CHERN CLASS FORMULAS FOR DEGENERACY LOCI

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ABSTRACT. We study a new class of polynomials describing degeneracy loci for a sequence of vector bundles with maps between them and arbitrary rank conditions on the maps. These polynomials generalize all known types of Schubert polynomials. We give explicit formulas for the polynomials as linear combinations with integer coefficients of products of Schur determinants. We furthermore conjecture that all coefficients are positive and given by counting tableaux. We prove the conjecture in the case corresponding to four vector bundles.

RÉSUMÉ. Nous étudions une nouvelle classe de polynômes qui décrit l'ensemble des singularités d'une sequence de fibrés vectoriels reliés par des fonctions et des conditions arbitraires sur le rang des fonctions. Ces polynômes généralisent tous les types connus de polynômes de Schubert. Nous donnons des formules qui expriment ces polynômes par des combinaisons linéaires, à coefficients entiers, de déterminants de Schur. De plus, nous conjecturons que tous ces coefficients sont positifs et qu'ils comptent certains tableaux. Nous démontrons cette conjecture dans le cas correspondant à quatre fibrés vectoriels.

1. INTRODUCTION

The purpose of this talk is to report on a joint geometric project with W. Fulton [3], in which we proved a formula for a general type of degeneracy locus, and conjectured another. We will furthermore describe a combinatorial continuation of the project, in which we have attempted to prove the conjecture and succeeded in a special case. Here we are extremely thankful to S. Fomin, who provided a vital involution on pairs of tableaux, and who has collaborated with us on the combinatorial aspects of this problem.

Chern class formulas for degeneracy loci have in the past proven to reveal very interesting polynomials. Schur polynomials and Schubert polynomials are all examples of this. In this talk we will consider a very general class of degeneracy loci and describe their formulas.

Let $E_0 \to E_1 \to \cdots \to E_n$ be a sequence of vector bundles with maps between them over an algebraic variety X. We will study the degeneracy loci obtained by putting arbitrary rank conditions on these maps and their compositions. Given a collection of non-negative integers $r = (r_{ij})$ for $0 \le i < j \le n$, define Ω_r to be the subset

$$\Omega_r = \{x \in X \mid \mathrm{rank}(E_i(x) \to E_j(x)) \le r_{ij} \ \forall \ i < j\}$$

of X. The goal is to find a formula for the cohomology class of Ω_r in terms of the Chern classes of the bundles E_i. The numbers r_{ij} are called rank conditions. For simplicity, we will assume here that the maps of the vector bundles are sufficiently general, so that Ω_r has the expected codimension. For convenience we set $r_{ii} = \operatorname{rank}(E_i)$.

A classical case is that of two bundles. To describe the solution in this case we need some notation. Let $I = (a_1, \ldots, a_m)$ be a sequence of integers, and let F and G be two vector bundles with Chern roots $\alpha_1, \ldots, \alpha_p$, and $\beta_1, \ldots, \beta_q$. We define the Schur polynomial

$$S_I(G - F) = \det(h_{a_i+j-i})$$

to be the determinant of the $m \times m$ matrix whose (i,j)'th entry is h_{a_i+j-i}, where the h_k are defined by the equation

$$\sum_k h_k t^k = \frac{\prod_{i=1}^p (1 - \alpha_i t)}{\prod_{i=1}^q (1 - \beta_i t)} \,.$$

If I is a partition, this is just the Schur polynomial $s_I(\beta/\alpha)$ defined in Macdonald [10], applied to the Chern roots of F and G. The Chern roots are in general independent, and can be thought of as variables.

In case of two bundles, the degeneracy locus Ω_r is now described by Thom-Porteous-Giambelli's Theorem:

$$[\Omega_r] = S_\lambda(E_1 - E_0)$$

where $\lambda = (r_{00} - r_{01})^{r_{11} - r_{01}}$ is the partition with $r_{00} - r_{01}$ repeated $r_{11} - r_{01}$ times.

For the general problem, many choices of rank conditions $r = (r_{ij})$ are redundant, i.e. the locus Ω_r can be written as the union of finitely many smaller loci $\Omega_{r'}$. It was shown by Abeasis and Del Fra [1] that the the non-redundant rank conditions are those that can be obtained in a concrete choice of vector spaces and linear maps. In other words a set of rank conditions $r = (r_{ij})$ is non-redundant iff there is a sequence of vector spaces with linear maps between them

$$V_0 \to V_1 \to \cdots \to V_n$$

such that $r_{ij} = \operatorname{rank}(V_i \to V_j)$ for all $i \le j$.

Abeasis and Del Fra classified the allowable rank conditions as those arising from diagrams of dots and lines as follows. For each vector space V_i, make $\dim V_i$ dots on a vertical line (column). Then connect some of the dots in each column to dots of the next, such that no two dots in the same column are connected to the same dot.

Example 1.
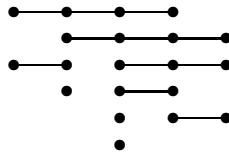
The dots represent basis vectors for the vector spaces V_i, and the lines represent how each basis vector of one space is mapped into the next space. r_{ij} is now obtained as the number of lines going from dots in column i to dots in column j.

Rank conditions $r = (r_{ij})$ may conveniently be arranged in a rank diagram as follows:

$$
\begin{array}{ccccccc}
E_0 & \rightarrow & E_1 & \rightarrow & E_2 & \cdots & E_n \\[4pt]
r_{00} & & r_{11} & & r_{22} & \cdots & r_{nn} \\[4pt]
& r_{01} & & r_{12} & \cdots & r_{n-1,n} \\[4pt]
& & \ddots & & & \\[4pt]
& & & r_{0n} & & &
\end{array}
$$

It follows from Abeasis and Del Fra's classification that a rank diagram is non-redundant if and only if

1. $r_{ii} = \operatorname{rank}(E_i)$,
2. $r_{ij} \le \min(r_{i,j-1}, r_{i+1,j})$, and
3. $r_{ij} - r_{i-1,j} - r_{i,j+1} + r_{i-1,j+1} \ge 0$.

From now on we will only consider rank diagrams satisfying these conditions.

Example 2. The line diagram of Example 1 corresponds to the following rank diagram:

$$
\begin{array}{ccccccccc}
E_0 & \rightarrow & E_1 & \rightarrow & E_2 & \rightarrow & E_3 & \rightarrow & E_4 \\[4pt]
2 & & 4 & & 6 & & 5 & & 3 \\[4pt]
& 2 & & 2 & & 4 & & 3 & \\[4pt]
& & 1 & & 2 & & 2 & & \\[4pt]
& & & 1 & & 1 & & & \\[4pt]
& & & & 0 & & & &
\end{array}
$$

If $r = (r_{ij})$ is a rank diagram, we may obtain a diagram of rectangles from r by replacing each triangle of integers

$$
\begin{array}{cc}
s & t \\
& u
\end{array}
$$

by a rectangle R with $t - u$ rows and $s - u$ columns. We will sometimes identify R with the partition $(s - u)^{t-u}$. However, even when R is empty, we still need to know both the number of rows and columns of R, and this information is not encoded in the zero-partition. It turns out that the rectangle diagram contains key information about the degeneracy locus Ω_r. For example the expected codimension of Ω_r in X is the total number of cells in all of the rectangles.

Example 3. The rank diagram in Example 2 gives the following rectangle
diagram:

PSfrag replacements

T
B
S_1
R_2
Only entries from T.
$W_2 =$
$W_i =$
Q_{i-1}
P_i
A_i
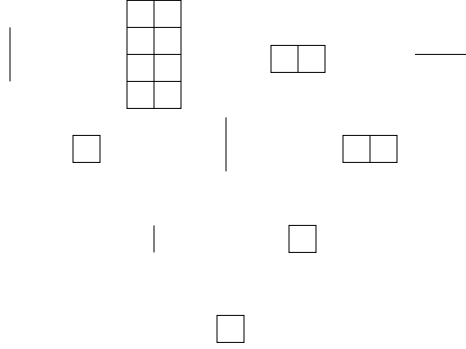
If the vector bundle maps are general, then the codimension of Ω_r is 15.

It follows from the axiom (3) for rank diagrams, that the rectangles always
get shorter when one goes southeast in a rectangle diagram, while they get
narrower when one goes southwest.

2. A FORMULA FOR Ω_r

For each sequence of partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})$, define a symbol
$S(\lambda) = S(\lambda^{(1)}, \ldots, \lambda^{(p)})$. A formal sum of symbols with $p = n$ represents a
cohomology class on X by identifying $S(\lambda^{(1)}, \ldots, \lambda^{(n)})$ with

$$\prod_{i=1}^{n} S_{\lambda^{(i)}}(E_i - E_{i-1}).$$

The main result of [3] is that the cohomology class of Ω_r is a linear combi-
nation with integer coefficients of such symbols:

$$[\Omega_r] = \sum_{\lambda} c_r^{\lambda} S(\lambda).$$

If I is a sequence of integers then $s_I(\beta/\alpha)$ is either zero or equal to
$\pm s_\lambda(\beta/\alpha)$ for a unique partition λ. For example, $s_{(1,2)}(\beta/\alpha) = 0$ and
$s_{(1,3)}(\beta/\alpha) = -s_{(2,2)}(\beta/\alpha)$. If $I^{(1)}, \ldots, I^{(p)}$ are sequences of integers, we
define $S(I^{(1)}, \ldots, I^{(p)})$ to be either zero or $\pm S(\lambda^{(1)}, \ldots, \lambda^{(p)})$ for a sequence
of partitions $\lambda^{(j)}$. If $s_{I^{(j)}}(\beta/\alpha) = 0$ for any j, then $S(I^{(1)}, \ldots, I^{(p)}) = 0$.
Otherwise write $s_{I^{(j)}}(\beta/\alpha) = \epsilon_j s_{\lambda^{(j)}}(\beta/\alpha)$ for each j with $\epsilon_j = \pm 1$, and
define

$$S(I^{(1)}, \ldots, I^{(p)}) = \left(\prod_j \epsilon_j\right) S(\lambda^{(1)}, \ldots, \lambda^{(p)}).$$

Let $R = (a)^b$ be a rectangle with b rows and a columns. Let μ and ν be
partitions, such that μ is no taller than R, i.e. the length of μ is at most b. We
then define $\frac{R|\mu}{\nu}$ to be the the sequence $(a + \mu_1, a + \mu_2, \ldots, a + \mu_b, \nu_1, \nu_2, \ldots)$.
Note that b is significant even when a is zero.

Example 4. If $R = (3)^4$, $\mu = (4, 3, 1)$, and $\nu = (5, 4, 1)$, then the entries of $\frac{R|\mu}{\nu}$ are the numbers of cells in the rows of the diagram:
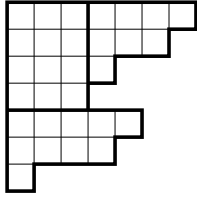
We now define polynomials P_r inductively. If the rectangle diagram has only one row, we put

$$P_r = S(R)$$

where R is the only rectangle in the diagram (regarded as a partition).

If the rectangle diagram has more than one row, let $\bar{r}$ denote the bottom n rows of the rank diagram. Then $\bar{r}$ is itself a valid rank diagram, and its rectangle diagram consists of the bottom $n-1$ rows of the rectangle diagram for r. By induction $P_{\bar{r}}$ is a well defined linear combination of symbols $S(\lambda^{(1)}, \dots, \lambda^{(n-1)})$.

Let the top row of the rectangle diagram for r consist of the rectangles $R_1, \dots, R_n$. Then P_r is obtained from $P_{\bar{r}}$ by replacing each symbol $S(\lambda^{(1)}, \dots, \lambda^{(n-1)})$ by the sum

$$\sum_{\mu_i, \nu_i} \left(\prod_{i=1}^{n-1} c_{\mu_i, \nu_i}^{\lambda^{(i)}} \right) S(\frac{R_1|\mu_1}{\emptyset}, \frac{R_2|\mu_2}{\nu_1}, \dots, \frac{R_n|\emptyset}{\nu_{n-1}}).$$

The sum is over all partitions μ_i and ν_i for $1 \leq i \leq n-1$ such that μ_i is no taller than R_i. The constants $c_{\mu\nu}^{\lambda}$ denote Littlewood-Richardson numbers, and $\emptyset$ is the zero-partition.

Theorem. *The cohomology class of Ω_r is equal to P_r.*

It is clear from the definition that P_r can be written in the form

$$P_r = \sum c_r^{\lambda} S(\lambda)$$

where the c_r^{λ} are integer constants depending on λ and the rectangle diagram. The definition furthermore gives an explicit way of calculating these constants.

Example 5. Let r be the rank diagram of Example 2. Then we have

The polynomial P_r is a sum of 72 different symbols, each with coefficient one.

Example 6. Given a permutation $w \in S_n$, one may construct a line diagram with $2n$ columns, here shown for $w = 2413 \in S_4$:
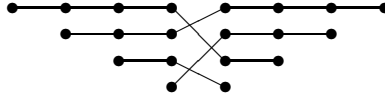
The connections of the dots in the middle is given by w; the i'th dot on the right side is connected to the $w(i)$'th dot on the left. If r is the corresponding rank diagram, then P_r is equal to the universal Schubert polynomial $\mathfrak{S}_w$ defined by Fulton [6], although the only proof of this now is from geometry. Universal Schubert polynomials specialize to the double Schubert polynomials defined by Lascoux and Schützenberger [9], as well as quantum Schubert polynomials defined by Fomin, Gelfand, and Postnikov [4]. The polynomials P_r therefore give new formulas for the previously known Schubert polynomials.
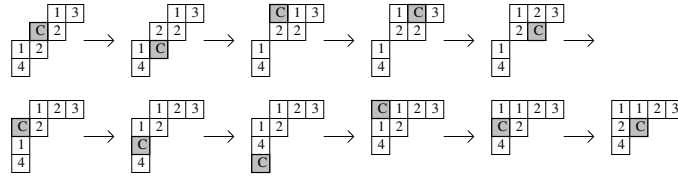
We have calculated a large number of examples of the polynomials P_r, but never found any negative coefficients c_r^λ. It is in our attempts to prove that all these coefficients are non-negative, that some very interesting combinatorics has appeared.

3. Conjectured behavior of P_r

Recall that a semistandard Young tableau on a partition λ is a filling of the cells in the Young diagram of λ with integers, such that the entries are strictly increasing down the columns, and weakly increasing along the rows. The partition λ is called the shape of the tableau. By a tableau we will always mean a semistandard Young tableau.

Two tableaux P and Q can be multiplied using the jeu de taquin algorithm. Here one arranges P and Q in a skew diagram such that the upper right corner of P is attached to the lower left corner of Q. Then one performs jeu de taquin slides on the diagram until a tableau is reached. For each slide, one chooses an inner corner C of the skew diagram. Then one slides the smaller of the cells below and to the right of C into C. If these cells are equal, then the cell below C is used. C is then replaced by the "hole" that was made by the slide, and the operation is repeated until C is no longer in the diagram. If the diagram is still not a tableau, a new inner corner is chosen, and the whole process is repeated. It is a theorem that the tableau obtained in this way is independent of the choices of inner corners. It follows that tableau multiplication is associative.

Example 7. To multiply T_i $\boxed{4}$ with A_i $\boxed{2}$, one does the sequence of slides:

$$\begin{array}{ccc}
\boxed{\begin{smallmatrix}1&3\\C&2\end{smallmatrix}} \\
\boxed{\begin{smallmatrix}1&2\\4\end{smallmatrix}}
\end{array}
\rightarrow
\begin{array}{c}
\boxed{\begin{smallmatrix}1&3\\2&2\end{smallmatrix}} \\
\boxed{\begin{smallmatrix}1&C\end{smallmatrix}}
\end{array}
\rightarrow
\begin{array}{c}
\boxed{\begin{smallmatrix}C&1&3\\2&2\end{smallmatrix}} \\
\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}
\end{array}
\rightarrow
\begin{array}{c}
\boxed{\begin{smallmatrix}1&C&3\\2&2\end{smallmatrix}} \\
\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}
\end{array}
\rightarrow
\begin{array}{c}
\boxed{\begin{smallmatrix}1&2&3\\2&C\end{smallmatrix}} \\
\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}
\end{array}
\rightarrow$$

$$\begin{array}{c}
\boxed{\begin{smallmatrix}1&2&3\\1&2\end{smallmatrix}} \\
\boxed{C}
\end{array}
\rightarrow
\begin{array}{c}
\boxed{\begin{smallmatrix}1&2&3\\1&2\\4\end{smallmatrix}}
\end{array}
\rightarrow
\begin{array}{c}
\boxed{\begin{smallmatrix}C&1&2&3\\1&2\\4\end{smallmatrix}}
\end{array}
\rightarrow
\begin{array}{c}
\boxed{\begin{smallmatrix}1&1&2&3\\C&2\\4\end{smallmatrix}}
\end{array}
\rightarrow
\begin{array}{c}
\boxed{\begin{smallmatrix}1&1&2&3\\2&C\\4\end{smallmatrix}}
\end{array}$$

We find that
$$\boxed{\begin{smallmatrix}1&2\\4\end{smallmatrix}} \cdot \boxed{\begin{smallmatrix}1&3\\2\end{smallmatrix}} = \boxed{\begin{smallmatrix}1&1&2&3\\2\\4\end{smallmatrix}}$$

By a tableau diagram we will mean a filling of the cells of a rectangle diagram with integers such that each rectangle becomes a tableau, and so that all of the cells in each tableau T are strictly larger than all of the cells in the tableaux appearing above T within 45 degree angles as shown.
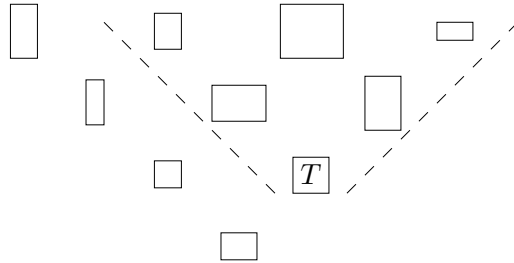

In the tableau diagram any empty rectangles are reduced to empty tableaux, and their dimensions are forgotten.

Example 8.

$$\boxed{\begin{smallmatrix}1&1\\2&3\\3&5\\4&6\end{smallmatrix}} \qquad \boxed{\begin{smallmatrix}2&4\end{smallmatrix}} \qquad \cdot$$

$$\boxed{8} \qquad \cdot \qquad \boxed{\begin{smallmatrix}5&5\end{smallmatrix}}$$

$$\cdot \qquad \boxed{7}$$

$$\boxed{9}$$
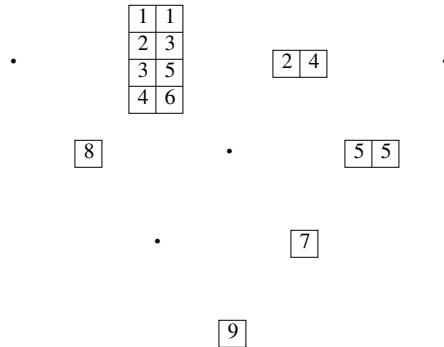
Next we will describe the notion of a factor sequence for a tableau diagram.

Example 9. Consider a tableau diagram with three rows:

$$\begin{array}{ccccc}
A & & B & & C \\
& R & & S & \\
& & T & &
\end{array}$$
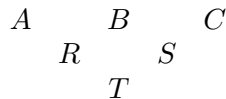
Take any factorization $T = T_1 \cdot T_2$ of T. Then form the products $R \cdot T_1$ and $T_2 \cdot S$ and take arbitrary factorizations $R \cdot T_1 = R_1 \cdot R_2$ and $T_2 \cdot S = S_1 \cdot S_2$. Then

$$(A \cdot R_1,\ R_2 \cdot B \cdot S_1,\ S_2 \cdot C)$$

is a factor sequence for the diagram, and all factor sequences are obtained in this way.

Now consider a general tableau diagram with n rows.

$$
\begin{array}{cccccc}
A_1 & & A_2 & A_3 & \cdots & A_n \\
 & B_1 & & B_2 & \cdots & B_{n-1} \\
 & & C_1 & \cdots & C_{n-2} & \\
 & & & \ddots & & \\
 & & & D_1 & &
\end{array}
$$

(1)

All factor sequences for this diagram will be sequences of n tableaux. If $n = 1$, then the only factor sequence is the sequence (A_1), consisting of the only tableau in the diagram. If $n > 1$, then a sequence of tableaux

$$(W_1, \ldots, W_n)$$

is a factor sequence if there exist tableaux $P_1, Q_1, \ldots, P_{n-1}, Q_{n-1}$ such that $Q_{i-1} \cdot A_i \cdot P_i = W_i$ for $i = 1, \ldots, n$, and

$$(R_2 \cdot B \cdot S_1, \ldots, P_{n-1} \cdot Q_n \cdot S_1)$$

is a factor sequence for the lower $n-1$ rows of the diagram. Here Q_0 and P_n are by convention empty tableaux.

Example 10. For the tableau diagram of Example 8, the lower three rows have the following six factor sequences:

$$
A_i \left(\begin{array}{|c|c|}\hline 8 & 9 \\\hline\end{array},\ \begin{array}{|c|c|}\hline 7 & 5 \\\hline 5 & \\\hline\end{array}\right),\ A_i \left(\begin{array}{|c|c|}\hline 8 & 9 \\\hline\end{array},\ \begin{array}{|c|}\hline 7 \\\hline\end{array}\right),\ A_i \left(\begin{array}{|c|c|}\hline 8 & 9 \\\hline\end{array},\ \begin{array}{|c|c|}\hline 5 & 5 \\\hline\end{array}\right),
$$

$$
A_i \left(\begin{array}{|c|c|}\hline 8 & 9 \\\hline\end{array},\ \begin{array}{|c|c|}\hline 7 \\\hline 5 & 5 \\\hline\end{array}\right),\ A_i \left(\begin{array}{|c|}\hline 8 \\\hline\end{array},\ \begin{array}{|c|}\hline 9 \\\hline\end{array}\right)
$$

The whole diagram has 72 different factor sequences.

Conjecture. *Let $r = (r_{ij})$ be a rank diagram. Then c_r^λ is the number of different factor sequences $(W_1, \ldots, W_n)$ for any fixed tableau diagram for r, such that W_i has shape $\lambda^{(i)}$ for each i.*

This conjecture will first of all imply that all the coefficients c_r^λ are positive. Another consequence is that the polynomials P_r will be independent of the side lengths of any empty rectangles in the rectangle diagram. Finally, the conjecture implies that the number of factor sequences is independent of the choices of tableaux in the diagram, which is not clear.

One reason that the conjectured formula is hard to work with is that a factor sequence can arise in many ways. According to the definition, to

check if a sequence $(W_1, \ldots, W_n)$ of tableaux is a factor sequence, one would have to try all factorizations $W_i = Q_{i-1} \cdot A_i \cdot P_i$ of each W_i, and check if $(P_1 \cdot Q_1, \ldots, P_{n-1} \cdot Q_{n-1})$ is a factor sequence for the lower diagram. However, the following proposition gives a direct criterion. Note that a first necessary condition is that each W_i must contain A_i as a subtableau in the upper-left corner.

Proposition. *Let P_i be the part of W_i lying to the right of the rectangular subtableau A_i, and let Q_i be everything lying below A_i and P_i:*

Only entries from T.

$$W_2 =$$

$$W_i = \begin{array}{|c|c|} \hline A_i & P_i \\ \hline Q_{i-1} & \\ \hline \end{array}$$

Then $(W_1, \ldots, W_n)$ is a factor sequence for (1) if and only if Q_0 and P_n are empty and $(P_1 \cdot Q_1, \ldots, P_{n-1} \cdot Q_{n-1})$ is a factor sequence for the lower diagram.

Let r' be the rank diagram obtained by mirroring r in a vertical line, i.e. $r'_{ij} = r_{n-j,n-i}$, and let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ be a sequence of partitions. Then it is clear from the geometric description that $c_{r'}^{\lambda} = c_r^{\tilde{\lambda}}$, where $\tilde{\lambda} = (\tilde{\lambda}^{(n)}, \ldots, \tilde{\lambda}^{(1)})$ is the sequence of conjugate partitions in reverse order. While this symmetry is not obvious from the definition of the polynomials P_r given in Section 2, it may easily be deduced from the conjecture.

We can prove the conjecture when the tableau diagram has at most three rows, or equivalently the number of vector bundles is at most four. When the tableau diagram has two rows, the conjecture follows from the fact that if T is any tableau on shape λ, and μ and ν are partitions, then there are $c_{\mu\nu}^{\lambda}$ different factorizations $T = T_1 \cdot T_2$ such that T_1 has shape μ and T_2 has shape ν. The proof in the case of three rows relies on the following two Lemmas.

Lemma 1. *Let $\mathcal{D}$ be a tableau diagram with three rows*

$$\begin{array}{ccccc} A & & B & & C \\ & R & & S & \\ & & T & & \end{array}$$

and let $\mathcal{L}$ be the diagram of the bottom two rows. Assume that the shape of B is the rectangle $(a)^b$. Let R_1, R_2, S_1, and S_2 be tableaux. Then the following are equivalent:

1. *$(R_1 \cdot R_2, S_1 \cdot S_2)$ is a factor sequence for $\mathcal{L}$.*
2. *$(A \cdot R_1, R_2 \cdot B \cdot S_1, S_2 \cdot C)$ is a factor sequence for $\mathcal{D}$, and only entries from T appear beyond the a'th column of R_2 and below the b'th row of S_1.*

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Only entries from T.
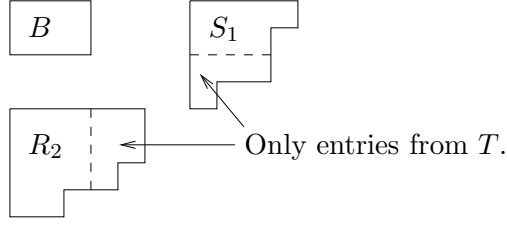
Fix a positive integer b. If (P,Q) is a pair of tableaux of shapes μ and ν such that Q has at most b rows, we define $S(\frac{Q}{P}) = S(I)$ where $I = (\nu_1, \dots, \nu_b, \mu_1, \mu_2, \dots)$. Let $\mathcal{P}_b$ be the set of all pairs (P,Q) such that $S(\frac{Q}{P}) \neq 0$, and so that P and Q do not fit together as a tableau with Q in the top b rows and P below. This means that the b'th row of Q must be shorter than the first row of P, or some cell in the first row of P must be smaller than the cell in the same position on the b'th row of Q.

Lemma 2 (Fomin's involution). *There exists an involution of $\mathcal{P}_b$ with the property that if (P,Q) is mapped to (P',Q'), then*

1. $P' \cdot Q' = P \cdot Q$,
2. $S(\frac{Q'}{P'}) = -S(\frac{Q}{P})$, *and*
3. *the first column of P' is equal to the first column of P.*

The proof of Lemma 2 was provided by S. Fomin in the form of an explicit algorithm for carrying out the involution. This nice algorithm will be described in [2].

Now the proof of the conjecture for rectangle diagrams with at most three rows relies on a calculation. We will abuse notation and denote the shape of a tableau W also by W when it appears in an S-symbol.

Let r be a rank diagram for four bundles and let $\mathcal{D}$ and $\mathcal{L}$ be as in Lemma 1. Using the conjecture on $\mathcal{L}$ we have

$$P_{\bar{r}} = \sum_{(W_1, W_2)} S(W_1, W_2),$$

the sum is over all factor sequences for $\mathcal{L}$. By the definition of P_r we get

$$P_r = \sum_{(W_1, W_2)} \sum_{W_1 = R_1 \cdot R_2} \sum_{W_2 = S_1 \cdot S_2} S(\frac{A|R_1}{\emptyset}, \frac{B|S_1}{R_2}, \frac{C|\emptyset}{S_2}).$$

The second and third sums are over all factorizations of W_1 and W_2 such that S_1 is no taller than B. This expresses P_r as a sum over collections of tableaux R_1, R_2, S_1, S_2 satisfying the properties in Lemma 1. Using the second condition in the Lemma, we may rewrite this as

$$P_r = \sum_{(W_1, W_2, W_3)} \sum_{R_2, S_1} S(W_1, \frac{B|S_1}{R_2}, W_3)$$

where the first sum is over all factor sequences of $\mathcal{D}$ and the second is over all tableaux R_2, S_1 such that $W_2 = R_2 \cdot B \cdot S_1$ and so that only entries from T appear in the part of R_2 that is broader than B, and S_1 is no taller than B. It is enough to prove that

$$S(W_1, W_2, W_3) = \sum_{R_2, S_1} S\left(W_1, \frac{B|S_1}{R_2}, W_3\right).$$

Here we use Lemma 2 with b equal to the number of rows of B to cancel out all of the terms in the sum, except the one where R_2 and S_1 are as in the picture. Only entries from T.

$$W_i = Q_{i-1} W_2 = P_i A_i$$

For all other pairs R_2, S_1, let P be the part of R_2 that is broader than B. Let (P', S_1') be the pair corresponding to (P, S_1) by Fomin's involution, and let R_2' be R_2 with the part broader than B replaced by P'. Then $S\left(W_1, \frac{B|R_2}{S_1}, W_3\right)$ and $S\left(W_1, \frac{B|R_2'}{S_1'}, W_3\right)$ cancel each other out. This finishes the proof of the conjecture for the case of four bundles.

The full conjecture follows from an assertion that Fomin's involution preserves factor sequences. When $n = 3$, this assertion follows from Lemma 1. The assertion has furthermore been verified in 500,000 randomly generated examples for $n \leq 10$. We consider this as strong evidence for the conjecture.

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