_2.P_X$ be the Richardson variety in $X = \mathrm{SG}(2, 6)$ from Example 3.5. We have $Y \cong \mathrm{Fl}(3)$, so Y contains no triangles of T-stable curves, and Corollary 5.6 provides no information about $A_1(Y)$.

5.3. **The first Chow group of a product.** We next compute the first Chow group of a product of varieties under suitable conditions. Let Γ be a connected solvable linear algebraic group, and let X be a projective Γ-variety. A result of Hirschowitz [Hir73] states that each equivariant Chow group $A_d^\Gamma(X)$, defined as the group of Γ-stable cycles modulo rational equivalence given by semi-invariant rational functions, is isomorphic to the usual Chow group $A_d(X)$. This result was generalized to algebraic schemes in [FMSS95]. We need the following consequence.

Proposition 5.9. *Let Γ be a connected solvable linear algebraic group, and let X be a projective Γ-variety. Then $A_d(X)$ is generated by Γ-stable cycles for all d.*

Given projective varieties X and Y, the pushforward along the projections from $X \times Y$ defines a natural group homomorphism

$$\rho : A_1(X \times Y) \to A_1(X) \oplus A_1(Y)\,. \tag{6}$$

Lemma 5.10. *Let X and Y be projective varieties satisfying $A_0(X) \cong \mathbb{Z} \cong A_0(Y)$. Assume that a connected solvable linear algebraic group acts algebraically on X with finitely many fixed points. Then the map (6) is an isomorphism.*

Proof. Let $p \in A_0(X)$ and $q \in A_0(Y)$ denote the point classes of X and Y, and let $\phi : A_1(X) \oplus A_1(Y) \to A_1(X \times Y)$ be the group homomorphism defined by $\phi(\sigma \oplus \tau) = \sigma \times q + p \times \tau$, see [Ful98, §1.10]. Since $\rho \circ \phi$ is the identity, it suffices to show that ϕ is surjective. Let Γ be the solvable group acting on X, and extend the action to $X \times Y$ by $g.(x, y) = (g.x, y)$. Let $C \subset X \times Y$ be a 1-dimensional closed irreducible subvariety that is stable under this action. By Proposition 5.9 it is enough to show that the class $[C] \in A_1(X \times Y)$ belongs to the image of ϕ. Let $C' = \pi_X(C) \subset X$ be the image of C under the projection $\pi_X : X \times Y \to X$. Then C' is a Γ-stable irreducible closed subvariety of X. If $\dim(C') = 1$, then choose $x_0 \in C'$ such that x_0 is not a Γ-fixed point in X, and choose $y_0 \in Y$ such that $(x_0, y_0) \in C$. Using that C contains the orbit $\Gamma.(x_0, y_0)$, we obtain $C = C' \times \{y_0\}$ and $[C] = \phi([C'] \oplus 0)$. Otherwise $C' = \{x_0\}$ is a single point in X. In this case we obtain $C = \{x_0\} \times C''$, where $C'' = \pi_Y(C)$ is the projection of C to Y, and therefore $[C] = \phi(0 \oplus [C''])$, as required. □

We will say that a projective variety X is *strongly indecomposable* if $A_0(X) \cong \mathbb{Z}$, $A_1(X) \cong \mathbb{Z}$, and some connected solvable linear algebraic group acts on X with finitely many fixed points. This implies that X is indecomposable by Lemma 3.14.

Corollary 5.11. *Let $X_1, \dots, X_r$ and $Y_1, \dots, Y_s$ be strongly indecomposable projective varieties, and assume that $\phi : X_1 \times \cdots \times X_r \to Y_1 \times \cdots \times Y_s$ is an isomorphism of algebraic varieties. Then $r = s$, and ϕ is given by $\phi(x_1, \dots, x_r) = (\phi_1(x_{\sigma(1)}), \dots, \phi_r(x_{\sigma(r)}))$ for some permutation $\sigma \in S_r$ and isomorphisms $\phi_j : X_{\sigma(j)} \to Y_j$ for $1 \le j \le r$.*

Proof. Identify $A_1(X_i) = \mathbb{Z}$ and $A_1(Y_j) = \mathbb{Z}$ such that curve classes in X_i and Y_j correspond to positive integers, and set $X = X_1 \times \cdots \times X_r$ and $Y = Y_1 \times \cdots \times Y_s$. By Lemma 5.10 we have $A_1(X) = \mathbb{Z}^r$ and $A_1(Y) = \mathbb{Z}^s$. Since $\phi_* : A_1(X) \to A_1(Y)$ is an isomorphism, we obtain $r = s$. In addition, both ϕ_* and its inverse $(\phi^{-1})_*$ are given by matrices of non-negative integers. This implies that the matrix of ϕ_* is a permutation matrix, hence each composition $\pi_j \phi : X \to Y_j$ of ϕ with the projection to Y_j depends only on $X_{\sigma(j)}$ for some permutation $\sigma \in S_r$. This shows that ϕ has the claimed form. □

Let X_u^v be a Richardson variety in the flag variety $X = G/P_X$. Since the maximal torus T acts on X_u^v with finitely many fixed points, the following observation shows that Lemma 5.10 applies to products of Richardson varieties.

Lemma 5.12. *For any Richardson variety X_u^v we have $A_0(X_u^v) = \mathbb{Z}$.*

Proof. By Proposition 5.9 we must show that all T-fixed points in X_u^v are rationally equivalent. Any such point has the form $w.P_X$, where $w \in W^X$ and $v \le w \le u$.

There exists a chain in the Bruhat order on W^X, $v = v_0 < v_1 < \cdots < v_r = w$, such that $\ell(v_i) = \ell(v) + i$ for $1 \leq i \leq r$. Using that each Richardson variety $X_{v_i}^{v_{i-1}}$ is isomorphic to $\mathbb{P}^1$ and contained in X_u^v, we obtain $[v_{i-1}.P_X] = [v_i.P_X] \in A_0(X_u^v)$ for all $i \in [1, r]$. This implies $[w.P_X] = [v.P_X]$ in $A_0(X_u^v)$, as required. □

5.4. **The first Chow group of a cominuscule Richardson variety.** We can now compute the first Chow group of any cominuscule Richardson variety. Given a cominuscule flag variety X and two straight shapes $\tau \subset \sigma \subset \mathcal{P}_X$, we will say that σ and τ are *adjacent* if the corresponding points $w_\tau.P_X, w_\sigma.P_X \in X^T$ are adjacent. Equivalently, we have $\tau \neq \sigma = s_\alpha \bullet \tau$ for some $\alpha \in \Phi^+$. In this case we let $C_{\tau,\sigma}$ denote the unique T-stable curve through $w_\tau.P_X$ and $w_\sigma.P_X$. Since any Richardson variety X_λ^μ is T-convex, we have $C_{\tau,\sigma} \subset X_\lambda^\mu$ if and only if $\mu \subset \tau \subset \sigma \subset \lambda$.

Lemma 5.13. *Let X be a cominuscule flag variety, and let $\mu \subset \lambda \subset \mathcal{P}_X$ be straight shapes. Assume that $\theta = \lambda/\mu$ is connected and $\delta_X(\theta) = \delta_X(\mathcal{P}_X)$. Then pushforward along the inclusion $X_\lambda^\mu \subset X$ is an isomorphism of groups $A_1(X_\lambda^\mu) \cong A_1(X)$.*

Proof. It follows from Lemma 4.6(c) that $\mathcal{P}_X$ is connected, so we have $A_1(X) \cong \mathbb{Z}$ by Lemma 4.2. In view of Lemma 5.1, it suffices to show that $A_1(X_\lambda^\mu) = \mathbb{Z}[X_{\{\alpha\}}]$ holds for any long box $\alpha \in \theta$. Proposition 5.9 implies that $A_1(X_\lambda^\mu)$ is generated by the classes $[C_{\sigma,\tau}]$ for which σ and τ are adjacent and $\mu \subset \tau \subset \sigma \subset \lambda$. If $|\sigma/\tau| \geq 2$, then Proposition 5.7 shows that there exists a straight shape κ such that $\tau \subset \kappa \subset \sigma$ and κ is adjacent to both τ and σ. Corollary 5.6 then implies that $[C_{\tau,\sigma}]$ is a multiple of either $[C_{\tau,\kappa}]$ or $[C_{\kappa,\sigma}]$. It follows by induction on $|\sigma/\tau|$ that $A_1(X_\lambda^\mu)$ is generated by the classes $[C_{\tau,\sigma}]$ for which $\sigma/\tau = \{\alpha\}$ contains a single box. In this case $C_{\tau,\sigma} = X_\sigma^\tau$ is a 1-dimensional Richardson variety contained in X_λ^μ, and we have $[C_{\tau,\sigma}] = [X_{\{\alpha\}}]$. If all boxes of θ are long, then the claim follows from Lemma 5.2 and the assumption that θ is connected, so assume that θ contains both long and short boxes. If $\alpha' \in \theta$ is any short box, then since θ is connected, there exists a path from α' to some long box $\alpha \in \theta$, in which case Lemma 5.2 implies that $[X_{\{\alpha'\}}] = 2[X_{\{\alpha\}}]$. It follows that $A_1(X_\lambda^\mu)$ is generated by the classes $[X_{\{\alpha\}}]$ defined by all long boxes $\alpha \in \theta$. It remains to prove that these classes are all the same. Since X is either an odd quadric or a Lagrangian Grassmannian, an inspection of Table 1 shows that the long boxes in $\mathcal{P}_X$ are linearly ordered. Let the long boxes in θ be $\{\alpha_1 < \alpha_2 < \cdots < \alpha_\ell\}$. If $\alpha_i < \alpha_{i+1}$ is a covering in $\mathcal{P}_X$, then Lemma 5.2 shows $[X_{\{\alpha_i\}}] = [X_{\{\alpha_{i+1}\}}] \in A_1(X_\lambda^\mu)$. Otherwise Table 1 shows that $\delta_X(\alpha_i) = \delta_X(\alpha_{i+1})$, and we have $\alpha_i < \beta < \alpha_{i+1}$ for a unique short box $\beta \in \mathcal{P}_X$, such that $\{\alpha_i, \beta, \alpha_{i+1}\}$ is a skew shape in $\mathcal{P}_X$. Choose straight shapes $\tau, \sigma \subset \mathcal{P}_X$ such that $\mu \subset \tau \subset \sigma \subset \lambda$ and $\sigma/\tau = \{\alpha_i, \beta, \alpha_{i+1}\}$, and set $S = \delta_X(\sigma/\tau)$ and $Y = G_S w_\sigma.P_X$. We deduce from Proposition 4.9 that $Y = X_\sigma^\tau \cong Q^3$, hence $[X_{\{\alpha_i\}}] = [X_{\{\alpha_{i+1}\}}]$ holds in $A_1(X_\sigma^\tau)$ by Lemma 5.1. Since $X_\sigma^\tau \subset X_\lambda^\mu$, we obtain $[X_{\{\alpha_i\}}] = [X_{\{\alpha_{i+1}\}}]$ in $A_1(X_\lambda^\mu)$, which completes the proof. □

Theorem 5.14. *Let X be a cominuscule flag variety, and let $\theta \subset \mathcal{P}_X$ be a skew shape. Let $\theta_1, \ldots, \theta_r$ be the connected components of θ.*

(a) *The Richardson variety X_θ is indecomposable if and only if θ is connected.*

(b) *We have $X_\theta \cong X_{\theta_1} \times \cdots \times X_{\theta_r}$ as algebraic varieties; the associated projections $X_\theta \to X_{\theta_i}$ are uniquely determined by the isomorphism class of X_θ as a variety, up to reordering and automorphisms of the targets.*

(c) *The pushforwards along the projections define an isomorphism of first Chow groups* $A_1(X_\theta) \cong A_1(X_{\theta_1}) \oplus \cdots \oplus A_1(X_{\theta_r})$.

(d) *The first Chow group* $A_1(X_\theta)$ *is a free abelian group of rank* r.

(e) *Assume that* X *is the minimal embedding of* X_θ. *Then proper pushforward along the inclusion* $i : X_\theta \to X$ *is an isomorphism of groups* $i_* : A_1(X_\theta) \xrightarrow{\cong} A_1(X)$.

Proof. The isomorphism $X_\theta \cong X_{\theta_1} \times \cdots \times X_{\theta_r}$ follows from Corollary 4.15, after which part (c) follows from Lemma 5.10 and Lemma 5.12. For the remaining claims we may assume that X is the minimal embedding of X_θ, so that $\delta_X(\theta) = \delta_X(\mathcal{P}_X)$. If θ is connected, then so is $\mathcal{P}_X$ by Lemma 4.6(c), so we have $A_1(X_\theta) \cong A_1(X) \cong \mathbb{Z}$ by Lemma 5.13 and Lemma 4.2. Part (a) now follows from Lemma 3.14. The uniqueness of the factorization in part (b) follows from Corollary 5.11, and part (d) follows from (c) since $A_1(X_{\theta_i}) \cong \mathbb{Z}$ for $1 \leq i \leq r$. Finally, part (e) follows from part (c) and Lemma 5.13, noting that $\widehat{\theta}_1, \ldots, \widehat{\theta}_r$ are the connected components of $\mathcal{P}_X$. This completes the proof. □

5.5. **The Picard group of a cominuscule Richardson variety.** Given any complete variety Z, we let $A_1(Z)^*$ denote the dual of the first Chow group of Z. The *degree map* of Z is the natural group homomorphism $\mathrm{Pic}(Z) \to A_1(Z)^*$ defined by the intersection pairing

$$\mathrm{Pic}(Z) \otimes A_1(Z) \to \mathbb{Z} \quad ; \quad \mathcal{L} \otimes [C] \mapsto \int_C c_1(\mathcal{L}) = \deg(\mathcal{L}|_C)\,.$$

Our computation of the Picard group $\mathrm{Pic}(X_\theta)$ requires the following fact.

Theorem 5.15. *Let* X_θ *be any cominuscule Richardson variety. There exists a desingularization* $\pi : \widetilde{X_\theta} \to X_\theta$ *such that the degree map* $\mathrm{Pic}(\widetilde{X_\theta}) \to A_1(\widetilde{X_\theta})^*$ *is injective.*

We prove this statement in Section 6 when the minimal embedding of X_θ is a product of cominuscule flag varieties of classical Lie types, see Corollary 6.9. In response to our questions, the general case of Theorem 5.15 has been proved by David Anderson and Sam Payne [AP26], by showing that Brion's *brick resolution* from the proof of [Bri05, Thm. 4.2.1] (see also [KLS14, Appendix A]) satisfies the required property (and much more!).

Theorem 5.16. *Let* X *be a cominuscule flag variety, and let* $\theta \subset \mathcal{P}_X$ *be a skew shape. Let* $\theta_1, \ldots, \theta_r$ *be the connected components of* θ.

(a) *The degree map is an isomorphism of groups* $\mathrm{Pic}(X_\theta) \xrightarrow{\cong} A_1(X_\theta)^*$.

(b) *We have* $\mathrm{Pic}(X_\theta) \cong \mathrm{Pic}(X_{\theta_1}) \oplus \cdots \oplus \mathrm{Pic}(X_{\theta_r})$.

(c) *The Picard group* $\mathrm{Pic}(X_\theta)$ *is a free abelian group of rank* r.

(d) *Assume that* X *is the minimal embedding of* X_θ. *Then pullback along the inclusion* $i : X_\theta \to X$ *is an isomorphism of groups* $i^* : \mathrm{Pic}(X) \xrightarrow{\cong} \mathrm{Pic}(X_\theta)$.

Proof. We may assume that X is the minimal embedding of X_θ, so that the pushforward map $i_* : A_1(X_\theta) \to A_1(X)$ is an isomorphism by Theorem 5.14(e). Let $\pi : \widetilde{X_\theta} \to X_\theta$ be the desingularization of Theorem 5.15. Since X_θ is normal, we have $\pi_*(\mathcal{O}_{\widetilde{X_\theta}}) = \mathcal{O}_{X_\theta}$ by Zariski's main theorem. Given any line bundle $\mathcal{L}$ on X_θ, we obtain

$$\pi_*(\pi^*\mathcal{L}) = \pi_*(\pi^*\mathcal{L} \otimes \mathcal{O}_{\widetilde{X_\theta}}) = \mathcal{L} \otimes \pi_*(\mathcal{O}_{\widetilde{X_\theta}}) = \mathcal{L}\,.$$

It follows that the pullback map $\pi^* : \mathrm{Pic}(X_\theta) \to \mathrm{Pic}(\widetilde{X_\theta})$ is injective. We obtain the following commutative diagram, where the vertical maps are degree maps.

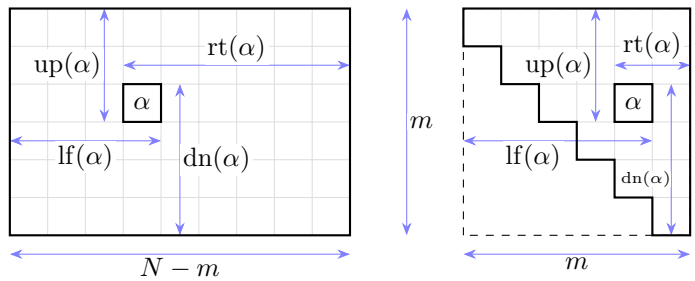
$$\begin{tikzcd}
\mathrm{Pic}(X) \arrow[r, "i^*"'] \arrow[d, "\cong"'] & \mathrm{Pic}(X_\theta) \arrow[r, hook, "\pi^*"'] \arrow[d] & \mathrm{Pic}(\widetilde{X_\theta}) \arrow[d, hook] \\
A_1(X)^* \arrow[r, "(i_*)^*", "\cong"'] & A_1(X_\theta)^* \arrow[r, "(\pi_*)^*"] & A_1(\widetilde{X_\theta})^*
\end{tikzcd}$$

The right square of this diagram implies that the degree map $\mathrm{Pic}(X_\theta) \to A_1(X_\theta)^*$ is injective, after which the left square shows that this map as well as the pullback map $i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X_\theta)$ are isomorphisms. The result therefore follows from Theorem 5.14. □

6. Desingularization of Richardson varieties in Grassmannians

In this section we construct an explicit desingularization of any Richardson variety in a Grassmannian of type A or a maximal isotropic Grassmannian, in the style of Zelevinsky's desingularization of Schubert varieties [Zel83]. This desingularization is obtained as a tower of (isotropic) Grassmann bundles and is used to prove that Theorem 5.15 holds for products of cominuscule Richardson varieties of classical Lie types when $\mathrm{char}(\mathbb{K}) \neq 2$. Our construction is of independent interest, as the known desingularization of Richardson varieties in [Bri02] involves a fiber product of Bott–Samelson varieties, which is less explicit. We give a unified treatment of the Grassmannian $\mathrm{Gr}(m, N)$ of type A, the Lagrangian Grassmannian $\mathrm{LG}(m, 2m)$ of type C, and the maximal orthogonal Grassmannian $\mathrm{OG}(m, 2m+1)$ of type B, noting that $\mathrm{OG}(m, 2m+1)$ is isomorphic to the cominuscule orthogonal Grassmannian $\mathrm{OG}(m+1, 2m+2)$ of type D. The chosen Grassmannian will be denoted X, and we use the construction of X given in Section 2.2. In particular, we assume $\mathrm{char}(\mathbb{K}) \neq 2$ when X is an orthogonal Grassmannian, and we set $N = 2m$ for $X = \mathrm{LG}(m, 2m)$, and $N = 2m+1$ for $X = \mathrm{OG}(m, 2m+1)$.

The partially ordered set $\mathcal{P}_X$ is identified with a diagram of boxes in the plane; see Table 1. For $X = \mathrm{Gr}(m, N)$, $\mathcal{P}_X$ is a rectangle with m rows and $N-m$ columns, and for $X = \mathrm{LG}(m, 2m)$ and $X = \mathrm{OG}(m, 2m+1)$, $\mathcal{P}_X$ is the upper-triangular part of an $m \times m$ rectangle. Given any box $\alpha \in \mathcal{P}_X$, let $\mathrm{up}(\alpha)$ be the row number of α counted from the top of $\mathcal{P}_X$ and let $\mathrm{dn}(\alpha)$ be the row number counted from the bottom, so that $\mathrm{up}(\alpha) + \mathrm{dn}(\alpha) = m+1$. Let $\mathrm{lf}(\alpha)$ be the column number of α counted from the left and let $\mathrm{rt}(\alpha)$ be the column number counted from the right side of $\mathcal{P}_X$, so that $\mathrm{lf}(\alpha) + \mathrm{rt}(\alpha) = N - m + 1$ for $X = \mathrm{Gr}(m, N)$, and $\mathrm{lf}(\alpha) + \mathrm{rt}(\alpha) = m+1$ for $X = \mathrm{LG}(m, 2m)$ and for $X = \mathrm{OG}(m, 2m+1)$. We also set $\mathrm{ur}(\alpha) = \mathrm{up}(\alpha) + \mathrm{rt}(\alpha) - 1$. The partial order on $\mathcal{P}_X$ is given by $\alpha' \leq \alpha''$ if and only if $\mathrm{up}(\alpha') \leq \mathrm{up}(\alpha'')$ and $\mathrm{lf}(\alpha') \leq \mathrm{lf}(\alpha'')$.



Let $E_\bullet \subset \mathbb{K}^N$ be the B-stable flag defined by $E_i = \langle e_1, \dots, e_i \rangle$ for $0 \le i \le N$, and let $F_\bullet \subset \mathbb{K}^N$ be the B^--stable flag defined by $F_i = \langle e_{N+1-i}, \dots, e_N \rangle$. In types B and C we have $E_i^\perp = E_{N-i}$ and $F_i^\perp = F_{N-i}$, where $U^\perp$ denotes the kernel of the linear map $\mathbb{K}^N \to U^*$ defined by the nonzero bilinear form ω on $\mathbb{K}^N$, for any subspace $U \subset \mathbb{K}^N$. Each box $\alpha \in \mathcal{P}_X$ defines a B-stable Schubert variety

$$X_\alpha = \{V \in X \mid \dim(V^\perp + E_{N-\mathrm{ur}(\alpha)}) \le N - \mathrm{rt}(\alpha)\}\,,$$

where we write $V^\perp = V$ in type A, and a B^--stable Schubert variety

$$X^\alpha = \{V \in X \mid \dim(V \cap F_{\mathrm{ur}(\alpha)}) \ge \mathrm{up}(\alpha)\}\,.$$

Proposition 6.1. *The Schubert varieties in X defined by a straight shape $\lambda \subset \mathcal{P}_X$ are given by*

$$X_\lambda = \bigcap_{\alpha \notin \lambda} X_\alpha \qquad \textit{and} \qquad X^\lambda = \bigcap_{\alpha \in \lambda} X^\alpha\,.$$

The first intersection is over all boxes of $\mathcal{P}_X \smallsetminus \lambda$; it suffices to include all minimal boxes of $\mathcal{P}_X \smallsetminus \lambda$ in this intersection. Similarly, it suffices to include all maximal boxes of λ in the second intersection.

Proof. The definition of X^λ matches known definitions of Schubert varieties, noting that all Schubert conditions follow from those given by maximal boxes of λ; see [Ful97, §9.4], [FP98, §3.1], and [BKT15, Prop. A1]. Let $\lambda^\vee = w_0^X.(\mathcal{P}_X \smallsetminus \lambda)$ be the Poincaré dual shape of λ; see [BS16, §2]. Using that $X_{\lambda^\vee} = w_0.X^\lambda$, it suffices to show that $X_\alpha = w_0.X^{w_0^X.\alpha}$ for all $\alpha \in \mathcal{P}_X$. The effect of applying w_0 is to replace $F_\bullet$ with $E_\bullet = w_0.F_\bullet$, and the action of w_0^X on $\mathcal{P}_X$ is the order-reversing involution that rotates by 180 degrees in type A and reflects across a southwest-to-northeast diagonal in types B and C. In type A we have $\mathrm{ur}(w_0^X.\alpha) = N - \mathrm{ur}(\alpha)$, $\mathrm{up}(w_0^X.\alpha) = \mathrm{dn}(\alpha)$, and $m + N - \mathrm{ur}(\alpha) - \mathrm{dn}(\alpha) = N - \mathrm{rt}(\alpha)$, which implies

$$\begin{aligned} w_0.X^{w_0^X.\alpha} &= \{V \in X \mid \dim(V \cap E_{N-\mathrm{ur}(\alpha)}) \ge \mathrm{dn}(\alpha)\} \\ &= \{V \in X \mid \dim(V + E_{N-\mathrm{ur}(\alpha)}) \le N - \mathrm{rt}(\alpha)\}\,. \end{aligned}$$

In types B and C we have $(V \cap E_{\mathrm{ur}(\alpha)})^\perp = V^\perp + E_{N-\mathrm{ur}(\alpha)}$, $\mathrm{ur}(w_0^X.\alpha) = \mathrm{ur}(\alpha)$, and $\mathrm{up}(w_0^X.\alpha) = \mathrm{rt}(\alpha)$, which implies

$$\begin{aligned} w_0.X^{w_0^X.\alpha} &= \{V \in X \mid \dim(V \cap E_{\mathrm{ur}(\alpha)}) \ge \mathrm{rt}(\alpha)\} \\ &= \{V \in X \mid \dim(V^\perp + E_{N-\mathrm{ur}(\alpha)}) \le N - \mathrm{rt}(\alpha)\}\,. \end{aligned}$$

The proposition follows from this. □

Given a Richardson variety $X_\lambda^\mu \subset X$, we define the sets of integers

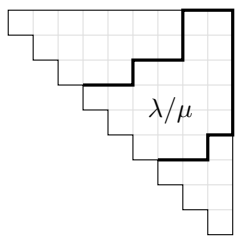
$$a^-(\mu) = \{\mathrm{up}(\alpha) : \alpha \in \max(\mu)\} \quad \text{and} \quad a^+(\lambda) = \{N - \mathrm{rt}(\alpha) : \alpha \in \min(\mathcal{P}_X \smallsetminus \lambda)\}\,.$$

Let $\mathrm{Fl}(a^-(\mu) \cup a^+(\lambda); N)$ be the variety of partial flags in $\mathbb{K}^N$ of the indicated dimensions; this flag variety has Lie type A regardless of the type of X. Let $\widehat{X}^\mu_\lambda \subset \mathrm{Fl}(a^-(\mu) \cup a^+(\lambda); N)$ be the Richardson variety of partial flags $V_\bullet$ satisfying

$$V_{N-\mathrm{rt}(\alpha)} \supset E_{N-\mathrm{ur}(\alpha)} \quad \text{for each minimal box } \alpha \text{ in } \mathcal{P}_X \smallsetminus \lambda, \text{ and}$$
$$V_{\mathrm{up}(\alpha)} \subset F_{\mathrm{ur}(\alpha)} \quad \text{for each maximal box } \alpha \text{ in } \mu.$$

Let $\widetilde{X}^\mu_\lambda \subset \mathrm{Fl}(a^-(\mu) \cup \{m\} \cup a^+(\lambda); N)$ be the subvariety of partial flags $V_\bullet$ satisfying the above conditions defining $\widehat{X}^\mu_\lambda$, in addition to $V_m \in X$; for $X = \mathrm{OG}(m, 2m+1)$ we further require $V_m^\perp \subset V_{N-\mathrm{rt}(\alpha)}$ for $\alpha \in \min(\mathcal{P}_X \smallsetminus \lambda)$. We prove in Theorem 6.7 below that the projection $\widetilde{X}^\mu_\lambda \to X^\mu_\lambda$ defined by $V_\bullet \mapsto V_m$ is a resolution of singularities.

Example 6.2. Let $X = \mathrm{LG}(9, 18)$ and consider the Richardson variety X^μ_λ defined by the skew shape:



Since $V^\perp = V$ holds for $V \in X$, we obtain

$$X^\mu = \{V \in X \mid \dim(V \cap F_4) \geq 2 \text{ and } \dim(V \cap F_7) \geq 3\} \quad \text{and}$$
$$X_\lambda = \{V \in X \mid \dim(V + E_9) \leq 15 \text{ and } \dim(V + E_{12}) \leq 17\}\,.$$

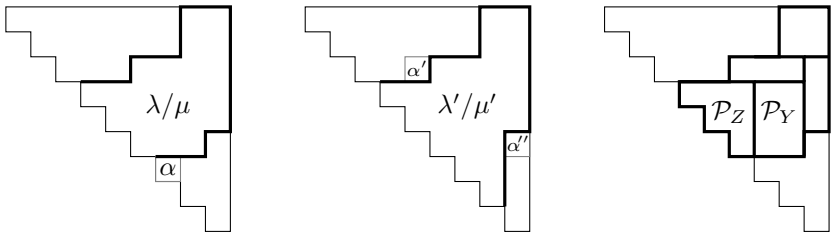
The Richardson variety $\widehat{X}^\mu_\lambda \subset \mathrm{Fl}(2, 3, 15, 17; 18)$ is defined by the conditions

$$V_2 \subset F_4\,, \quad V_3 \subset F_7\,, \quad V_{15} \supset E_9\,, \quad V_{17} \supset E_{12}\,.$$

Using that $\widehat{X}^\mu_\lambda$ is an intersection of non-singular opposite Schubert varieties in $\mathrm{Fl}(a^-(\mu) \cup a^+(\lambda); N)$, it follows from Kleiman's transversality theorem [Kle74] that $\widehat{X}^\mu_\lambda$ is non-singular in characteristic zero. The same applies to $\widetilde{X}^\mu_\lambda$ in type A. To prove that $\widehat{X}^\mu_\lambda$ is non-singular in arbitrary characteristic, we show that this variety can be constructed as a tower of Grassmann bundles.

Choose $\alpha \in \max(\mu) \cup \min(\mathcal{P}_X \smallsetminus \lambda)$ with $\mathrm{ur}(\alpha)$ maximal. Let $\mu' \subset \mathcal{P}_X$ and $\lambda' \subset \mathcal{P}_X$ be the unique straight shapes defined by $\max(\mu') = \max(\mu) \smallsetminus \{\alpha\}$ and $\min(\mathcal{P}_X \smallsetminus \lambda') = \min(\mathcal{P}_X \smallsetminus \lambda) \smallsetminus \{\alpha\}$. We have $\mu' = \mu$ or $\lambda' = \lambda$.

Example 6.3. For the Richardson variety of Example 6.2 we obtain the following box α and straight shapes $\mu' \subset \lambda' \subset \mathcal{P}_X$. The boxes α' and α'' are used in the proof of Lemma 6.4, and the shapes $\mathcal{P}_Y$ and $\mathcal{P}_Z$ are discussed in Remark 6.5.



Lemma 6.4. *The natural projection $\widehat{X}^\mu_\lambda \to \widehat{X}^{\mu'}_{\lambda'}$ is a Grassmann bundle.*

Proof. Choose $\alpha' \in \max(\mu')$ with $\mathrm{ur}(\alpha')$ maximal, and choose $\alpha'' \in \min(\mathcal{P}_X \smallsetminus \lambda')$ with $\mathrm{ur}(\alpha'')$ maximal. If μ' is empty, so that α' does not exist, we use the notation $\mathrm{up}(\alpha') = \mathrm{ur}(\alpha') = 0$. Similarly, we write $\mathrm{rt}(\alpha'') = \mathrm{ur}(\alpha'') = 0$ when $\lambda' = \mathcal{P}_X$. Let $\mathcal{V}_\bullet$ denote the tautological flag on $\widehat{X}^{\mu'}_{\lambda'}$.

Assume first that $\alpha \in \mu$. Given a point $V_\bullet \in \widehat{X}^{\mu'}_{\lambda'} = \widehat{X}^{\mu'}_{\lambda}$, points in the fiber over $V_\bullet$ are obtained by adjoining a subspace $V_{\mathrm{up}(\alpha)}$ of dimension $\mathrm{up}(\alpha)$ to $V_\bullet$ satisfying $V_{\mathrm{up}(\alpha')} \subset V_{\mathrm{up}(\alpha)} \subset V_{N-\mathrm{rt}(\alpha'')} \cap F_{\mathrm{ur}(\alpha)}$. By our choice of α, we have $V_{N-\mathrm{rt}(\alpha'')} + F_{\mathrm{ur}(\alpha)} \supset E_{N-\mathrm{ur}(\alpha'')} + F_{\mathrm{ur}(\alpha)} = \mathbb{K}^N$, which implies that $\mathcal{V}_{N-\mathrm{rt}(\alpha'')} \cap F_{\mathrm{ur}(\alpha)}$ is a well-defined vector subbundle of $\widehat{X}^{\mu'}_{\lambda} \times \mathbb{K}^N$ of rank $\mathrm{ur}(\alpha) - \mathrm{rt}(\alpha'')$. We obtain

$$\widehat{X}^{\mu}_{\lambda} \cong \mathrm{Gr}\left(\mathrm{up}(\alpha) - \mathrm{up}(\alpha'), (\mathcal{V}_{N-\mathrm{rt}(\alpha'')} \cap F_{\mathrm{ur}(\alpha)})/\mathcal{V}_{\mathrm{up}(\alpha')}\right) \to \widehat{X}^{\mu'}_{\lambda}\,.$$

Otherwise we have $\alpha \in \mathcal{P}_X \smallsetminus \lambda$, and points in the fiber over $V_\bullet \in \widehat{X}^{\mu'}_{\lambda'} = \widehat{X}^{\mu}_{\lambda'}$ are obtained by adjoining a subspace $V_{N-\mathrm{rt}(\alpha)}$ of dimension $N - \mathrm{rt}(\alpha)$ to $V_\bullet$ satisfying $V_{\mathrm{up}(\alpha')} + E_{N-\mathrm{ur}(\alpha)} \subset V_{N-\mathrm{rt}(\alpha)} \subset V_{N-\mathrm{rt}(\alpha'')}$. By our choice of α, we have $V_{\mathrm{up}(\alpha')} \cap E_{N-\mathrm{ur}(\alpha)} \subset F_{\mathrm{ur}(\alpha')} \cap E_{N-\mathrm{ur}(\alpha)} = 0$, which implies that $\mathcal{V}_{\mathrm{up}(\alpha')} \oplus E_{N-\mathrm{ur}(\alpha)}$ is a vector subbundle of $\widehat{X}^{\mu}_{\lambda'} \times \mathbb{K}^N$. Using that $\mathrm{ur}(\alpha) - \mathrm{rt}(\alpha) = \mathrm{up}(\alpha) - 1$, we obtain

$$\widehat{X}^{\mu}_{\lambda} \cong \mathrm{Gr}\left(\mathrm{up}(\alpha) - \mathrm{up}(\alpha') - 1, \mathcal{V}_{N-\mathrm{rt}(\alpha'')}/(\mathcal{V}_{\mathrm{up}(\alpha')} \oplus E_{N-\mathrm{ur}(\alpha)})\right) \to \widehat{X}^{\mu}_{\lambda'}\,.$$

This completes the proof. □

Remark 6.5. Let $[\lambda/\mu]$ denote the subset of boxes in λ/μ that are in the same row as a box from μ or in the same column as a box from $\mathcal{P}_X \smallsetminus \lambda$. It follows from the proof of Lemma 6.4 that $\dim(\widehat{X}^{\mu}_{\lambda}) = \#[\lambda/\mu]$. In fact, $[\lambda/\mu] \smallsetminus [\lambda'/\mu']$ is a rectangle that can be identified with $\mathcal{P}_Y$, where Y is the fiber of the projection $\widehat{X}^{\mu}_{\lambda} \to \widehat{X}^{\mu'}_{\lambda'}$. Moreover, the shape $\lambda/\mu \smallsetminus [\lambda/\mu]$ is either a rectangle or an upper-triangular shape. The proof of Lemma 6.6 shows that this shape can be identified with $\mathcal{P}_Z$, where Z is the fiber of the projection $\widetilde{X}^{\mu}_{\lambda} \to \widehat{X}^{\mu}_{\lambda}$. The desingularization $\widetilde{X}^{\mu}_{\lambda}$ of the Richardson variety from Example 6.2 is a tower of four Grassmann bundles followed by one Lagrangian Grassmannian bundle, where the fibers correspond to the rectangular and upper-triangular shapes in the last diagram of Example 6.3.

Lemma 6.6. *The natural projection $\widetilde{X}^{\mu}_{\lambda} \to \widehat{X}^{\mu}_{\lambda}$ is a Grassmann bundle when $X = \mathrm{Gr}(m, N)$ has type A, is a Lagrangian Grassmannian bundle when $X = \mathrm{LG}(m, 2m)$ has type C, and is a locally trivial bundle of maximal orthogonal Grassmannians when $X = \mathrm{OG}(m, 2m+1)$ has type B.*

Proof. Choose $\alpha' \in \max(\mu)$ with $\mathrm{ur}(\alpha')$ maximal, and choose $\alpha'' \in \min(\mathcal{P}_X \smallsetminus \lambda)$ with $\mathrm{ur}(\alpha'')$ maximal. Let $\mathcal{V}_\bullet$ be the tautological flag on $\widehat{X}^{\mu}_{\lambda}$. Points in the fiber over $V_\bullet \in \widehat{X}^{\mu}_{\lambda}$ are obtained by adjoining a subspace V_m of dimension m to $V_\bullet$ satisfying $V_{\mathrm{up}(\alpha')} \subset V_m \subset V_m^{\perp} \subset V_{N-\mathrm{rt}(\alpha'')}$. In type A we obtain

$$\widetilde{X}^{\mu}_{\lambda} \cong \mathrm{Gr}\left(m - \mathrm{up}(\alpha'), \mathcal{V}_{N-\mathrm{rt}(\alpha'')}/\mathcal{V}_{\mathrm{up}(\alpha')}\right) \to \widehat{X}^{\mu}_{\lambda}\,.$$

Assume that X has type B or C. For $V_\bullet \in \widehat{X}^{\mu}_{\lambda}$ we have $V_{\mathrm{up}(\alpha')} \subset F_{\mathrm{ur}(\alpha')}$ and $V^{\perp}_{N-\mathrm{rt}(\alpha'')} \subset E^{\perp}_{N-\mathrm{ur}(\alpha'')} = E_{\mathrm{ur}(\alpha'')}$. It follows that $V_{\mathrm{up}(\alpha')}$ and $V^{\perp}_{N-\mathrm{rt}(\alpha'')}$ are isotropic subspaces of $\mathbb{K}^N$, and also that $V_{\mathrm{up}(\alpha')} \cap V^{\perp}_{N-\mathrm{rt}(\alpha'')} = 0$. Therefore, $V_{\mathrm{up}(\alpha')} + V^{\perp}_{N-\mathrm{rt}(\alpha'')} = V_{\mathrm{up}(\alpha')} \oplus V^{\perp}_{N-\mathrm{rt}(\alpha'')}$ is a direct sum in $\mathbb{K}^N$. Since $\mathrm{up}(\alpha') \leq m \leq N - \mathrm{rt}(\alpha'')$, we obtain $V_{\mathrm{up}(\alpha')} \subset V_{N-\mathrm{rt}(\alpha'')}$ and $V^{\perp}_{N-\mathrm{rt}(\alpha'')} \subset V^{\perp}_{\mathrm{up}(\alpha')}$.

It follows that $V_{\mathrm{up}(\alpha')} + V^{\perp}_{N-\mathrm{rt}(\alpha'')} \subset V^{\perp}_{\mathrm{up}(\alpha')} \cap V_{N-\mathrm{rt}(\alpha'')} = (V_{\mathrm{up}(\alpha')} + V^{\perp}_{N-\mathrm{rt}(\alpha'')})^{\perp}$. The form on $\mathbb{K}^N$ therefore defines a non-degenerate bilinear form on the vector bundle

$$\mathcal{H} = (\mathcal{V}^{\perp}_{\mathrm{up}(\alpha')} \cap \mathcal{V}_{N-\mathrm{rt}(\alpha'')})/(\mathcal{V}_{\mathrm{up}(\alpha')} + \mathcal{V}^{\perp}_{N-\mathrm{rt}(\alpha'')})\,.$$

When $X = \mathrm{LG}(m, 2m)$ has type C, $\widetilde{X}^{\mu}_{\lambda}$ is the Lagrangian Grassmannian bundle

$$\widetilde{X}^{\mu}_{\lambda} = \mathrm{LG}(r, \mathcal{H}) \to \widehat{X}^{\mu}_{\lambda}\,,$$

where $r = m - \mathrm{up}(\alpha') - \mathrm{rt}(\alpha'')$. This bundle is automatically locally trivial, as any symplectic vector bundle has a local basis $\{b_1, \ldots, b_{2r}\}$ satisfying $\omega(b_i, b_j) = \delta_{i+j,2r+1}$ for $1 \le i \le j \le 2r$.

Assume $X = \mathrm{OG}(m, 2m+1)$ has type B. Using that $F_{m+1} \subset F^{\perp}_{\mathrm{ur}(\alpha')} \subset V^{\perp}_{\mathrm{up}(\alpha')}$ and $E_{m+1} \subset E_{N-\mathrm{ur}(\alpha'')} \subset V_{N-\mathrm{rt}(\alpha'')}$, it follows that

$$V^{\perp}_{\mathrm{up}(\alpha')} \cap V_{N-\mathrm{rt}(\alpha'')} = (E_m \cap V^{\perp}_{\mathrm{up}(\alpha')}) \oplus \mathbb{K}e_{m+1} \oplus (F_m \cap V_{N-\mathrm{rt}(\alpha'')})\,,$$

hence $\mathcal{H}$ factors as the direct sum

$$\mathcal{H} = (E_m \cap \mathcal{V}^{\perp}_{\mathrm{up}(\alpha')})/\mathcal{V}^{\perp}_{N-\mathrm{rt}(\alpha'')} \oplus \mathbb{K}e_{m+1} \oplus (F_m \cap \mathcal{V}_{N-\mathrm{rt}(\alpha'')})/\mathcal{V}_{\mathrm{up}(\alpha')}\,.$$

Each point $V_\bullet \in \widehat{X}^{\mu}_{\lambda}$ therefore has an open neighborhood on which the orthogonal bundle $\mathcal{H}$ can be trivialized; that is, $\mathcal{H}$ has a local basis of sections $\{b_1, \ldots, b_{2r+1}\}$ satisfying $\omega(b_i, b_j) = \delta_{i+j,2r+2}$ for $1 \le i, j \le 2r+1$, where $r = m - \mathrm{up}(\alpha') - \mathrm{rt}(\alpha'')$. We deduce that $\widetilde{X}^{\mu}_{\lambda} \to \widehat{X}^{\mu}_{\lambda}$ is a locally trivial bundle with fiber $\mathrm{OG}(r, 2r+1)$. □

Theorem 6.7. *The map $\widetilde{X}^{\mu}_{\lambda} \to X^{\mu}_{\lambda}$ defined by $V_\bullet \mapsto V_m$ is a desingularization.*

Proof. It follows from Lemma 6.4 and Lemma 6.6 that $\widetilde{X}^{\mu}_{\lambda}$ is non-singular and irreducible. We must show that the projection $\pi : \widetilde{X}^{\mu}_{\lambda} \to X^{\mu}_{\lambda}$ is birational. The definition of $\widetilde{X}^{\mu}_{\lambda}$ implies that $V_m \in X^{\mu}_{\lambda}$ for each $V_\bullet \in \widetilde{X}^{\mu}_{\lambda}$, so π is a well-defined morphism of varieties. For each point V in the open Richardson variety $\mathring{X}^{\mu}_{\lambda} = Bw_\lambda.P_X \cap B^- w_\mu.P_X$, we have $\dim(V \cap F_{\mathrm{ur}(\alpha)}) = \mathrm{up}(\alpha)$ for $\alpha \in \max(\mu)$, and $\dim(V^{\perp} + E_{N-\mathrm{ur}(\alpha)}) = N - \mathrm{rt}(\alpha)$ for $\alpha \in \min(\mathcal{P}_X \smallsetminus \lambda)$. We therefore obtain a point $V_\bullet \in \widetilde{X}^{\mu}_{\lambda}$ by setting $V_{\mathrm{up}(\alpha)} = V \cap F_{\mathrm{ur}(\alpha)}$ for $\alpha \in \max(\mu)$, $V_m = V$, and $V_{N-\mathrm{rt}(\alpha)} = V^{\perp} + E_{N-\mathrm{ur}(\alpha)}$ for $\alpha \in \min(\mathcal{P}_X \smallsetminus \lambda)$. The assignment $V \mapsto V_\bullet$ defines a morphism $\mathring{X}^{\mu}_{\lambda} \to \widetilde{X}^{\mu}_{\lambda}$ that is inverse to π. □

When X has type A or C, it follows immediately from our construction that the degree map $\mathrm{Pic}(\widetilde{X}^{\mu}_{\lambda}) \to A_1(\widetilde{X}^{\mu}_{\lambda})^*$ is injective, thanks to the explicit descriptions of the Chow rings of Grassmann bundles [Ful98, Prop. 14.6.3] and Lagrangian Grassmannian bundles [AF24, Prop. 13.4.1]. Analogous results for bundles of orthogonal Grassmannians seem to be missing from the literature (see [EG95] for some related results). To establish Theorem 5.15 for cominuscule Richardson varieties of classical Lie types, we use the following result.

Proposition 6.8. *Let $\rho : Z \to Y$ be a locally trivial fibration of non-singular projective varieties with fiber F. Assume that Y is rational and the degree maps $\mathrm{Pic}(Y) \to A_1(Y)^*$ and $\mathrm{Pic}(F) \to A_1(F)^*$ are injective. Then the degree map $\mathrm{Pic}(Z) \to A_1(Z)^*$ is also injective. Moreover, we have an exact sequence*

$$0 \to \mathrm{Pic}(Y) \xrightarrow{\rho^*} \mathrm{Pic}(Z) \xrightarrow{i^*} \mathrm{Pic}(F) \to 0\,,$$

where $i : F = \rho^{-1}(y_0) \subset Z$ is the inclusion of the fiber over a general point $y_0 \in Y$.

Proof. Let $U \subset Y$ be an open neighborhood of y_0 such that $\rho^{-1}(U) \cong U \times F$, with ρ identified with the first projection. Since Y is rational, we may assume that U is isomorphic to an open subset of affine space. Set $d = \dim(Y)$ and $e = \dim(Z)$. The rows of the following commutative diagram are exact by [Ful98, Prop. 1.8].

$$\begin{tikzcd}
A_{e-1}(\rho^{-1}(Y \smallsetminus U)) \arrow[r] & \mathrm{Pic}(Z) \arrow[r] & \mathrm{Pic}(\rho^{-1}(U)) \arrow[r] & 0 \\
A_{d-1}(Y \smallsetminus U) \arrow[r] \arrow[u, "\rho^*", "\cong"'] & \mathrm{Pic}(Y) \arrow[r] \arrow[u, "\rho^*"] & \mathrm{Pic}(U) = 0 &
\end{tikzcd}$$

This diagram implies that the image of $\rho^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(Z)$ is the kernel of the restriction map $\mathrm{Pic}(Z) \to \mathrm{Pic}(\rho^{-1}(U))$. Using that $i^* : \mathrm{Pic}(\rho^{-1}(U)) \to \mathrm{Pic}(F)$ is an isomorphism, we deduce that the top row of the following commutative diagram is right exact.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(Y) \arrow[r, "\rho^*"] \arrow[d, hook] & \mathrm{Pic}(Z) \arrow[r, "i^*"] \arrow[d] & \mathrm{Pic}(F) \arrow[r] \arrow[d, hook] & 0 \\
& A_1(Y)^* \arrow[r, "(\rho_*)^*", "\subset"'] & A_1(Z)^* \arrow[r, "(i_*)^*"] & A_1(F)^* &
\end{tikzcd}$$

Since $\rho_* : A_1(Z) \to A_1(Y)$ is surjective, it follows that $(\rho_*)^* : A_1(Y)^* \to A_1(Z)^*$ and $\rho^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(Z)$ are injective. This establishes the short exact sequence of the proposition. Finally, the injectivity of the map $\mathrm{Pic}(Z) \to A_1(Z)^*$ follows from the second diagram. □

Corollary 6.9. *Let X_θ be a cominuscule Richardson variety whose indecomposable factors have classical Lie types A, B, C, and D. There exists a desingularization $\widetilde{X_\theta} \to X_\theta$ such that $\widetilde{X_\theta}$ is a tower of locally trivial bundles, where the fibers are Grassmannians of type A, Lagrangian Grassmannians, or maximal orthogonal Grassmannians. Moreover, the degree map $\mathrm{Pic}(\widetilde{X_\theta}) \to A_1(\widetilde{X_\theta})^*$ is injective.*

Proof. If $\widetilde{X_\theta}$ is any variety constructed as a tower of locally trivial bundles of flag varieties, then Proposition 6.8 shows that the degree map $\mathrm{Pic}(\widetilde{X_\theta}) \to A_1(\widetilde{X_\theta})^*$ is injective. To see that the required desingularization exists, we may assume that X_θ is indecomposable with minimal embedding X. If X is a quadric hypersurface, then X_θ is a Schubert variety in X, and the Bott–Samelson desingularization is a tower of $\mathbb{P}^1$-bundles. Otherwise we use the desingularization of Theorem 6.7. □

7. Hilbert Polynomials

The determination of the Picard group of a cominuscule Richardson variety allows us to define additional geometric invariants. A first example is the Hilbert polynomial of the minimal ample line bundle, which we study in this section.

Let $X = G/P_X$ be a cominuscule flag variety, let $\theta \subset \mathcal{P}_X$ be a skew shape, and let $X_\theta \subset X$ be the associated cominuscule Richardson variety. It follows from Theorem 5.16 that the Picard group $\mathrm{Pic}(X_\theta)$ contains a unique minimal ample line bundle, which we denote by $\mathcal{O}_{X_\theta}(1)$. Let $H_\theta(r) \in \mathbb{Q}[r]$ be the associated Hilbert polynomial defined by

$$H_\theta(r) = \chi(X_\theta, \mathcal{O}_{X_\theta}(r)) = \sum_{i \geq 0} (-1)^i \dim H^i(X_\theta, \mathcal{O}_{X_\theta}(r)) \,,$$

where $\mathcal{O}_{X_\theta}(r) = \mathcal{O}_{X_\theta}(1)^{\otimes r}$. The Hilbert polynomial $H_\theta(r)$ depends only on the isomorphism class of X_θ as an algebraic variety.

More generally, let $H_\theta^X(r) \in \mathbb{Q}[r]$ denote the Hilbert polynomial defined by

$$H_\theta^X(r) = \chi(X_\theta, \mathcal{O}_X(r)|_{X_\theta}),$$

where $\mathcal{O}_X(1)$ is the minimal ample line bundle in $\mathrm{Pic}(X)$. This polynomial depends on the embedding $X_\theta \subset X$, but has the advantage that it satisfies a recursive formula, which we prove in Corollary 7.3 below. When X is the minimal embedding of X_θ, we have $\mathcal{O}_{X_\theta}(1) = \mathcal{O}_X(1)|_{X_\theta}$ and $H_\theta(r) = H_\theta^X(r)$ by Theorem 5.16(d), so our formula for $H_\theta^X(r)$ also covers $H_\theta(r)$.

Let $K(X)$ be the K-theory ring of X. Given straight shapes $\lambda, \mu \subset \mathcal{P}_X$, we denote the corresponding Schubert classes in $K(X)$ by $\mathcal{O}_\lambda = [\mathcal{O}_{X_\lambda}]$ and $\mathcal{O}^\mu = [\mathcal{O}_{X^\mu}]$. Let $\chi : K(X) \to \mathbb{Z}$ be the sheaf Euler characteristic map. We have

$$\chi(\mathcal{O}_\lambda \cdot \mathcal{O}^\mu) = \chi([\mathcal{O}_{X_\lambda \cap X^\mu}]) = \begin{cases} 1 & \text{if } \mu \subset \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

A skew shape $\theta \subset \mathcal{P}_X$ will be called *short* if it consists of short boxes. The skew shape θ is called a *rook strip* if no two boxes of θ are comparable. A *short rook strip* is a rook strip consisting of short boxes. Given straight shapes $\lambda, \nu \subset \mathcal{P}_X$, define the constant

$$C_\nu^\lambda(X) = \#\{\mu \subset \mathcal{P}_X \mid \lambda \subset \mu \subset \nu \text{ and } \mu/\lambda \text{ is a short skew shape}\}.$$

We have $C_\nu^\nu(X) = 1$, and $C_\nu^\lambda(X)$ is nonzero only if $\lambda \subset \nu$. When Φ is simply-laced, we have $C_\nu^\lambda(X) = 1$ whenever $\lambda \subset \nu$.

The following statement is a special case of the Pieri–Chevalley formula from [FL94, PR99]. We give an expository proof based on identities from [BCMP18].

Lemma 7.1. *Multiplication by $[\mathcal{O}_X(1)]$ in $K(X)$ is given by*

$$[\mathcal{O}_X(1)] \cdot \mathcal{O}_\nu = \sum_{\lambda \subset \nu} C_\nu^\lambda(X)\, \mathcal{O}_\lambda,$$

where the sum is over all straight shapes λ contained in ν.

Proof. Define linear endomorphisms ϕ and ψ of $K(X)$ by

$$\phi(\mathcal{O}^\lambda) = \sum_{\mu/\lambda \text{ short rook strip}} (-1)^{|\mu/\lambda|}\, \mathcal{O}^\mu,$$

where the sum is over all $\mu \subset \mathcal{P}_X$ such that $\lambda \subset \mu$ and μ/λ is a short rook strip, and

$$\psi(\mathcal{O}^\mu) = \sum_{\nu/\mu \text{ rook strip}} (-1)^{|\nu/\mu|}\, \mathcal{O}^\nu,$$

the sum over all $\nu \subset \mathcal{P}_X$ such that $\mu \subset \nu$ and ν/μ is any rook strip. By [BCMP18, Lemma 3.5 and Theorem 3.6] we have

$$\chi(\psi(\mathcal{O}^\lambda) \cdot \mathcal{O}_\nu) = \delta_{\lambda,\nu} \quad \text{and} \quad [\mathcal{O}_X(-1)] \cdot \mathcal{O}^\lambda = \psi(\phi(\mathcal{O}^\lambda)).$$

Notice that ϕ is invertible with inverse

$$\phi^{-1}(\mathcal{O}^\lambda) = \sum_{\mu/\lambda \text{ short}} \mathcal{O}^\mu,$$

where the sum is over straight shapes $\mu \subset \mathcal{P}_X$ such that $\lambda \subset \mu$ and μ/λ is a short skew shape. The coefficient of $\mathcal{O}_\lambda$ in $[\mathcal{O}_X(1)] \cdot \mathcal{O}_\nu$ is now given by

$$\begin{aligned}\chi\left(\psi(\mathcal{O}^\lambda) \cdot [\mathcal{O}_X(1)] \cdot \mathcal{O}_\nu\right) &= \chi\left([\mathcal{O}_X(-1)] \cdot \phi^{-1}(\mathcal{O}^\lambda) \cdot [\mathcal{O}_X(1)] \cdot \mathcal{O}_\nu\right) \\ &= \chi\left(\phi^{-1}(\mathcal{O}^\lambda) \cdot \mathcal{O}_\nu\right) = C^\lambda_\nu(X)\,,\end{aligned}$$

as required. □

Remark 7.2. Although both the Pieri–Chevalley formula from [FL94, PR99] and the identities from [BCMP18] were proved for flag varieties defined over $\mathbb{C}$, they hold in arbitrary characteristic. One way to see this is to observe that the Graham-Willems formula [Gra02, Wil06] for the restrictions of equivariant K-theoretic Schubert classes to T-fixed points is independent of characteristic; this implies that the Schubert structure constants of the equivariant K-theory ring $K_T(G/P)$ are independent of characteristic. Indeed, Graham's proof of the formula uses only that the Bott–Samelson resolution of a Schubert variety is a rational resolution, which holds in any characteristic [Ram85]. We thank William Graham for a very helpful discussion about this.

Given straight shapes $\lambda \subset \nu \subset \mathcal{P}_X$, we set $\Delta H^X_{\nu/\lambda}(r) = H^X_{\nu/\lambda}(r+1) - H^X_{\nu/\lambda}(r)$.

Corollary 7.3. *The Hilbert polynomial $H^X_{\nu/\lambda}$ is determined by $H^X_{\nu/\lambda}(0) = 1$ and*

$$\Delta H^X_{\nu/\lambda} = \sum_{\lambda \subset \mu \subsetneq \nu} C^\mu_\nu(X)\, H^X_{\mu/\lambda}\,,$$

where the sum is over all straight shapes μ satisfying $\lambda \subset \mu \subsetneq \nu$.

Proof. Using that $[\mathcal{O}_X(r)|_{X^\lambda_\nu}] = [\mathcal{O}_X(1)]^r \cdot \mathcal{O}_\nu \cdot \mathcal{O}^\lambda$ holds in $K(X)$, we obtain from Lemma 7.1 that $[\mathcal{O}_{X^\lambda_\nu}(r+1)] = \sum_{\lambda \subset \mu \subset \nu} C^\mu_\nu(X)\,[\mathcal{O}_{X^\lambda_\mu}(r)]$. The result follows by subtracting $[\mathcal{O}_{X^\lambda_\nu}(r)]$ from both sides of this identity and applying χ. □

Corollary 7.4. *The coefficients a_d in the binomial expansion*

$$H^X_{\nu/\lambda}(r) = \sum_{d=0}^{|\nu/\lambda|} a_d \binom{r}{d}$$

are positive integers.

Proof. We have $a_0 = H^X_{\nu/\lambda}(0) = 1$, and the remaining coefficients in

$$\Delta H^X_{\nu/\lambda}(r) = \sum_{d=1}^{|\nu/\lambda|} a_d \binom{r}{d-1}$$

are positive integers by Corollary 7.3 and induction on $|\nu/\lambda|$. □

Example 7.5. The minuscule Richardson varieties defined by the shapes

$\theta_1 =$ and $\theta_2 =$

of types A_6 and E_6 have the same Hilbert polynomial

$$H_{\theta_1}(r) = H_{\theta_2}(r) = 1 + 11\,r + 43\binom{r}{2} + 82\binom{r}{3} + 83\binom{r}{4} + 43\binom{r}{5} + 9\binom{r}{6}\,.$$

The Richardson varieties X_{θ_1} and X_{θ_2} are not isomorphic as algebraic varieties, for example because the singular locus of X_{θ_1} is isomorphic to $\mathbb{P}^2$, whereas the singular locus of X_{θ_2} has two components isomorphic to $\mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^2$.

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