SEIDEL AND PIERI PRODUCTS IN COMINUSCULE QUANTUM K-THEORY

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Abstract. We prove a collection of formulas for products of Schubert classes in the quantum K-theory ring $\mathrm{QK}(X)$ of a cominuscule flag variety X. This includes a K-theory version of the Seidel representation, stating that the quantum product of a Seidel class with an arbitrary Schubert class is equal to a single Schubert class times a power of the deformation parameter q. We also prove new Pieri formulas for the quantum K-theory of maximal orthogonal Grassmannians and Lagrangian Grassmannians, and give a new proof of the known Pieri formula for the quantum K-theory of Grassmannians of type A. Our formulas have simple statements in terms of quantum shapes that represent the natural basis elements $q^d[\mathcal{O}_{X^u}]$ of $\mathrm{QK}(X)$. Along the way we give a simple formula for K-theoretic Gromov-Witten invariants of Pieri type for Lagrangian Grassmannians, and prove a rationality result for the points in a Richardson variety in a symplectic Grassmannian that are perpendicular to a point in projective space.

Contents

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1. Introduction

In this paper we prove a collection of explicit formulas for products of Schubert classes in the quantum K-theory ring $\mathrm{QK}(X)$ of a cominuscule flag variety. These formulas include a K-theory version of the Seidel representation of the quantum cohomology ring $\mathrm{QH}(X)$ [Sei97, Bel04, CMP09], as well as Pieri formulas for products with special Schubert classes of classical Grassmannians that generalize earlier Pieri formulas in quantum cohomology [Ber97, KT03, KT04] and in K-theory [Len00, BR12]. The Pieri formula for $\mathrm{QK}(X)$ is known from [BM11] when X is a Grassmannian of type A, but is new for maximal orthogonal Grassmannians and Lagrangian Grassmannians. Our formulas have simple expressions in terms of *quantum shapes* that encode the natural basis elements $q^d\mathcal{O}^u = q^d[\mathcal{O}_{X^u}]$ of $\mathrm{QK}(X)$, generalizing the familiar identification of cominuscule Schubert classes with diagrams of boxes [Pro84].

Let $X = G/P_X$ be a flag variety defined by a complex semisimple linear algebraic group G and a parabolic subgroup P_X. Let Φ be the root system of G, W the Weyl group, and let B be a Borel subgroup contained in P_X. A simple root γ is called *cominuscule* if, when the highest root is expressed in the basis of simple roots, the coefficient of γ is one. The flag variety X is called *cominuscule* if P_X is a maximal parabolic subgroup defined by a cominuscule simple root. Let $w_0^X \in W$ be the minimal representative of the longest element w_0 modulo the Weyl group W_X of P_X. The minimal representatives w_0^F defined by all cominuscule flag varieties of G, together with the identity, form a subgroup of the Weyl group:

$$W^{\mathrm{comin}} = \{w_0^F \mid F = G/P_F \text{ is cominuscule}\} \cup \{1\} \leq W\,.$$

Each element $u \in W$ defines the Schubert varieties $X_u = \overline{Bu.P_X}$ and $X^u = \overline{B^-u.P_X}$ in X. The Schubert classes $[X^w]$ for $w \in W^{\mathrm{comin}}$ will be called *Seidel classes*. It was proved in [Bel04] and also in [CMP09] that quantum cohomology products with Seidel classes have only one term. More precisely, for $w \in W^{\mathrm{comin}}$ and $u \in W$ we have $[X^w] \star [X^u] = q^{\omega^\vee - u^{-1}.\omega^\vee}[X^{wu}]$ in $\mathrm{QH}(X)$, where $\omega^\vee$ is the unique fundamental coweight such that $w.\omega^\vee = w_0.\omega^\vee$. This defines a representation of W^{comin} on $\mathrm{QH}(X)/\langle q-1\rangle$ called the *Seidel representation*. Our first result generalizes the Seidel representation to the quantum K-theory ring when X is itself cominuscule. We denote the Schubert classes in $K(X)$ by $\mathcal{O}_u = [\mathcal{O}_{X_u}]$ and $\mathcal{O}^u = [\mathcal{O}_{X^u}]$.

Theorem 1.1 (Seidel representation)**.** *Let $X = G/P_X$ be a cominuscule flag variety, and let $w \in W^{\mathrm{comin}}$ and $u \in W$. We have in $\mathrm{QK}(X)$ that*

$$\mathcal{O}^w \star \mathcal{O}^u = q^d\, \mathcal{O}^{wu}\,,$$

where d is determined by $\int_d c_1(T_X) + \mathrm{codim}(X^{wu}) = \mathrm{codim}(X^w) + \mathrm{codim}(X^u)$.

When $X = G/P_X$ is a cominuscule flag variety, the subset $W^X \subset W$ of minimal representatives of the cosets in W/W_X can be represented by generalized Young diagrams [Pro84, Per07, BS16]. Set $\mathcal{P}_X = \{\alpha \in \Phi \mid \alpha \geq \gamma\}$, where γ is the cominuscule simple root defining X, and give $\mathcal{P}_X$ the partial order $\alpha' \leq \alpha$ if and only if $\alpha - \alpha'$ is a sum of positive roots. The inversion set $I(u) = \{\alpha \in \Phi^+ \mid u.\alpha < 0\}$ of any element $u \in W^X$ is a lower order ideal in $\mathcal{P}_X$. The set $\mathcal{P}_X$ can be identified with a set of boxes in the plane, which in turn identifies $I(u)$ with a diagram of boxes that we call the *shape* of u. This defines a bijection between the set of shapes in $\mathcal{P}_X$ and the Schubert basis $\{[X^u]\}$ of $H^*(X,\mathbb{Z})$.

More generally, let $\mathcal{B} = \{q^d[X^u] \mid u \in W^X, d \in \mathbb{Z}\}$ be the natural $\mathbb{Z}$-basis of $\mathrm{QH}(X)_q = \mathrm{QH}(X) \otimes \mathbb{Z}[q, q^{-1}]$. It was shown in [BCMP22] that $\mathcal{B}$ has a natural partial order defined by $q^e[X^v] \leq q^d[X^u]$ if and only if X_u and X^v can be connected by a rational curve of degree at most $d - e$. Moreover, this partial order is a distributive lattice when X is cominuscule. Let $\widehat{\mathcal{P}}_X \subset \mathcal{B}$ be the subset of join-irreducible elements. Then $\widehat{\mathcal{P}}_X$ is an infinite partially ordered set that contains $\mathcal{P}_X$ as an interval. When $X = \mathrm{Gr}(m,n)$ is a Grassmannian of type A, $\widehat{\mathcal{P}}_X = \mathbb{Z}^2/\mathbb{Z}(m, m-n)$ is Postnikov's *cylinder* from [Pos05]. This poset was also defined in [Hag04]. The posets $\widehat{\mathcal{P}}_X$ defined by other cominuscule flag varieties are isomorphic to certain *full heaps* of affine Dynkin diagrams that were constructed in [Gre13] and used to study minuscule representations.

Define a *quantum shape* to be any (non-empty, proper, lower) order ideal $\lambda \subset \widehat{\mathcal{P}}_X$. A quantum shape will also be called a *shape* when it cannot be misunderstood to be a classical shape in $\mathcal{P}_X$. The assignment

$$I(q^d[X^u]) \,=\, \{\widehat{\alpha} \in \widehat{\mathcal{P}}_X \mid \widehat{\alpha} \leq q^d[X^u]\}$$

defines an order isomorphism from $\mathcal{B}$ to the set of shapes in $\widehat{\mathcal{P}}_X$, where shapes are ordered by inclusion. We write $\mathcal{O}^\lambda = q^d\mathcal{O}^u$ when $\lambda = I(q^d[X^u])$ is the quantum shape of $q^d[X^u]$.

Quantum multiplication by any Seidel class σ defines an order automorphism of $\mathcal{B}$, which restricts to an order automorphism of $\widehat{\mathcal{P}}_X$. If $\lambda \subset \widehat{\mathcal{P}}_X$ is any quantum shape, then $\sigma \star \lambda = \{\sigma \star \widehat{\alpha} \mid \widehat{\alpha} \in \lambda\}$ defines a new quantum shape such that

$$\sigma \star \mathcal{O}^\lambda \,=\, \mathcal{O}^{\sigma \star \lambda}\,.$$

Here we have abused notation and identified σ with the corresponding K-theory class $\mathcal{O}^{I(\sigma)} \in \mathrm{QK}(X)$. The poset $\widehat{\mathcal{P}}_X$ can be identified with an infinite set of boxes in the plane, such that each automorphism defined by a Seidel class is represented by a translation of the plane, possibly combined with a reflection. This gives a simple description of products with Seidel classes in terms of quantum shapes.

Let $X = G/P_X$ be a cominuscule classical Grassmannian, that is, a Grassmannian $\mathrm{Gr}(m,n)$ of type A, a maximal orthogonal Grassmannian $\mathrm{OG}(n,2n)$, or a Lagrangian Grassmannian $\mathrm{LG}(n,2n)$. The Chern classes of the tautological vector bundles over X are represented by the *special* Schubert varieties $X^p \subset X$, with $p \in \mathbb{N}$. Formulas for products with the special Schubert classes $[X^p]$ are known as *Pieri formulas.* Our Pieri formula for $\mathrm{QK}(X)$ takes the form

$$\mathcal{O}^p \star \mathcal{O}^\lambda \,=\, \sum_{\nu} c(\nu/\lambda, p)\, \mathcal{O}^\nu\,,$$

where the sum is over all quantum shapes ν containing λ. The coefficient $c(\nu/\lambda, p)$ depends on p as well as the *skew shape* $\nu/\lambda := \nu \smallsetminus \lambda \subset \widehat{\mathcal{P}}_X$. For Grassmannians of type A and maximal orthogonal Grassmannians, these coefficients $c(\nu/\lambda, p)$ are identical to those appearing in the Pieri formulas for the ordinary K-theory ring. These coefficients are signed binomial coefficients in type A [Len00], and are signed counts of KOG-tableaux of shape ν/λ for maximal orthogonal Grassmannians [BR12]. In fact, in these cases the Pieri formula for $\mathrm{QK}(X)$ is an easy consequence of Theorem 1.1, the Pieri formula for $K(X)$, and a bound on the q-degrees in cominuscule quantum products proved in [BCMP22].

Assume now that $X = \mathrm{LG}(n,2n)$ is a Lagrangian Grassmannian. In this case our Pieri formula for $\mathrm{QK}(X)$ is more difficult to state and prove. While the coefficients of the Pieri formula for $K(X)$ are expressed as signed counts of KLG-tableaux in [BR12], we need to amend the definition of KLG-tableau with additional conditions in the quantum case. The tableaux satisfying these conditions will be called *QKLG-tableaux*. Another difference is that the Lagrangian Grassmannian X does not have enough Seidel classes to translate the Pieri formula for $K(X)$ to one for $\mathrm{QK}(X)$. We must therefore prove our quantum Pieri formula 'from scratch', starting with a geometric computation of the relevant K-theoretic Gromov-Witten invariants, and then use combinatorics to translate these Gromov-Witten invariants to the structure constants $c(\nu/\lambda, p)$ of Pieri products. While both parts resemble the proof of the Pieri formula from [BR12], the technical challenges are harder for several reasons, and many steps rely on results proved in [BCMP22].

Our computation of Gromov-Witten invariants targets those of the form

$$I_d(\mathcal{O}^p, \mathcal{O}^v, \mathcal{O}_u) \,=\, \chi(\mathrm{ev}_1^*(\mathcal{O}^p)\cdot \mathrm{ev}_2^*(\mathcal{O}^v)\cdot \mathrm{ev}_3^*(\mathcal{O}_u))\,,$$

where $\mathrm{ev}_1, \mathrm{ev}_2, \mathrm{ev}_3 : \overline{\mathcal{M}}_{0,3}(X,d) \to X$ are the evaluation maps from the Kontsevich modulo space. By [BCMP18b], these can be computed as

$$I_d(\mathcal{O}^p, \mathcal{O}^v, \mathcal{O}_u) \,=\, \chi_X([\mathcal{O}_{\Gamma_d(X_u,X^v)}]\cdot \mathcal{O}^p)\,,$$

where the *curve neighborhood* $\Gamma_d(X_u, X^v) \subset X$ is defined as the union of all stable curves of degree d connecting X_u and X^v. Let $\widehat{X} = \mathrm{SF}(1, n; 2n)$ be the variety of two-step isotropic flags in the symplectic vector space $\mathbb{C}^{2n}$, and let $\pi : \widehat{X} \to X$ and $\eta : \widehat{X} \to \mathbb{P}^{2n-1}$ be the projections. We then have $\mathcal{O}^p = \pi_*\eta^*([\mathcal{O}_L])$ for any linear subspace $L \subset \mathbb{P}^{2n-1}$ of dimension $n-p$. The projection formula therefore gives

$$I_d(\mathcal{O}^p, \mathcal{O}^v, \mathcal{O}_u) \,=\, \chi_{\mathbb{P}^{2n-1}}(\eta_*\pi^*[\mathcal{O}_{\Gamma_d(X_u,X^v)}]\cdot[\mathcal{O}_L])\,.$$

We compute the right hand side by showing that the restricted map

$$\eta : \pi^{-1}(\Gamma_d(X_u, X^v)) \,\to\, \eta(\pi^{-1}(\Gamma_d(X_u, X^v))) \tag{1}$$

is cohomologically trivial, and that its image is a complete intersection in $\mathbb{P}^{2n-1}$ defined by explicitly determined equations. More precisely, define the skew shape $\theta = I(q^d[X^u])/I([X^v])$ in $\widehat{\mathcal{P}}_X$, let $N(\theta)$ be the number of components of θ that are disjoint from the two diagonals in $\widehat{\mathcal{P}}_X$ (Section 7), and let $R(\theta)$ be the size of a maximal rim contained in θ. Assuming that $R(\theta) \le n$, we show that $\eta(\pi^{-1}(\Gamma_d(X_u, X^v)))$ is a complete intersection in $\mathbb{P}^{2n-1}$ defined by $N(\theta)$ quadratic equations and $n - R(\theta) - N(\theta)$ linear equations. This gives the formula

$$I_d(\mathcal{O}^p, \mathcal{O}^v, \mathcal{O}_u) \,=\, \chi(\mathcal{O}_{L\,\cap\,\eta(\pi^{-1}(\Gamma_d(X_u,X^v)))}) \,=\, \sum_{j=0}^{R(\theta)-p} (-1)^j\, 2^{N(\theta)-j}\binom{N(\theta)}{j}\,. \tag{2}$$

In the special case $d = 0$ we have $\Gamma_d(X_u, X^v) = X_u \cap X^v$, so (1) is the projection of a Richardson variety in $\widehat{X}$. This map was proved to be cohomologically trivial in [BR12] by showing that its general fibers are themselves Richardson varieties. This result has been generalized to arbitrary projections of Richardson varieties, see [BC12, KLS14] and [BCMP22, Thm. 2.10]. However, the variety $\pi^{-1}(\Gamma_d(X_u, X^v))$ for $d > 0$ is not a Richardson variety, and it is difficult to determine the fibers of the projection (1).

Let $Y_d = \mathrm{SG}(n-d, 2n)$ be the symplectic Grassmannian of isotropic subspaces of dimension $n-d$ in $\mathbb{C}^{2n}$, set $Z_d = \mathrm{SF}(n-d, n; 2n)$, and let $p_d : Z_d \to X$ and $q_d : Z_d \to Y_d$ be the projections. By the quantum-to-classical construction (see [BCMP22, §5] and references therein) we have $\Gamma_d(X_u, X^v) = p_d(q_d^{-1}(R))$, where $R = q_d(p_d^{-1}(X_u)) \cap q_d(p_d^{-1}(X^v))$ is a Richardson variety in Y_d. Define the *perpendicular incidence variety*

$$S = \{(K, L) \in Y_d \times \mathbb{P}^{2n-1} \mid K \subset L^\perp\},$$

and let $f : S \to \mathbb{P}^{2n-1}$ and $g : S \to Y_d$ be the projections. We then have $f(g^{-1}(R)) = \eta(\pi^{-1}(\Gamma_d(X_u, X^v)))$.

We prove that for any Richardson variety $R \subset Y_d$, the general fibers of the map $f : g^{-1}(R) \to f(g^{-1}(R))$ are rational, and the image $f(g^{-1}(R))$ is a complete intersection in $\mathbb{P}^{2n-1}$ defined by explicitly given linear and quadratic equations. The required properties of the projection (1) are deduced from this result. Our results about perpendicular incidences of Richardson varieties in Y_d are stronger than required for this paper, but of independent interest. For example, the fibers of $f : g^{-1}(R) \to f(g^{-1}(R))$ is a plausible definition of Richardson varieties in the odd symplectic Grassmannian $\mathrm{SG}(n-d, 2n-1)$. Notice also that S is not a flag variety, so $f(g^{-1}(R))$ is not a projected Richardson variety.

A final step in our proof of the Pieri formula for $\mathrm{QK}(X)$ is to translate the formula (2) for Gromov-Witten invariants of Pieri type to a formula for the Pieri coefficients $c(\nu/\lambda, p)$. We first show that the structure constants $I_d(\mathcal{O}^p, \mathcal{O}^v, \mathcal{I}_u)$ of the undeformed product $\mathcal{O}^p \odot \mathcal{O}^v$ (see Section 2.5) are determined by recursive identities. These identities are used to prove that the Pieri coefficients $c(\nu/\lambda, p)$ satisfy analogous recursive identities. The Pieri formula for $\mathrm{QK}(X)$ then follows by checking that the signed counts of QKLG-tableaux satisfy the same identities.

This paper is organized as follows. In Section 2 we fix our notation for flag varieties and discuss preliminaries. Section 3 contains the proof of Theorem 1.1. In Section 4 we define quantum shapes in the partially ordered set $\widehat{\mathcal{P}}_X$, and explain how quantum multiplication by Seidel classes correspond to order automorphisms of this set. The Pieri formulas for $\mathrm{QK}(X)$ are given in Section 5 for Grassmannians of type A, in Section 6 for maximal orthogonal Grassmannians, and in Section 7 for Lagrangian Grassmannians. These sections also explain in detail how the posets $\widehat{\mathcal{P}}_X$ for the classical Grassmannians are identified with sets of boxes in the plane. While the Pieri formulas for $\mathrm{Gr}(m, n)$ and $\mathrm{OG}(n, 2n)$ have short proofs given after their statements, the proof of the Pieri formula for Lagrangian Grassmannians is given in the last three sections. Section 8 proves that the map $f : g^{-1}(R) \to f(g^{-1}(R))$ is cohomologically trivial and identifies its image as a complete intersection in $\mathbb{P}^{2n-1}$. Section 9 uses this result to prove the formula (2) for Gromov-Witten invariants $I_d(\mathcal{O}^p, \mathcal{O}^v, \mathcal{O}_u)$ of Pieri type. Finally, Section 10 proves the recursive identities that determine the invariants $I_d(\mathcal{O}^p, \mathcal{O}^v, \mathcal{I}_u)$ and the Pieri coefficients $c(\nu/\lambda, p)$.

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2. Cominuscule flag varieties

In this section we summarize some basic notation and definitions. We follow the notation of [BCMP22].

2.1. **Flag varieties.** Let G be a connected semisimple linear algebraic group over $\mathbb{C}$, and fix a Borel subgroup B and a maximal torus T such that $T \subset B \subset G$. The opposite Borel subgroup $B^- \subset G$ is determined by $B \cap B^- = T$. Let W be the Weyl group of G and let Φ be the root system, with simple roots $\Delta \subset \Phi^+$.

A *flag variety* of G is a projective variety with a transitive G-action. Given a flag variety X of G, we let $P_X \subset G$ denote the stabilizer of the unique B-fixed point in X. We obtain the identification $X = G/P_X = \{g.P_X \mid g \in G\}$, where $g.P_X$ is the g-translate of the B-fixed point.

Let $W_X \subset W$ be the Weyl group of P_X and let $W^X \subset W$ be the set of minimal representatives of the cosets in W/W_X. Each element $w \in W$ defines the Schubert varieties

$$X_w = \overline{Bw.P_X} \qquad \text{and} \qquad X^w = \overline{B^- w.P_X}\,,$$

and for $w \in W^X$ we have $\dim(X_w) = \operatorname{codim}(X^w, X) = \ell(w)$. The Bruhat order on W^X is defined by $v \leq u$ if and only if $X_v \subset X_u$.

Any element $u \in W$ has a unique parabolic factorization $u = u^X u_X$, where $u^X \in W^X$ and $u_X \in W_X$. The parabolic factorization of the longest element $w_0 \in W$ is $w_0 = w_0^X w_{0,X}$, where w_0^X is the longest element in W^X and $w_{0,X}$ is the longest element in W_X. We have $w_0.X^u = X_{u^\vee}$ for any $u \in W^X$, where $u^\vee = w_0\, u\, w_{0,X} \in W^X$ denotes the Poincare dual basis element.

Lemma 2.1. *Let $Z = G/P_Z$ be any flag variety with $P_Z \subset P_X$, and let $p : Z \to X$ be the projection. Let $F = p^{-1}(1.P_X) = P_X/P_Z$ denote the fiber over $1.P_X$, considered as a flag variety of P_X. Let $u \in W^X$ and $w \in W^Z$.*

(a) *We have $p(Z_w) = X_w = X_{w^X}$, and the general fibers of $p : Z_w \to X_w$ are translates of $F_{w_X} = Z_{w_X}$.*

(b) *We have $p(Z^w) = X^w = X^{w^X}$, and the general fibers of $p : Z^w \to X^w$ are translates of F^{w_X}.*

(c) *The map $p : Z^w \to X^w$ is birational if and only if $w_X = w_{0,X}^Z := (w_{0,X})^Z$.*

(d) *We have $p^{-1}(X_u) = Z_{u\, w_{0,X}^Z}$, and $u\, w_{0,X}^Z \in W^Z$.*

(e) *We have $p^{-1}(X^u) = Z^u$, and $u \in W^Z$.*

Proof. Parts (a) and (b) are [BCMP22, Thm. 2.8 and Remark 2.9], and part (c) follows from (b). Parts (d) and (e) hold because the T-fixed points in $p^{-1}(u.P_X)$ are the points of the form $ut.P_Z$, with $t \in W_X$. ◻

Proposition 2.2. *Let $Y = G/P_Y$ and $X = G/P_X$ be flag varieties, let $u \in W^Y$, and assume that $(P_X.P_Y) \cap Y^u \neq \emptyset$. Then $(P_X.P_Y) \cap Y^u = (w_0^X)^{-1}.Y^v$, where $v = w_0^X u\, ((w_{0,Y})^X)^{-1} \in W^Y$. In particular, $(P_X.P_Y) \cap Y^u$ is a Schubert variety in Y.*

Proof. Set $Z = G/(P_X \cap P_Y)$, let $p : Z \to X$ and $q : Z \to Y$ be the projections, and set $F = p^{-1}(1.P_X) = P_X/P_Z$. Let $t = w_0 u w_{0,Z}$ be the Poincare dual element of u in W^Z. By [BCMP22, Thm. 2.8] we have $t.F \cap Z_t = t^X.Z_{t_X}$. The assumption $P_X.P_Y \cap Y^u \neq \emptyset$ implies that $p(Z^u) = X$, hence $t^X = w_0^X$ and $t_X = (w_0^X)^{-1} t$. We obtain

$$F \cap Z^u = w_0.(t.F \cap Z_t) = w_0.(t^X.Z_{t_X}) = (w_0 t^X w_0).Z^{w_0 t_X w_{0,Z}} = (w_0^X)^{-1}.Z^{w_0^X u}\,,$$

where $w_0 t_X w_{0,Z} = w_0^X u$ belongs to W^Z. Since $q : F \cap Z^u \to (P_X.P_Y) \cap Y^u$ is an isomorphism, it follows from Lemma 2.1(c) that $(w_0^X u)_Y = w_{0,Y}^Z = (w_{0,Y})^X$ and $(w_0^X u)^Y = w_0^X u((w_{0,Y})^X)^{-1}$. The result now follows from Lemma 2.1(b). ◻

2.2. **Cominuscule flag varieties.** A simple root $\gamma \in \Delta$ is called *cominuscule* if the coefficient of γ is one when the highest root of Φ is expressed in the basis of simple roots. The flag variety $X = G/P_X$ is called cominuscule if P_X is a maximal parabolic subgroup corresponding to a cominuscule simple root γ, that is, s_γ is the unique simple reflection in W^X. A cominuscule flag variety X is also called *minuscule* if the root system Φ is simply laced. In the remainder of this section we assume that $X = G/P_X$ is the cominuscule flag variety defined by the cominuscule simple root $\gamma \in \Delta$.

The Bruhat order on W^X is a distributive lattice [Pro84] with meet and join operations defined by $X_{u\cap v} = X_u \cap X_v$ and $X^{u\cup v} = X^u \cap X^v$ for $u, v \in W^X$. The minimal representatives in W^X can be identified with shapes of boxes as follows [Pro84, Per07, BS16]. The root system Φ has a natural partial order defined by $\alpha' \leq \alpha$ if and only if $\alpha - \alpha'$ is a sum of positive roots. Let $\mathcal{P}_X \subset \Phi^+$ be the subset

$$\mathcal{P}_X = \{\alpha \in \Phi \mid \alpha \geq \gamma\}\,,$$

with the induced partial order (see Table 1). A lower order ideal $\lambda \subset \mathcal{P}_X$ is called a *shape* in $\mathcal{P}_X$. There is a natural bijection between W^X and the set of shapes in $\mathcal{P}_X$ that sends $w \in W^X$ to its inversion set

$$I(w) = \{\alpha \in \Phi^+ \mid w.\alpha \in \Phi^-\}\,.$$

This correspondence is compatible with the Bruhat order, so that $v \leq u$ holds in W^X if and only if $I(v) \subset I(u)$. In addition, we have $\ell(w) = |I(w)|$. The elements of $\mathcal{P}_X$ will frequently be called *boxes*. There exists a natural labeling $\delta : \mathcal{P}_X \to \Delta$ defined by $\delta(\alpha) = w.\alpha$, where $w \in W^X$ is the unique element with shape $I(w) = \{\alpha' \in \mathcal{P}_X : \alpha' < \alpha\}$. Given $u \in W^X$, write $I(u) = \{\gamma = \alpha_1, \alpha_2, \dots, \alpha_\ell\}$, where the boxes of $I(u)$ are listed in non-decreasing order, that is, $\alpha_i \leq \alpha_j$ implies $i \leq j$. Then $u = s_{\delta(\alpha_\ell)} \cdots s_{\delta(\alpha_2)} s_{\delta(\alpha_1)}$ is a reduced expression for u.

If $\lambda \subset \mathcal{P}_X$ is any shape and $w \in W^X$ is the corresponding element with $I(w) = \lambda$, then the Schubert varieties defined by w will also be denoted by

$$X_\lambda = X_w \qquad \text{and} \qquad X^\lambda = X^w\,.$$

2.3. **Curve neighborhoods.** Let $M_d = \overline{\mathcal{M}}_{0,3}(X, d)$ denote the Kontsevich moduli space of 3-pointed stable maps to X of degree d and genus zero, see [FP97]. The evaluation maps are denoted $\mathrm{ev}_i : M_d \to X$, for $1 \leq i \leq 3$. Given opposite Schubert varieties X_u and X^v in X and a degree $d \geq 0$, let

$$M_d(X_u, X^v) = \mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v)$$

be the *Gromov-Witten variety* of stable maps that send the first two marked points to X_u and X^v, respectively. This variety is empty or unirational with rational singularities [BCMP13, §3]. The *curve neighborhood*

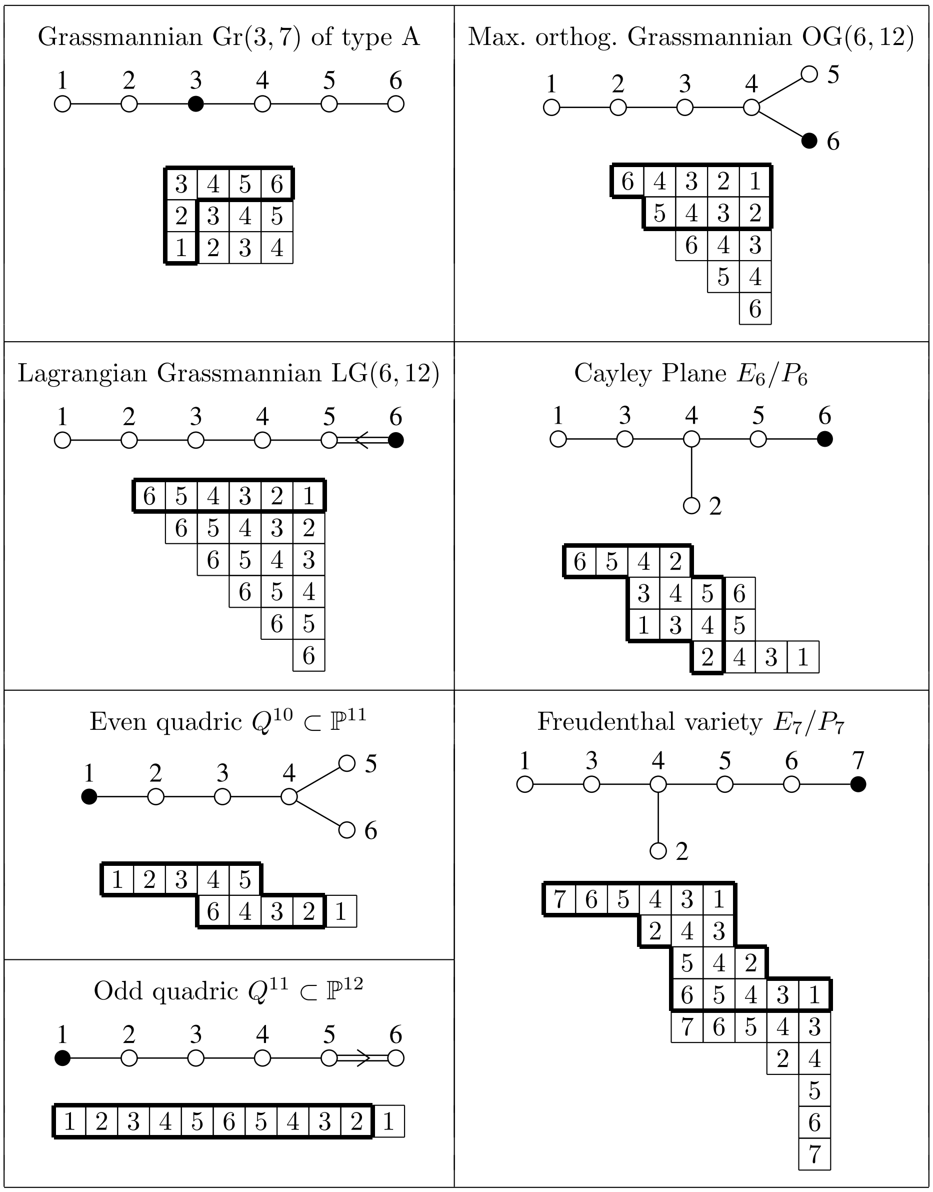
$$\Gamma_d(X_u, X^v) = \mathrm{ev}_3(M_d(X_u, X^v))$$

is the union of all stable curves of degree d in X that connect X_u and X^v. In particular, $\Gamma_d(X_u) = \Gamma_d(X_u, X)$ is the union of all stable curves of degree d that pass through X_u. Since this variety is a Schubert variety in X [BCMP13, Prop. 3.2(b)], we can define elements $u(d), v(-d) \in W^X$ by

$$\Gamma_d(X_u) = X_{u(d)} \qquad \text{and} \qquad \Gamma_d(X^v) = X^{v(-d)}\,.$$

Define $z_d \in W^X$ by $\Gamma_d(1.P_X) = X_{z_d}$.

TABLE 1. Partially ordered sets $\mathcal{P}_X$ of cominuscule varieties with $I(z_1)$ highlighted. In each case the partial order is given by $\alpha' \leq \alpha$ if and only if α' is weakly north-west of α.



The curve neighborhood $\Gamma_d(X_u, X^v)$ can be constructed as a projected Richardson variety as follows [BCMP18b]. Given $x, y \in X$, let $\operatorname{dist}(x, y) \in H_2(X, \mathbb{Z}) = \mathbb{Z}$ denote the minimal degree of a rational curve in X that meets both x and y. The *diameter* of X is the distance $d_X(2) = \operatorname{dist}(1.P_X, w_0.P_X)$ between two general points. For $0 \leq d \leq d_X(2)$, we can choose points $x, y \in X$ with $\operatorname{dist}(x, y) = d$. Let $\Gamma_d(x, y)$ be the union of all stable curves of degree d that pass through x and y.

Then $\Gamma_d(x,y)$ is a non-singular Schubert variety, whose stabilizer P_{Y_d} is a parabolic subgroup of G. The set of all G-translates of $\Gamma_d(x,y)$ can therefore be identified with the flag variety $Y_d = G/P_{Y_d}$. Let $Z_d = G/P_{Z_d}$ be the flag variety defined by $P_{Z_d} = P_X \cap P_{Y_d}$, and let $p_d : Z_d \to X$ and $q_d : Z_d \to Y_d$ be the projections. Set

$$Y_d(X_u, X^v) = q_d(p_d^{-1}(X_u)) \cap q_d(p_d^{-1}(X^v)) \qquad \text{and}$$
$$Z_d(X_u, X^v) = q_d^{-1}(Y_d(X_u, X^v))\,.$$

These varieties are Richardson varieties in Y_d and Z_d. By [BCMP18b, Thm. 4.1] and [BCMP22, Thm. 10.1] we then have $\Gamma_d(X_u, X^v) = p_d(Z_d(X_u, X^v))$, and the restricted projection

$$p_d : Z_d(X_u, X^v) \to \Gamma_d(X_u, X^v) \tag{3}$$

is cohomologically trivial. We let $\kappa_d = (w_{0,Y_d})^X = w_{0,Y_d}^{Z_d} \in W^X$ be the unique element such that $X_{\kappa_d} = p(q^{-1}(1.P_{Y_d}))$ is a translate of $\Gamma_d(x,y)$. A combinatorial description of the elements $\kappa_d, z_d \in W^X$ can be found in [BCMP22, Def. 5.2].

2.4. **Quantum cohomology.** The (small) *quantum cohomology ring* $\mathrm{QH}(X)$ is a $\mathbb{Z}[q]$-algebra, which is defined by $\mathrm{QH}(X) = H^*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a $\mathbb{Z}[q]$-module. When X is cominuscule, the multiplicative structure is given by

$$[X_u] \star [X^v] \;=\; \sum_{d \geq 0} (p_d)_* [Z_d(X_u, X^v)]\, q^d\,.$$

This follows from the *quantum equals classical* theorem [Buc03, BKT03, CMP08, BM11, CP11, BCMP18b]. A mostly type-uniform proof was given in [BCMP22]. Notice that we have

$$(p_d)_*[Z_d(X_u, X^v)] \;=\; \begin{cases} [\Gamma_d(X_u, X^v)] & \text{if } \dim \Gamma_d(X_u, X^v) = \dim Z_d(X_u, X^v), \\ 0 & \text{otherwise,} \end{cases}$$

for example because the projection (3) is cohomologically trivial. Let

$$\mathrm{QH}(X)_q = \mathrm{QH}(X) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q, q^{-1}]$$

be the localization of $\mathrm{QH}(X)$ at the deformation parameter q. The set $\mathcal{B} = \{q^d\,[X^u] \mid u \in W^X \text{ and } d \in \mathbb{Z}\}$ is a natural $\mathbb{Z}$-basis of $\mathrm{QH}(X)_q$.

2.5. **Quantum K-theory.** Let $K(X)$ denote the K-theory ring of algebraic vector bundles on X. Given $u \in W$, we let $\mathcal{O}_u = [\mathcal{O}_{X_u}]$ and $\mathcal{O}^u = [\mathcal{O}_{X^u}]$ denote the corresponding K-theoretic Schubert classes. For any shape $\lambda \subset \mathcal{P}_X$, we similarly write $\mathcal{O}_\lambda = [\mathcal{O}_{X_\lambda}]$ and $\mathcal{O}^\lambda = [\mathcal{O}_{X^\lambda}]$.

The *quantum K-theory ring* $\mathrm{QK}(X)$ is an algebra over the power series ring $\mathbb{Z}[\![q]\!]$, which is given by $\mathrm{QK}(X) = K(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\![q]\!]$ as a $\mathbb{Z}[\![q]\!]$-module. An *undeformed product* on $\mathrm{QK}(X)$ is defined by

$$\mathcal{O}_u \odot \mathcal{O}^v \;=\; \sum_{d \geq 0} (p_d)_* [\mathcal{O}_{Z_d(X_u, X^v)}]\, q^d \;=\; \sum_{d \geq 0} [\mathcal{O}_{\Gamma_d(X_u, X^v)}]\, q^d\,.$$

This product $\mathcal{O}_u \odot \mathcal{O}^v$ is not associative. Let $\psi : \mathrm{QK}(X) \to \mathrm{QK}(X)$ be the *line neighborhood operator*, defined as the $\mathbb{Z}[\![q]\!]$-linear extension of the map $\psi = (\mathrm{ev}_2)_*(\mathrm{ev}_1)^* : K(X) \to K(X)$, where ev_1 and ev_2 are the evaluation maps from $\overline{\mathcal{M}}_{0,2}(X,1)$. Equivalently, we have $\psi(\mathcal{O}^u) = \mathcal{O}^{u(-1)}$ for $u \in W^X$. Givental's associative quantum product on $\mathrm{QK}(X)$ is then given by [BCMP18a, Prop. 3.2]

$$\mathcal{O}_u \star \mathcal{O}^v \;=\; (1 - q\psi)(\mathcal{O}_u \odot \mathcal{O}^v)\,.$$

Let $\mathrm{QK}(X)_q$ be the localization obtained by adjoining the inverse of q to $\mathrm{QK}(X)$. The set $\mathcal{B}' = \{q^d\, \mathcal{O}^u \mid u \in W^X \text{ and } d \in \mathbb{Z}\}$ is a $\mathbb{Z}$-basis of $\mathrm{QK}(X)_q$, in the sense that every element of $\mathcal{F} \in \mathrm{QK}(X)_q$ can be uniquely expressed as an infinite linear combination

$$\mathcal{F} = \sum_{d \geq d_0} \sum_{u \in W^X} a_{u,d}\, q^d\, \mathcal{O}^u$$

of $\mathcal{B}'$, with $a_{u,d} \in \mathbb{Z}$ and the degrees d bounded below.

3. The Seidel representation on quantum K-theory

Let $X = G/P_X$ be a fixed cominuscule flag variety. In this section we prove that certain products $\mathcal{O}^u \star \mathcal{O}^v$ in $\mathrm{QK}(X)$ are equal to a single element $q^d\, \mathcal{O}^w$ from $\mathcal{B}'$. The same statement was proved in [Bel04, CMP09] for products of Schubert classes in the quantum cohomology ring $\mathrm{QH}(M)$ of any flag variety $M = G/P_M$. For $u, v \in W$ we let $d_{\min}(u,v)$ and $d_{\max}(u,v)$ denote the minimal and maximal powers of q in the quantum cohomology product $[X^u] \star [X^v] \in \mathrm{QH}(X)$. Let $d_{\max}(u) = d_{\max}(u, w_0^X)$ be the maximal power of q in $[X^u] \star [1.P_X]$.

Lemma 3.1. *Let $u \in W^X$ and $d_{\max}(u) \leq d \leq d_X(2)$. Then $\Gamma_d(1.P_X, X^u) = (w_0^X)^{-1}.X^v$, where $v = w_0^X(u \cup \kappa_d)(z_d\kappa_d)^{-1} \in W^X$.*

Proof. Using that $\kappa_d \in W_{Y_d}$, we obtain $q_d(p_d^{-1}(X^u)) = q_d(p_d^{-1}(X^{u\cup\kappa_d}))$, and hence $\Gamma_d(1.P_X, X^u) = \Gamma_d(1.P_X, X^{u\cup\kappa_d})$, so we may replace u with $u \cup \kappa_d$ and assume that $d = d_{\max}(u)$ (see [BCMP22, §7.1]). We have $q_d({p_d}^{-1}(1.P_X)) = P_X.P_{Y_d}$ and $q_d(p_d^{-1}(X^u)) = (Y_d)^{u\kappa_d}$ by Lemma 2.1, and since $\kappa_d \leq u_{Y_d} \leq w_{0,Y_d}^{Z_d} = \kappa_d = \kappa_d^{-1}$, we obtain $u\kappa_d \in W^{Y_d}$. It therefore follows from Proposition 2.2 that $Y_d(1.P_X, X^u) = (P_X.P_{Y_d}) \cap Y_d^{u\kappa_d} = (w_0^X)^{-1}.Y_d^v$, where $v = w_0^X(u\kappa_d)\kappa_d^{-1} = w_0^X u \in W^{Y_d}$. The result follows from this and Lemma 2.1, using that $p_d : Z_d(1.P_X, X^u) \to \Gamma_d(1.P_X, X^u)$ is birational [BCMP22, Prop. 7.1] and $w_{0,X}^{Z_d} = z_d\kappa_d$ [BCMP22, Lemma 6.1]. $\square$

Corollary 3.2. *For $u \in W$ we have $[1.P_X] \star [X^u] = q^{d_{\max}(u)}\, [X^{w_0^X u}]$ in $\mathrm{QH}(X)$ and $[\mathcal{O}_{1.P_X}] \star \mathcal{O}^u = q^{d_{\max}(u)}\, \mathcal{O}^{w_0^X u}$ in $\mathrm{QK}(X)$.*

Proof. This follows from Lemma 3.1 together with [BCMP22, Prop. 7.1, Thm. 8.3, and Thm. 8.10]. Notice that the product $[\mathcal{O}_{1.P_X}] \star \mathcal{O}^u$ has no exceptional degree by the inequality in [BCMP22, Def. 8.2]. $\square$

Let $W^{\mathrm{comin}} \subset W$ be the subset of point representatives of cominuscule flag varieties of G, together with the identity element:

$$W^{\mathrm{comin}} = \{w_0^F \mid F \text{ is a cominuscule flag variety of } G\} \cup \{1\}\,.$$

Remarkably, this is a subgroup of W, which is also isomorphic to the quotient of the coweight lattice of Φ by the coroot lattice. The isomorphism sends w_0^F to the class of the fundamental coweight corresponding to F.

The classes $q^d[X^w] \in \mathrm{QH}(X)_q$ and $q^d\mathcal{O}^w \in \mathrm{QK}(X)_q$ given by $w \in W^{\mathrm{comin}}$ and $d \in \mathbb{Z}$ are called *Seidel classes*. The cohomological Seidel classes $q^d[X^w]$ form a subgroup of the group of units $\mathrm{QH}(X)_q^\times$ by [Bel04, CMP09]. We will see in Corollary 3.7 below that the K-theoretic Seidel classes similarly form a subgroup of $\mathrm{QK}(X)_q^\times$.

The following lemma shows that $[X^u]$ is a Seidel class if and only if the dual class $[X_u]$ is a Seidel class (when X is cominuscule).

Lemma 3.3. *Let $X = G/P_X$ and $F = G/P_F$ be flag varieties. The dual element of $(w_0^F)^X$ in W^X is $((w_0^F)^{-1} w_0^X)^X$.*

Proof. Using that $w_0 = w_0^F w_{0,F}$, we obtain $(w_0^F)^{-1} w_0 = w_{0,F} = (w_{0,F})^{-1} = w_0 w_0^F$, so the dual element of $(w_0^F)^X$ is $(w_0 w_0^F)^X = ((w_0^F)^{-1} w_0)^X = ((w_0^F)^{-1} w_0^X)^X$. $\square$

The following combinatorial lemma is justified with a case-by-case argument. We hope to give a type-independent proof in later work.

Lemma 3.4. *Let X be a cominuscule flag variety, let $\alpha \in I(z_1) \smallsetminus \{\gamma\}$, and define $u \in W^X$ by $I(u) = \{\alpha' \in \mathcal{P}_X \mid \alpha' \leq (z_1 s_\gamma).\alpha\}$. The following are equivalent.*

(a) *$u = w^X$ for some $w \in W^{\mathrm{comin}}$.*

(b) *$\delta(\alpha)$ is a cominuscule simple root.*

(c) *$\alpha \not\leq (z_1 s_\gamma).\alpha$.*

(d) *$\mathcal{P}_X \smallsetminus I(u) = \{\alpha' \in \mathcal{P}_X \mid \alpha' \geq \alpha\}$.*

When these conditions hold we have $u^\vee = (w_0^F)^X$, where $F = G/P_F$ is the cominuscule flag variety defined by $\delta(\alpha)$.

Proof. The action of $w_{0,X}$ restricts to an order-reversing involution of $\mathcal{P}_X$, and $z_1 s_\gamma : I(z_1) \smallsetminus \{\gamma\} \to w_{0,X}.(I(z_1) \smallsetminus \{\gamma\})$ is an order isomorphism, see [BCMP22, Lemma 4.4 and Prop. 5.10]. This uniquely determines $(z_1 s_\gamma).\alpha$ for most cominuscule flag varieties. In this proof we will identify shapes labeled by simple root numbers with the product of the corresponding simple reflections in south-east to north-west order. For example, the set $\mathcal{P}_X$ labeled by simple root numbers, as in Table 1, is identified with w_0^X.

Assume first that the root system Φ has type A_{n-1}, with simple roots $\Delta = \{\beta_1, \dots, \beta_{n-1}\}$. All simple roots are cominuscule. Let $X = \mathrm{Gr}(m, n)$ be defined by $\gamma = \beta_m$. Then $\mathcal{P}_X$ is a rectangle with m rows and $n - m$ columns, and $I(z_1) \smallsetminus \{\gamma\}$ consists of the top row and leftmost column of $\mathcal{P}_X$, except for the minimal box γ. Let $\alpha \in I(z_1) \smallsetminus \{\gamma\}$ be the box in column c of the top row of $\mathcal{P}_X$. Then $(z_1 s_\gamma).\alpha$ is the box in column $c-1$ of the bottom row of $\mathcal{P}_X$, and $I(u)$ is a rectangle with m rows and $c-1$ columns. We also have $\delta(\alpha) = \beta_{m+c-1}$, which defines $F = \mathrm{Gr}(m+c-1, n)$. The shape of $(w_0^F)^X$ is a rectangle with m rows and $n - m - c + 1$ columns; this follows because the top part of $I(w_0^F)$ cancels when w_0^F is reduced modulo W_X. For example, for $X = \mathrm{Gr}(3, 8)$ and $c = 4$, we obtain $F = \mathrm{Gr}(6, 8)$ and

$$w_0^X = \begin{array}{|c|c|c|c|c|}\hline 3&4&5&\boxed{6}&7\\\hline 2&3&4&5&6\\\hline 1&2&3&4&5\\\hline\end{array}\,, \quad w_0^F = \begin{array}{|c|c|}\hline 6&7\\\hline 5&6\\\hline 4&5\\\hline 3&4\\\hline 2&3\\\hline 1&2\\\hline\end{array}\,, \quad \text{and} \quad (w_0^F)^X = \begin{array}{|c|c|}\hline 3&4\\\hline 2&3\\\hline 1&2\\\hline\end{array} = s_2 s_1 s_3 s_2 s_4 s_3\,.$$

The marked box is α. It follows that u is dual to $(w_0^F)^X$ in W^X, and conditions (a)-(d) are satisfied. A symmetric argument applies when α belongs to the leftmost column of $\mathcal{P}_X$.

We next assume that Φ has type D_n, with simple roots $\Delta = \{\beta_1, \dots, \beta_n\}$. The three cominuscule flag varieties of this type are $Q = D_n/P_1$, $X' = D_n/P_{n-1}$, and $X'' = D_n/P_n$. Here $Q \cong Q^{2n-2}$ is a quadric and $X' \cong X'' \cong \mathrm{OG}(n, 2n)$ are maximal orthogonal Grassmannians. For $n = 6$, the point representatives are

$$w_0^Q = \begin{array}{|c|c|c|c|c|cccc}\hline 1&2&3&4&\mathbf{5}&&&&\\ \hline &&&&\mathbf{6}&4&3&2&1\\ \cline{5-9}\end{array}\,, \quad w_0^{X'} = \begin{array}{ccccc} 5&4&3&2&\mathbf{1}\\ &\mathbf{6}&4&3&2\\ &&5&4&3\\ &&&6&4\\ &&&&5 \end{array}\,, \text{ and } \ w_0^{X''} = \begin{array}{ccccc} 6&4&3&2&\mathbf{1}\\ &\mathbf{5}&4&3&2\\ &&6&4&3\\ &&&5&4\\ &&&&6 \end{array}\,.$$

Let $X = Q$. The elements in W^Q representing Seidel classes other than 1 and $[1.P_X]$ are the two elements of length $n-1$. For $n = 6$, we obtain

$$(w_0^{X'})^Q = \begin{array}{ccccc} 1&2&3&4&5 \end{array} \quad \text{and} \quad (w_0^{X''})^Q = \begin{array}{cccc} 1&2&3&4\\ &&&6 \end{array}\,.$$

The set $I(z_1) \smallsetminus \{\gamma\}$ contains all boxes of $\mathcal{P}_Q$, except γ and the maximal box. The two incomparable boxes of $\mathcal{P}_Q$ are $\alpha' = \theta + \beta_{n-1}$ and $\alpha'' = \theta + \beta_n$, where $\theta = \beta_1 + \cdots + \beta_{n-2}$. Since $z_1 s_\gamma$ swaps α' and α'' and fixes all other boxes of $I(z_1) \smallsetminus \{\gamma\}$, it follows that (a)-(d) are satisfied if and only if $\alpha \in \{\alpha', \alpha''\}$. Assume that $\alpha = \alpha''$. We obtain $u = s_{n-1} s_{n-2} \cdots s_2 s_1$, $\delta(\alpha) = \beta_n$, and $F = X''$. If n is even, then the bottom label of $w_0^{X'}$ is $n-1$, hence $u = (w_0^{X'})^Q$, and otherwise $u = (w_0^{X''})^Q$. This is consistent with the lemma, since the elements $(w_0^{X'})^Q$ and $(w_0^{X''})^Q$ are dual to each other when n is even and self-dual when n is odd. A symmetric argument applies when $\alpha = \alpha'$.

Let $X = X'$. The shape of $(w_0^Q)^{X'}$ is a single row of $n-1$ boxes, and $(w_0^{X''})^{X'}$ is dual to $(w_0^Q)^{X'}$ in $W^{X'}$. For $n = 6$, we have

$$(w_0^Q)^{X'} = \begin{array}{ccccc} 5&4&3&2&1 \end{array} \quad \text{and} \quad (w_0^{X''})^{X'} = \begin{array}{cccc} 5&4&3&2\\ &6&4&3\\ &&5&4\\ &&&6 \end{array}\,.$$

The set $I(z_1) \smallsetminus \{\gamma\}$ consists of the first two rows of $\mathcal{P}_{X'}$, with γ removed. Let $\alpha_1, \alpha_n \in I(z_1) \smallsetminus \{\gamma\}$ be the unique boxes with labels $\delta(\alpha_1) = \beta_1$ and $\delta(\alpha_n) = \beta_n$. Then $(z_1 s_\gamma).\alpha_n = \alpha_1$, and $(z_1 s_\gamma).\alpha_1$ is the second to last diagonal box of $\mathcal{P}_X$. It follows that conditions (a)-(d) hold if and only if $\alpha \in \{\alpha_1, \alpha_n\}$, and the description of $u^\vee$ is accurate.

If X is a Lagrangian Grassmannian $\mathrm{LG}(n, 2n)$, an odd quadric Q^{2n-1}, or the Freudenthal variety E_7/P_7, then no boxes of $I(z_1) \smallsetminus \{\gamma\}$ satisfy conditions (a)–(d). The Cayley plane E_6/P_6 is similar to the cases of type D_n and left to the reader. $\square$

Lemma 3.5. *Let X be a minuscule flag variety, let $u_1, u_2, \ldots, u_\ell \in W^X$, and assume that $\mathcal{O}^{u_1} \star \mathcal{O}^{u_2} \star \cdots \star \mathcal{O}^{u_\ell} = q^d$ for some $d \in \mathbb{Z}$. Then $\mathcal{O}^{u_i} \star \mathcal{B}' \subset \mathcal{B}'$ for each i, where $\mathcal{B}' = \{q^e \mathcal{O}^v \mid v \in W^X, d \in \mathbb{Z}\}$ is the $\mathbb{Z}$-basis of $\mathrm{QK}(X)_q$.*

Proof. It follows from [BCMP22, Thm. 8.4] that $\mathrm{QK}(X)_q$ has non-negative structure constants relative to the basis

$$\mathcal{B}'' = \{(-1)^{\ell(v) + \int_d c_1(T_X)}\, q^d \mathcal{O}^v \mid v \in W^X \text{ and } d \in \mathbb{Z}\}\,.$$

The lemma therefore follows from the proof of [BW21, Lemma 3]. Namely, if the expansion of $\mathcal{O}^{u_i} \star \mathcal{O}^v$ contains more than one term, then so does the expansion of $\mathcal{O}^{u_1} \star \cdots \star \mathcal{O}^{u_\ell} \star \mathcal{O}^v = q^d \mathcal{O}^v$, which is a contradiction. $\square$

Theorem 3.6. *Let X be a cominuscule flag variety and let $u \in W^X$. The following are equivalent.*

(S1) *$u = w^X$ for some $w \in W^{\mathrm{comin}}$.*

(S2) $[X^u]\star\mathcal{B}\subset\mathcal{B}$, *where* $\mathcal{B}=\{q^d\,[X^v]\mid v\in W^X,\, d\in\mathbb{Z}\}$ *is the* $\mathbb{Z}$*-basis of* $\mathrm{QH}(X)_q$.

(S3) $\mathcal{O}^u\star\mathcal{B}'\subset\mathcal{B}'$, *where* $\mathcal{B}'=\{q^d\,\mathcal{O}^v\mid v\in W^X,\, d\in\mathbb{Z}\}$ *is the* $\mathbb{Z}$*-basis of* $\mathrm{QK}(X)_q$.

(S4) $[X_u]\star[X^u]=[1.P_X]\,\in\,\mathrm{QH}(X)$.

(S5) $\mathcal{O}_u\star\mathcal{O}^u=[\mathcal{O}_{1.P_X}]\,\in\,\mathrm{QK}(X)$.

(S6) $d_{\max}(u^\vee,u)=0$.

(S7) *We have* $u\in\{1,w_0^X\}$, *or* $\exists\;\alpha\in I(z_1)$ *such that* $\alpha\notin I(u)$ *and* $(z_1s_\gamma).\alpha\in I(u)$.

Furthermore, if α *is as in condition* (S7)*, then* $I(u)=\{\alpha'\in\mathcal{P}_X\mid\alpha'\leq(z_1s_\gamma).\alpha\}$, $\delta(\alpha)$ *is a cominuscule simple root, and* $u^\vee=(w_0^F)^X$ *where* $F=G/P_F$ *is the cominuscule flag variety defined by* $\delta(\alpha)$.

Proof. We may assume $u\notin\{1,w_0^X\}$ by Corollary 3.2. The implications (S3) $\Rightarrow$ (S2) $\Rightarrow$ (S4) and (S3) $\Rightarrow$ (S5) $\Rightarrow$ (S4) are clear, noting that the quantum cohomology product $[X^u]\star[X^v]$ is the leading term of $\mathcal{O}^u\star\mathcal{O}^v$, and is non-zero by Corollary 3.2 since $[X_u]\star[X^u]\star[X^v]\neq0$. The implication (S4) $\Rightarrow$ (S6) is also clear. Using the notation $u_1,u^1\in W$ defined in [BCMP22, Def. 6.5], it follows from [BCMP22, Prop. 7.1 and Cor. 7.4] that $d_{\max}(u^\vee,u)=0$ is equivalent to $u_1\not\leq u^1$, noting that $d_{\max}(u)>0$ and $d_{\max}(u^\vee)>0$. The elements u_1 and u^1 are cominuscule minimal representatives, so $u_1\not\leq u^1$ is equivalent to $I(u_1)\not\subset I(u^1)$. By [BCMP22, Prop. 6.2 and Prop. 6.7(b)] these inversion sets are given by

$$I(u_1)=z_1^{-1}.(I(u)\cap(I(s_\gamma^\vee)\smallsetminus I(z_1^\vee)))\quad\text{and}\quad I(u^1)=s_\gamma.(I(u)\cap(I(z_1)\smallsetminus\{\gamma\}))\,.$$

Since $(z_1s_\gamma)^{-1}.(I(s_\gamma^\vee)\smallsetminus I(z_1^\vee))=I(z_1)\smallsetminus\{\gamma\}$ and $\gamma\in I(u)$, we deduce that $I(u_1)\not\subset I(u^1)$ holds if and only if $(z_1s_\gamma)^{-1}.I(u)\cap I(z_1)\not\subset I(u)$. This proves that (S6) is equivalent to (S7). Assume (S7), and let $\alpha\in I(z_1)$ satisfy $\alpha\notin I(u)$ and $(z_1s_\gamma).\alpha\in I(u)$. Then $\alpha\not\leq(z_1s_\gamma).\alpha$, so Lemma 3.4 implies that $\delta(\alpha)$ is a cominuscule simple root. This is only possible when X is minuscule. Using (S6), it follows from [BCMP22, Thm. 8.3] that $\mathcal{O}_u\star\mathcal{O}^u=[\mathcal{O}_{1.P_X}]$. By Corollary 3.2, this implies that $(\mathcal{O}_u\star\mathcal{O}^u)^{\star m}$ is a power of q for some positive integer m, so it follows from Lemma 3.5 that $\mathcal{O}^u\star\mathcal{B}'\subset\mathcal{B}'$. This proves the implication (S7) $\Rightarrow$ (S3). We finally show that (S1) is equivalent to (S7). The implication (S7) $\Rightarrow$ (S1) follows immediately from Lemma 3.4. If (S1) holds, then $u^\vee=(w_0^F)^X$, where $F=G/P_F$ is the cominuscule flag variety defined by some cominuscule simple root $\gamma'\in\Delta\smallsetminus\{\gamma\}$. Let $\alpha\in I(z_1)$ be any root for which $\delta(\alpha)=\gamma'$, and define $v\in W^X$ by $I(v)=\{\alpha'\in\mathcal{P}_X\mid\alpha'\leq(z_1s_\gamma).\alpha\}$. Then Lemma 3.4 shows that $u=v$, which proves the implication (S1) $\Rightarrow$ (S7). The last claims of the theorem also follow from Lemma 3.4. ☐

The following result provides the action of the subgroup of Seidel classes in $\mathrm{QK}(X)_q^\times$ on the basis $\mathcal{B}'$. The statement was proved for the quantum cohomology of arbitrary flag varieties in [Bel04, CMP09].

Corollary 3.7. *Let* X *be a cominuscule flag variety, and let* $w\in W^{\mathrm{comin}}$ *and* $v\in W$*. Then,* $\mathcal{O}^w\star\mathcal{O}^v=q^{d_{\min}(w,v)}\,\mathcal{O}^{wv}$ *holds in* $\mathrm{QK}(X)$.

Proof. It follows from [Bel04, CMP09] that $[X^w]\star[X^v]=q^{d_{\min}(w,v)}\,[X^{wv}]$ holds in the quantum cohomology ring $\mathrm{QH}(X)$. The result follows from this since $[X^w]\star[X^v]$ is the leading term of $\mathcal{O}^w\star\mathcal{O}^v$, and $\mathcal{O}^w\star\mathcal{O}^v$ is a power of q times a single Schubert class by Theorem 3.6. ☐

Example 3.8. Let $X=Q^{2n-2}$ be the quadric of type D_n, let $P\in H^{4n-4}(X)$ be the point class, and let $\sigma,\tau\in H^{2n-2}(X)$ be the two Schubert classes of middle degree. Since W^{comin} has order 4 and $\deg(q) = \deg(P)$, we deduce that the Seidel classes in $H^*(X)$ consist of 1, σ, τ, and P. If n is even, then $\sigma \cdot \tau = P$ and $\sigma^2 = \tau^2 = 0$ hold in $H^*(X)$. It follows that $\sigma \star \tau = P$, $\sigma^2 = \tau^2 = q$, $\sigma \star P = q\,\tau$, and $\tau \star P = q\,\sigma$ hold in $\mathrm{QH}(X)$. Similarly, if n is odd, then $\sigma^2 = \tau^2 = P$, $\sigma \star \tau = q$, $\sigma \star P = q\,\tau$, and $\tau \star P = q\,\sigma$ hold in $\mathrm{QH}(X)$. Any product of a Seidel class with a non-Seidel Schubert class in $\mathrm{QH}(X)$ is the unique element in $\mathcal{B}$ of the correct degree. This determines all products with Seidel classes in $\mathrm{QH}(X)$. Products of arbitrary Schubert classes in $\mathrm{QH}(X)$ and $\mathrm{QK}(X)$ are determined by this together with Corollary 3.7 and the quantum Chevalley formulas [FW04, BCMP18a].

Example 3.9. Let $X = \mathrm{Gr}(2,4)$. Then

$$[X^{\ydiagram{2}}] \star [X^{\ydiagram{2,1}}] \;=\; q\,[X^{\ydiagram{1}}]$$

holds in $\mathrm{QH}(X)$. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbb{C}^4$. We claim that

$$\Gamma_1(X_{\ydiagram{1}}, X^{\ydiagram{2}}) \;=\; \{V \in X \mid V \cap \langle e_1, e_4\rangle \neq 0\}\,,$$

that is, $\Gamma_1(X_{\ydiagram{1}}, X^{\ydiagram{2}})$ is a translate of the Schubert divisor $X^{\ydiagram{1}}$. The curve neighborhood $\Gamma_1(X_{\ydiagram{1}}, X^{\ydiagram{2}})$ is the union of all lines connecting the Schubert varieties

$$\begin{aligned} X_{\ydiagram{1}} \;&=\; \{A \in X \mid \langle e_1\rangle \subset A \subset \langle e_1, e_2, e_3\rangle\} \;\text{ and} \\ X^{\ydiagram{2}} \;&=\; \{B \in X \mid \langle e_4\rangle \subset B\}\,. \end{aligned}$$

Given $V \in \Gamma_1(X_{\ydiagram{1}}, X^{\ydiagram{2}})$, we can find $A \in X_{\ydiagram{1}}$ and $B \in X^{\ydiagram{2}}$ such that

$$0 \;\neq\; A \cap B \;\subset\; V \;\subset\; A + B \;\neq\; \mathbb{C}^4\,.$$

Since V and $\langle e_1, e_4\rangle$ are both contained in $A + B$, we obtain $V \cap \langle e_1, e_4\rangle \neq 0$. This proves the claim, since $\Gamma_1(X_{\ydiagram{1}}, X^{\ydiagram{2}})$ is a divisor in X.

Set $Y_1 = \mathrm{Fl}(1,3;4)$, $Z_1 = \mathrm{Fl}(4)$, and let $p_1 : Z_1 \to X$ and $q_1 : Z_1 \to Y_1$ be the projections. We have $q_1 p_1^{-1}(X_{\ydiagram{1}}) = (Y_1)_{3142}$ and $q_1 p_1^{-1}(X^{\ydiagram{2}}) = (Y_1)^{1243}$, so it follows from Monk's formula that

$$[Y_1(X_{\ydiagram{1}}, X^{\ydiagram{2}})] \;=\; [Y_1^{2143}] \cdot [Y_1^{1243}] \;=\; [Y_1^{3142}] + [Y_1^{2341}]\,.$$

We deduce that $Y_1(X_{\ydiagram{1}}, X^{\ydiagram{2}})$ is not a Schubert variety in Y_1.

Remark 3.10. Let $M = G/P_M$ be any flag variety of G. Recall that $H_2(M,\mathbb{Z})$ can be identified with the coroot lattice of G modulo the coroot lattice of P_M, by identifying each curve class $[M_{s_\beta}]$ with the simple coroot $\beta^\vee$ (see e.g. [BM15, §2]). Let $u \in W$, $w \in W^{\text{comin}}$, and let $\beta \in \Delta \cap I(w)$ be the cominuscule simple root defining the cominuscule flag variety corresponding to w. Set $d = \omega_\beta^\vee - u^{-1}.\omega_\beta^\vee \in H_2(M,\mathbb{Z})$, where $\omega_\beta^\vee$ is the fundamental coweight dual to β. It was proved in [Bel04, CMP09] that the identity

$$[M^w] \star [M^u] = q^d [M^{wu}]$$

holds in the small quantum cohomology ring $\mathrm{QH}(M)$. This is consistent with the following conjecture.

Conjecture 3.11. *Let $M = G/P_M$ be any flag variety. For $u \in W$, $w \in W^{\text{comin}}$, $I(w) \cap \Delta = \{\beta\}$, and $d = \omega_\beta^\vee - u^{-1}.\omega_\beta^\vee \in H_2(M,\mathbb{Z})$, we have*

$$\Gamma_d(M_{w_0 w}, M^u) = w^{-1}.M^{wu}\,.$$

This conjecture follows from Proposition 2.2 when $d = 0$, from Lemma 3.1 when M is cominuscule and $w = w_0^M$, and from [LLSY22, Cor. 4.6] when M is a Grassmannian of type A and $[M^w]$ is a special Seidel class. In response to this paper, it was proved in [Tar23] that Conjecture 3.11 is true for all flag varieties of type A, and the general conjecture follows from the special case where P_M is a maximal parabolic subgroup.

4. Quantum shapes

Let $X = G/P_X$ be a cominuscule flag variety. An infinite partially ordered set $\widehat{\mathcal{P}}_X$ extending $\mathcal{P}_X$ was constructed in [BCMP22], such that elements of the set $\mathcal{B} = \{q^d[X^u] \mid u \in W^X, d \in \mathbb{Z}\}$ correspond to order ideals in $\widehat{\mathcal{P}}_X$ that we call *quantum shapes*. Isomorphic partially ordered sets were constructed in [Hag04, Pos05, Gre13]. Products of Seidel classes with arbitrary Schubert classes have simple combinatorial descriptions in terms of quantum shapes, and our Pieri formulas also have their simplest expressions in terms of these shapes. In this section we summarize the facts we need. Proofs of our claims and more details can be found in [BCMP22, §7.2]. Some claims are justified by Proposition 4.4 proved at the end of this section.

Recall that $\mathcal{B}$ is a $\mathbb{Z}$-basis of $\mathrm{QH}(X)_q$. Define a partial order on $\mathcal{B}$ by

$$q^e[X^v] \leq q^d[X^u] \iff \Gamma_{d-e}(X_u, X^v) \neq \emptyset\,.$$

The condition $\Gamma_{d-e}(X_u, X^v) \neq \emptyset$ says that some rational curve in X of degree at most $d-e$ intersects both X_u and X^v. Equivalently, $q^e[X^v] \leq q^d[X^u]$ holds if and only if $q^d[X^u]$ occurs with non-zero coefficient in the expansion of $q^e[X^v] \star q^{d'}[X^w]$ in $\mathrm{QH}(X)_q$, for some $w \in W^X$ and $d' \geq 0$ [BCMP22, §7.2]. The following was proved in [BCMP22, Thm. 7.8].

Theorem 4.1. *Let $u, v \in W^X$ and $d \in \mathbb{Z}$. The power q^d occurs in $[X^u] \star [X^v]$ if and only if $[X^v] \leq q^d[X_u] \leq [\mathrm{point}] \star [X^v]$.*

Corollary 4.2. *Assume that $u, u', v, v' \in W^X$ satisfy $u' \leq u$ and $v' \leq v$. Then $d_{\min}(u', v') \leq d_{\min}(u, v)$ and $d_{\max}(u', v') \leq d_{\max}(u, v)$.*

Proof. Set $d = d_{\min}(u, v)$. Then $[X^{v'}] \leq [X^v] \leq q^d[X_u] \leq q^d[X_{u'}]$. Using that $[X^{u'}] \star [X^{v'}] \neq 0$, this shows that $d_{\min}(u', v') \leq d$. Similarly, if we set $d = d_{\max}(u', v')$, then $q^d[X_u] \leq q^d[X_{u'}] \leq [\mathrm{point}] \star [X^{v'}] \leq [\mathrm{point}] \star [X^v]$ and $[X^u] \star [X^v] \neq 0$ implies that $d \leq d_{\max}(u, v)$, as required. $\square$

The following special case is useful for showing that a quantum product $[X^u] \star [X^v]$ has only classical terms.

Corollary 4.3. *Let $u, v \in W^X$. Assume that $u \leq w$ and $v \leq w_0 w$ for some $w \in W^{\mathrm{comin}}$. Then $d_{\max}(u, v) = 0$.*

Proof. This follows from Corollary 4.2 and condition (S6) of Theorem 3.6. $\square$

The partially ordered set $\mathcal{B}$ is a distributive lattice by [BCMP22, Prop. 7.10]. Let $\widehat{\mathcal{P}}_X \subset \mathcal{B}$ be the subset of all join-irreducible elements. These elements will be called *boxes*. Define a *quantum shape* in $\widehat{\mathcal{P}}_X$ to be any non-empty proper lower order ideal $\lambda \subset \widehat{\mathcal{P}}_X$. A quantum shape will also be called a *shape* when it cannot be misunderstood to be a classical shape in $\mathcal{P}_X$. A *skew shape* in $\widehat{\mathcal{P}}_X$ is the difference

$\lambda/\mu := \lambda \smallsetminus \mu$ between two shapes $\mu \subset \lambda \subset \widehat{\mathcal{P}}_X$. All shapes in $\widehat{\mathcal{P}}_X$ are infinite, and all skew shapes in $\widehat{\mathcal{P}}_X$ are finite. Given $q^d[X^u] \in \mathcal{B}$, define

$$I(q^d[X^u]) = \{\widehat{\alpha} \in \widehat{\mathcal{P}}_X \mid \widehat{\alpha} \leq q^d[X^u]\}\,.$$

Notice that if $q^d[X^u] \in \widehat{\mathcal{P}}_X$, then $q^d[X^u]$ is the unique maximal box of $I(q^d[X^u])$. By [BCMP22, Thm. 7.13], the map I is an order isomorphism of $\mathcal{B}$ with the set of all shapes in $\widehat{\mathcal{P}}_X$, where shapes are ordered by inclusion. For any shape $\lambda \subset \widehat{\mathcal{P}}_X$ we will write $\mathcal{O}^\lambda = q^d\mathcal{O}^u$, where $q^d[X^u] \in \mathcal{B}$ is the unique element with shape $I(q^d[X^u]) = \lambda$.

Given $\alpha \in \mathcal{P}_X$, define $\xi(\alpha) \in W^X$ by $I(\xi(\alpha)) = \{\alpha' \in \mathcal{P}_X \mid \alpha' \leq \alpha\}$. Then the quantum shape $I([X^{\xi(\alpha)}]) \subset \widehat{\mathcal{P}}_X$ contains a unique maximal box $\tau(\alpha)$ distinct from $1 \in \mathcal{B}$, the identity element of $\mathrm{QH}(X)$. The map $\tau : \mathcal{P}_X \to \widehat{\mathcal{P}}_X$ is an order isomorphism of $\mathcal{P}_X$ onto an interval in $\widehat{\mathcal{P}}_X$ by [BCMP22, Thm. 7.13]. We identify $\mathcal{P}_X$ with the image $\tau(\mathcal{P}_X) \subset \widehat{\mathcal{P}}_X$. Given a classical shape $\lambda \subset \mathcal{P}_X$, we will abuse notation and also use λ to denote the corresponding quantum shape $I([X^\lambda]) = \tau(\lambda) \cup I(1) \subset \widehat{\mathcal{P}}_X$, see Proposition 4.4(c). Both of these shapes define the same class $\mathcal{O}^\lambda \in \mathrm{QK}(X)$.

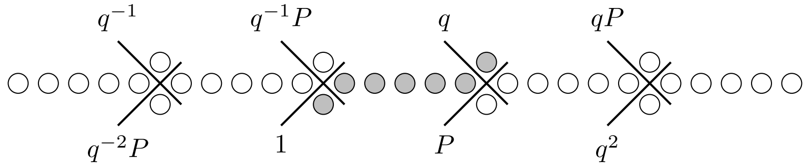
Quantum multiplication by any Seidel class $\sigma = q^d[X^w]$ in $\mathrm{QH}(X)_q$ defines an order automorphism of $\mathcal{B}$, which restricts to an order automorphism of $\widehat{\mathcal{P}}_X$. Since $1 \in \widehat{\mathcal{P}}_X$ by Proposition 4.4(a), it follows that all Seidel classes belong to $\widehat{\mathcal{P}}_X$. Given any shape $\lambda \subset \widehat{\mathcal{P}}_X$, we define a new quantum shape by $\sigma \star \lambda = \{\sigma \star \widehat{\alpha} \mid \widehat{\alpha} \in \lambda\}$. We then have

$$\mathcal{O}^{I(\sigma)} \star \mathcal{O}^\lambda \;=\; \mathcal{O}^{\sigma\star\lambda}$$

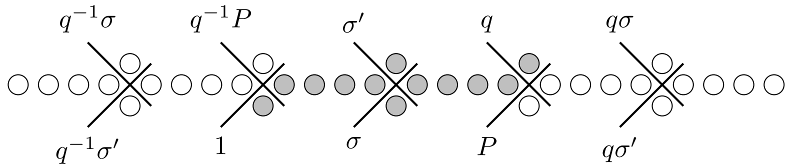
in $\mathrm{QK}(X)_q$, where $\mathcal{O}^{I(\sigma)} = q^d\mathcal{O}^w$ is the Seidel class in $\mathrm{QK}(X)_q$ corresponding to σ. The action of Seidel classes on $\widehat{\mathcal{P}}_X$ therefore determines arbitrary products with Seidel classes in $\mathrm{QH}(X)_q$ and $\mathrm{QK}(X)_q$. For multiplication by powers of q, we use the notation $\lambda[d] = q^d \star \lambda = \{q^d \star \widehat{\alpha} \mid \widehat{\alpha} \in \lambda\}$, so that $\mathcal{O}^{\lambda[d]} = q^d\mathcal{O}^\lambda$. The shifting operations on shapes in $\mathcal{P}_X$ (see [BCMP22, §6.2]) are then given by $\lambda(d) = \lambda[d] \cap \mathcal{P}_X$ (when $\lambda \subset \mathcal{P}_X$ is identified with the quantum shape $\lambda \cup I(1) \subset \widehat{\mathcal{P}}_X$).

The following figures show the partially ordered set $\widehat{\mathcal{P}}_X$ for the quadrics of dimensions 7 and 12, as well as the exceptional cominuscule flag varieties. Each set has the west-to-east order, where any node is covered by the nodes immediately northeast, east, or southeast of it. The elements of $\mathcal{P}_X$ are colored gray. Seidel classes are represented by lines marking the eastern borders of their quantum shapes. We use P to denote the point class, and σ and σ' are used to represent Seidel classes in $H^*(X,\mathbb{Z})$ that are not in the subgroup of $\mathrm{QH}(X)_q^\times$ generated by P and q. Multiplication by any Seidel class corresponds to the rigid transformation of $\widehat{\mathcal{P}}_X$ that moves the border of 1 to the border of the Seidel class. This rigid transformation is a horizontal translation, possibly combined with a reflection in a horizontal line.

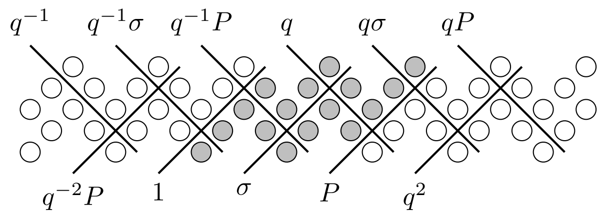
Q^7:



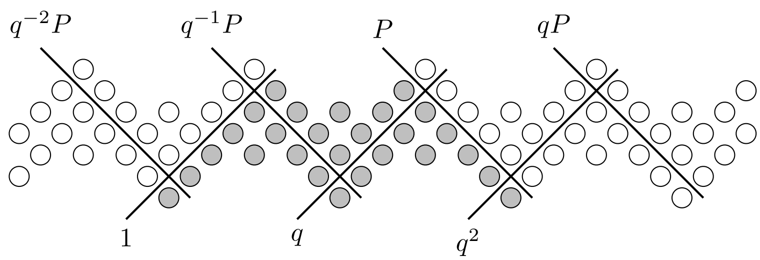
Q^{12}:



E_6/P_6:



E_7/P_7:



The following results will be used to describe the quantum posets $\widehat{\mathcal{P}}_X$ of classical Grassmannians in the next three sections.

Proposition 4.4. *Let $X = G/P_X$ be a cominuscule flag variety.*

(a) *We have $\widehat{\mathcal{P}}_X \cap H^*(X) = \{\tau(\alpha) \mid \alpha \in \mathcal{P}_X \smallsetminus I(z_1^\vee)\} \cup \{1\}$.*

(b) *The map $(\widehat{\mathcal{P}}_X \cap H^*(X)) \times \mathbb{Z} \to \widehat{\mathcal{P}}_X$ defined by $([X^u], d) \mapsto q^d[X^u]$ is bijective.*

(c) *We have $\tau(\mathcal{P}_X) = I([1.P_X]) \smallsetminus I(1) \subset \widehat{\mathcal{P}}_X$.*

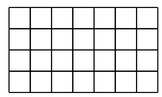
Proof. Parts (a) and (b) follow from [BCMP22, Def. 7.11 and Thm. 7.13], noting that $\tau(\alpha) = [X^{\xi(\alpha)}]$ holds if and only if $\alpha \in \mathcal{P}_X \smallsetminus I(z_1^\vee)$. Let $\alpha \in \mathcal{P}_X$. Then $\tau(\alpha) \leq \tau(\rho) = [1.P_X]$, where $\rho \in \mathcal{P}_X$ is the highest root. Since $[X^{\xi(\alpha)}] = \tau(\alpha) \cup 1$ by [BCMP22, Thm. 7.13(a)], and $[X^{\xi(\alpha)}] \neq 1$, we obtain $\tau(\alpha) \not\leq 1$. This proves that $\tau(\alpha) \in I([1.P_X]) \smallsetminus I(1)$. Given $\widehat{\alpha} \in I([1.P_X]) \smallsetminus I(1)$, we may write $\widehat{\alpha} = q^{-d}[X^{\xi(\alpha')}]$ for some $\alpha' \in \mathcal{P}_X \smallsetminus I(z_1^\vee)$ and $d \in \mathbb{Z}$. The condition $\widehat{\alpha} \leq [1.P_X]$ implies $d \geq 0$, and $\widehat{\alpha} \not\leq 1$ implies that $\alpha' \notin I(z_d)$ by [BCMP22, Lemma 7.12]. It therefore follows from [BCMP22, Prop. 5.9(a) and Cor. 5.11] that $\alpha = (z_1 s_\gamma)^{-d}.\alpha' \in \mathcal{P}_X$, and from [BCMP22, Def. 7.11] that $\tau(\alpha) = \widehat{\alpha}$. This proves part (c). $\square$

Lemma 4.5. *Let α be any non-minimal box in $\mathcal{P}_X \smallsetminus I(z_1^\vee)$, and let $\widehat{\alpha}' \lessdot \tau(\alpha)$ be a covering in $\widehat{\mathcal{P}}_X$. Then $\widehat{\alpha}' = \tau(\alpha')$ for some $\alpha' \in \mathcal{P}_X$, such that $\alpha' \lessdot \alpha$ is a covering in $\mathcal{P}_X$.*

Proof. Since α is not minimal in $\mathcal{P}_X \smallsetminus I(z_1^\vee)$, it follows from Proposition 4.4(a) that $\widehat{\alpha}' \not\leq 1$, hence $\widehat{\alpha}' = \tau(\alpha')$ for some $\alpha' \in \tau(\mathcal{P}_X)$ by Proposition 4.4(c). Proposition 4.4(c) also implies that $\alpha' \lessdot \alpha$ is a covering in $\mathcal{P}_X$, as required. $\square$

5. Pieri formula for Grassmannians of type A

5.1. **Quantum shapes.** Let $X = \mathrm{Gr}(m,n)$ be the Grassmannian of m-dimensional vector subspaces of $\mathbb{C}^n$. The quantum cohomology ring $\mathrm{QH}(X)$ was computed by Witten [Wit95] and Bertram [Ber97], and a Pieri formula for the ordinary K-theory ring $K(X)$ was obtained by Lenart [Len00]. The Grassmannian X is minuscule of type A_{n-1}, and the corresponding partially ordered set $\mathcal{P}_X$ is a rectangle of boxes with m rows and $n-m$ columns, endowed with the northwest-to-southeast order discussed below.

$\mathcal{P}_X =$ 

Each shape $\lambda \subset \mathcal{P}_X$ can be identified with a partition

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0)$$

with $\lambda_1 \leq n-m$, where λ_i is the number of boxes in the i-th row of λ. If $\lambda \subset \mathcal{P}_X$ consists of a single row of boxes, then λ will also be identified with the integer $p = |\lambda|$. The *special Schubert classes* in $K(X)$ are the classes $\mathcal{O}^p$ for $1 \leq p \leq n-m$. Another family of special classes consists of $\mathcal{O}^{(1)^r}$ for $1 \leq r \leq m$, where $(b)^a$ denotes a rectangle with a rows and b columns.

Let $\mathbb{Z}^2$ denote a grid of boxes (i,j) that fill the plane, where the row number i increases from north to south, and the column number j increases from west to east. We endow $\mathbb{Z}^2$ with the *northwest-to-southeast* partial order, defined by $(i',j') \leq (i,j)$ if and only if $i' \leq i$ and $j' \leq j$. The quotient $\mathbb{Z}^2/\mathbb{Z}(m,m-n)$ is ordered by $(i',j') + \mathbb{Z}(m,m-n) \ \leq \ (i,j) + \mathbb{Z}(m,m-n)$ if and only if $(i',j') \leq (i+am, j+am-an)$ for some $a \in \mathbb{Z}$. The cylinder $\mathbb{Z}^2/\mathbb{Z}(m,m-n)$ was used to study the quantum cohomology ring $\mathrm{QH}(X)$ in [Pos05, §3]. This partially ordered set was also defined in [Hag04, §8].

Proposition 5.1. *Let $X = \mathrm{Gr}(m,n)$ and set $\sigma = [X^{n-m}]$ and $\tau = [X^{(1)^m}]$.*

(a) *The group of Seidel classes in $\mathrm{QH}(X)_q^\times$ is generated by σ and τ.*

(b) *We have $\sigma^m = \tau^{n-m} = [1.P_X]$ and $\sigma \star \tau = q$ in $\mathrm{QH}(X)$.*

(c) *The map $\phi : \mathbb{Z}^2/\mathbb{Z}(m,m-n) \to \widehat{\mathcal{P}}_X$ defined by $\phi(i,j) = \sigma^i \star \tau^j \star [1.P_X]^{-1}$ is an order isomorphism, which identifies $\mathcal{P}_X$ with the rectangle $[1,m] \times [1,n-m]$.*

(d) *The actions of σ and τ on $\widehat{\mathcal{P}}_X$ are determined by $\sigma \star \phi(i,j) = \phi(i+1,j)$ and $\tau \star \phi(i,j) = \phi(i,j+1)$.*

Proof. Noting that $\sigma = [X^{w_0^F}]$ and $\tau = [X^{w_0^{F'}}]$, where $F = \mathrm{Gr}(1,n)$ and $F' = \mathrm{Gr}(n-1,n)$, it follows that σ and τ are Seidel classes in $\mathrm{QH}(X)$. Part (b) follows from Bertram's quantum Pieri formula [Ber97], and is also an easy consequence of Corollary 3.7. These results also show that

$$\sigma^i = [X^{(n-m)^i}] \quad \text{and} \quad \tau^j = [X^{(j)^m}]$$

for $1 \leq i \leq m$ and $1 \leq j \leq n-m$. Part (a) follows from this, noting that σ and τ generate n distinct Seidel classes in $H^*(X)$.

The map ϕ is well defined by part (b), and order-preserving since, if $(i',j') \leq (i,j)$, then $\phi(i,j)$ occurs in the expansion of the product $\phi(i',j') \star (\sigma^{i-i'} \star \tau^{j-j'})$. The maximal box of $(n-m)^i$ is the i-th box of the rightmost column of $\mathcal{P}_X$, and the maximal box of $(j)^m$ is the j-th box of the bottom row of $\mathcal{P}_X$. Since these maximal boxes include all boxes of $\mathcal{P}_X \smallsetminus I(z_1^\vee)$, it follows from Proposition 4.4 that ϕ is surjective. If $\sigma^i \star \tau^j = 1$ in $\mathrm{QH}(X)_q$, then since σ has order n and inverse τ in $\mathrm{QH}(X)/(q-1)$, we must have $j = i - an$ for some $a \in \mathbb{Z}$. Since $\sigma^i \star \tau^j = 1$ has degree $(n-m)i + mj = 0$ in $\mathrm{QH}(X)_q$, we obtain $(i,j) = a(m, m-n)$. This implies that ϕ is bijective. To show that ϕ is an order isomorphism, we must show that, if $\widehat{\alpha}' \lessdot \widehat{\alpha}$ is a covering in $\widehat{\mathcal{P}}_X$, then $\phi^{-1}(\widehat{\alpha}') < \phi^{-1}(\widehat{\alpha})$. If $X = \mathbb{P}^1$, then this follows because $\mathbb{Z}^2/\mathbb{Z}(1,-1)$ is totally ordered, so assume that $n \geq 3$. Using that ϕ is surjective and quantum multiplication by the Seidel classes σ and τ define order automorphisms of $\widehat{\mathcal{P}}_X$, we may assume that $\widehat{\alpha} = [1.P_X] = \phi(m, n-m)$ is the maximal box of $\mathcal{P}_X \subset \widehat{\mathcal{P}}_X$. We then deduce from Lemma 4.5 that $\widehat{\alpha}' = \phi(m-1, n-m)$ or $\widehat{\alpha}' = \phi(m, n-m-1)$, and in either case we obtain $\phi^{-1}(\widehat{\alpha}') < \phi^{-1}(\widehat{\alpha})$. Noting that $\phi(m, n-m) = [1.P_X]$ and $\phi(m,0) = \phi(0, n-m) = 1$, it follows from Proposition 4.4(c) that $\mathcal{P}_X$ is identified with the rectangle $[1,m] \times [1, n-m]$. This proves part (c). Part (d) follows from the definition of ϕ, which completes the proof. ☐

Example 5.2. Let $X = \mathrm{Gr}(2,5)$ and set $\sigma = [X^3]$, $\tau = [X^{(1,1)}]$, and $P = [1.P_X]$. The following figure shows the rectangle $[0,3] \times [0,4] \subset \mathbb{Z}^2$, with each box (i,j) labeled by $\phi(i,j)$. The framed 2×3 rectangle can be identified with $\mathcal{P}_X$.

P^{-1}	τ^{-2}	τ^{-1}	1	τ
σ^{-1}	$\sigma^{-1}\tau$	$\tau^{-1}\sigma$	σ	q
1	τ	τ^2	P	$q\sigma$
σ	q	$q\tau$	$q\tau^2$	qP

Remark 5.3. Let $X = \mathrm{Gr}(m,n)$. The map from Proposition 5.1(c) defines an order-preserving bijection $\phi : [1,m] \times \mathbb{Z} \to \widehat{\mathcal{P}}_X$, which is an order isomorphism if and only if $m = 1$. In particular, $\widehat{\mathcal{P}}_X$ does not have 'cylinder' behavior when $X = \mathbb{P}^{n-1}$ is projective space. A non-empty proper lower order ideal $\lambda \subset [1,m] \times \mathbb{Z}$ can be represented by the decreasing sequence $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$, where $\lambda_i \in \mathbb{Z}$ is maximal such that $(i, \lambda_i) \in \lambda$. The image $\phi(\lambda)$ is a shape in $\widehat{\mathcal{P}}_X$ if and only if $\lambda_1 - \lambda_m \leq n-m$, and any shape in $\widehat{\mathcal{P}}_X$ has this form. In this case the corresponding basis element $q^d[X^\mu]$ is obtained by removing rim-hooks from λ, see [BCFF99].

5.2. **Pieri formula.** Let $\theta \subset \widehat{\mathcal{P}}_X$ be a skew shape. A *row* of θ means a subset of the form $\theta \cap \phi(\{k\} \times \mathbb{Z})$, where $k \in \mathbb{Z}$ and ϕ is the map defined in Proposition 5.1, and a *column* of θ is a subset of the form $\theta \cap \phi(\mathbb{Z} \times \{k\})$. The skew shape $\theta \subset \widehat{\mathcal{P}}_X$ is called a *horizontal strip* if each column of θ contains at most one box. Let $r(\theta)$ denote the number of non-empty rows in θ. For $p \geq 1$ we define

$$\mathcal{A}(\theta, p) = \begin{cases} (-1)^{|\theta|-p}\binom{r(\theta)-1}{|\theta|-p} & \text{if } \theta \text{ is a horizontal strip,} \\ 0 & \text{otherwise.} \end{cases}$$

A Pieri formula for products of the form $\mathcal{O}^p \star \mathcal{O}^\lambda$ in $\mathrm{QK}(X)$ was proved in [BM11]. We proceed to show that this formula is an easy consequence of Corollary 3.7, Lenart's Pieri formula for $K(X)$ [Len00], and a bound on the q-degrees in quantum K-theory products proved in [BCMP22].

Theorem 5.4. *Let $X = \mathrm{Gr}(m,n)$, let $\lambda \subset \widehat{\mathcal{P}}_X$ be any quantum shape, and let $1 \leq p \leq n-m$. Then*

$$\mathcal{O}^p \star \mathcal{O}^\lambda = \sum_\nu \mathcal{A}(\nu/\lambda, p)\, \mathcal{O}^\nu$$

holds in $\mathrm{QK}(X)_q$, where the sum is over all quantum shapes $\nu \subset \widehat{\mathcal{P}}_X$ containing λ.

Proof. Set $\tau = \mathcal{O}^{(1)^m}$ and choose $k \in \mathbb{Z}$ maximal such that $\phi(m,k) \in \lambda$. By Corollary 3.7 and Proposition 5.1 we have $\tau^{-k} \star \mathcal{O}^\lambda = \mathcal{O}^\mu$, where $\mu \subset \mathcal{P}_X$ is a classical shape with $\mu_m = 0$. Corollary 4.3 then implies that $d_{\max}(p,\mu) = 0$, so [BCMP22, Cor. 8.3] shows that $\mathcal{O}^p \star \mathcal{O}^\mu$ agrees with the classical product $\mathcal{O}^p \cdot \mathcal{O}^\mu$ in $K(X)$. Notice that, if $\nu \supset \mu$ is any quantum shape such that ν/μ is a horizontal strip, then ν is a classical shape. It therefore follows from [Len00, Thm. 3.2] that

$$\mathcal{O}^p \star \mathcal{O}^\mu = \sum_\nu \mathcal{A}(\nu/\mu, p)\, \mathcal{O}^\nu$$

holds in $\mathrm{QK}(X)$, where the sum is over all shapes $\nu \subset \widehat{\mathcal{P}}_X$ containing μ. Since quantum multiplication by τ^k defines a module automorphism of $\mathrm{QK}(X)$ and defines an order automorphism of $\widehat{\mathcal{P}}_X$, this identity is equivalent to the theorem. $\square$

The following version of Theorem 5.4 is equivalent to the Pieri formula for $\mathrm{QK}(X)$ proved in [BM11].

Corollary 5.5. *Let $\lambda \subset \mathcal{P}_X$ be any shape and let $1 \leq p \leq n-m$. Then*

$$\mathcal{O}^p \star \mathcal{O}^\lambda = \sum_\mu \mathcal{A}(\mu/\lambda, p)\, \mathcal{O}^\mu + q \sum_\nu \mathcal{A}(\nu[1]/\lambda, p)\, \mathcal{O}^\nu$$

holds in $\mathrm{QK}(X)$, where the first sum is over all shapes $\mu \subset \mathcal{P}_X$ containing λ, and the second sum is over all shapes $\nu \subset \mathcal{P}_X$ for which $\nu[1]$ contains λ.

Proof. This is a direct translation of Theorem 5.4, using that $\mathcal{O}^{\nu[1]} = q\, \mathcal{O}^\nu$. $\square$

Example 5.6. Let $X = \mathrm{Gr}(3,7)$. By Remark 5.3 we can represent a shape $\lambda \subset \widehat{\mathcal{P}}_X$ by a non-empty proper lower order ideal $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3)$ in $[1,3] \times \mathbb{Z}$, such that $\lambda_1 - \lambda_3 \leq 4$. When $\lambda_3 \geq 0$, this order ideal will be displayed as a Young diagram with at most 3 rows. We will also identify the shape λ with the class $\mathcal{O}^\lambda$ in $\mathrm{QK}(X)$. With this notation we have

$$\mathcal{O}^3 \star \begin{ytableau} \ & \ & \ \\ \ & \ & \ \\ \ \end{ytableau} = \begin{ytableau} \ & \ & \ & + \\ \ & \ & \ \\ \ & + & + \end{ytableau} + \begin{ytableau} \ & \ & \ & + & + \\ \ & \ & \ \\ \ & + \end{ytableau} - \begin{ytableau} \ & \ & \ & + & + \\ \ & \ & \ \\ \ & + & + \end{ytableau},$$

where added boxes are indicated by pluses. This is equivalent to

$$\mathcal{O}^3 \star \begin{ytableau} \ & \ & \ \\ \ & \ & \ \\ \ \end{ytableau} = \begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{ytableau} + q \begin{ytableau} \ & \ \\ \ \end{ytableau} - q \begin{ytableau} \ & \ \\ \ & \ \end{ytableau}.$$

Notice that the shape

$$\begin{ytableau} \ & \ & \ & + & + & + \\ \ & \ & \ \\ \ & + \end{ytableau}$$

is not included, as the box added to the third row is in the same column of $\widehat{\mathcal{P}}_X$ as the rightmost box added to the first row.

6. Pieri formula for maximal orthogonal Grassmannians

6.1. **Quantum shapes.** Let $X = \mathrm{OG}(n, 2n)$ be the maximal orthogonal Grassmannian, parametrizing one component of the maximal isotropic subspaces of $\mathbb{C}^{2n}$ endowed with an orthogonal bilinear form. The quantum cohomology ring $\mathrm{QH}(X)$ was computed in [KT04], and a Pieri formula for the ordinary K-theory ring $K(X)$ was obtained in [BR12].

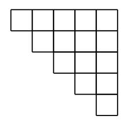
The orthogonal Grassmannian X is minuscule of type D_n. We identify the simple roots of type D_n with the vectors

$$\Delta = \{e_n - e_{n-1}, \dots, e_3 - e_2, e_2 - e_1, e_2 + e_1\},$$

where $\gamma = e_1 + e_2$ is the cominuscule simple root defining X. We then obtain

$$\mathcal{P}_X = \{e_i + e_j \mid 1 \le i < j \le n\},$$

where the partial order is given by $e_{i'} + e_{j'} \le e_i + e_j$ if and only if $i' \le i$ and $j' \le j$. We represent $\mathcal{P}_X$ as a staircase shape with $n-1$ rows, where $e_i + e_j$ is represented by the box in row i and column j:

$\mathcal{P}_{\mathrm{OG}(6,12)} =$ 

Each shape $\lambda \subset \mathcal{P}_X$ can be identified with a strict partition

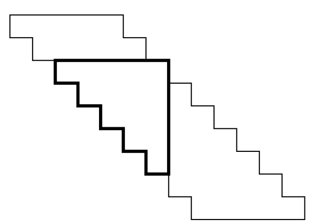
$$\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$$

with $\lambda_1 \le n-1$, where λ_i is the number of boxes in the i-th row of λ. If $\lambda \subset \mathcal{P}_X$ consists of a single row of boxes, then λ will also be identified with the integer $p = |\lambda|$. The special Schubert classes in $K(X)$ are the classes $\mathcal{O}^p$ for $1 \le p \le n-1$.

Define the set

$$\overline{\mathcal{P}}_X = \{(i,j) \in \mathbb{Z}^2 \mid i < j < i + n\},$$

and give $\overline{\mathcal{P}}_X$ the northwest-to-southeast order $(i', j') \le (i, j)$ if and only if $i' \le i$ and $j' \le j$. We represent $\overline{\mathcal{P}}_X$ as an infinite set of boxes (i, j) in the plane, where the row number i increases from north to south, and the column number j increases from west to east. Each row in $\overline{\mathcal{P}}_X$ contains $n-1$ boxes. The set $\mathcal{P}_X$ will be identified with the subset $\{(i,j) \in \mathbb{Z}^2 \mid 1 \le i < j \le n\} \subset \overline{\mathcal{P}}_X$.

$\overline{\mathcal{P}}_{\mathrm{OG}(6,12)} =$ 

Recall the map $\tau : \mathcal{P}_X \to \widehat{\mathcal{P}}_X$ from Section 4.

Proposition 6.1. *Let $X = \mathrm{OG}(n, 2n)$.*

(a) *The group of Seidel classes in $\mathrm{QH}(X)_q^\times$ is generated by $[1.P_X]$ and $[X^{n-1}]$.*

(b) *We have $[X^{n-1}]^2 = q$ and $[1.P_X]^2 = [X^{n-1}]^n$ in $\mathrm{QH}(X)$.*

(c) *The map $\phi : \overline{\mathcal{P}}_X \to \widehat{\mathcal{P}}_X$ defined by $\phi(i,j) = [X^{n-1}]^{j-n} \star \tau(e_{i+n-j} + e_n)$ is an order isomorphism which identifies $\mathcal{P}_X$ with the set $\{(i,j) \in \mathbb{Z}^2 \mid 1 \le i < j \le n\}$.*

(d) *The action of Seidel classes on* $\widehat{\mathcal{P}}_X$ *is given by* $[X^{n-1}] \star \phi(i,j) = \phi(i+1,j+1)$ *and* $[1.P_X] \star \phi(i,j) = \phi(j,i+n)$.

Proof. Let $F = Q^{2n-2}$ be the quadric of type D_n. Then we have the relation $w_0^F = s_1 \cdots s_{n-2} s_{n-1} s_n s_{n-2} \cdots s_1$, hence $(w_0^F)^X = s_1 \cdots s_{n-2} s_n$. This shows that $[X^{n-1}] = [X^{w_0^F}]$. Since $(w_0^F)^2 = 1$ holds in W, it follows from Corollary 3.7 that $[X^{n-1}]^2$ is a power of q. Using that $\deg(q) = 2n-2$, we obtain $[X^{n-1}]^2 = q$. Since W^{comin} has order 4, we have $(w_0^X)^2 \in \{1, w_0^F\}$, so Corollary 3.7 implies that either $[1.P_X]^2$ or $[X^{n-1}] \star [1.P_X]^2$ is a power of q. In either case, $[1.P_X]^2$ is a power of $[X^{n-1}]$, and since $\dim(X) = \binom{n}{2}$, we must have $[1.P_X]^2 = [X^{n-1}]^n$. Parts (a) and (b) follow from these observations.

For convenience we set $\alpha_i = e_i + e_n$ for $1 \leq i \leq n-1$ and $\alpha_i' = e_i + e_{n-1}$ for $1 \leq i \leq n-2$, so that $\mathcal{P}_X \smallsetminus I(z_1^\vee) = \{\alpha_1', \ldots, \alpha_{n-2}', \alpha_1, \ldots, \alpha_{n-1}\}$. Then $I(\tau(\alpha_i)) \cap \mathcal{P}_X$ consists of the top i rows of $\mathcal{P}_X$, and $I(\tau(\alpha_i')) \cap \mathcal{P}_X$ is obtained by removing the rightmost column in this shape. Notice also that $\tau(\alpha_1) = [X^{n-1}]$, $\tau(\alpha_{n-1}) = [1.P_X]$, and $\phi(i,j) = [X^{n-1}]^{j-n} \star \tau(\alpha_{i+n-j})$. It follows from [KT04] or Corollary 3.7 that $[X^{n-1}] \star \tau(\alpha_i') = \tau(\alpha_{i+1})$ holds in $\mathrm{QH}(X)$ for $1 \leq i \leq n-2$. Proposition 4.4 therefore implies that

$$\begin{aligned} \widehat{\mathcal{P}}_X \cap H^*(X) &= \{1, \tau(\alpha_1'), \ldots, \tau(\alpha_{n-2}'), \tau(\alpha_1), \ldots, \tau(\alpha_{n-1})\} \\ &= \{[X^{n-1}]^\epsilon \star \tau(\alpha_i) \mid 1 \leq i \leq n-1 \text{ and } \epsilon \in \{0,-1\}\} \end{aligned}$$

and that ϕ is bijective. Since $\alpha_i < \alpha_{i+1}$ holds in $\mathcal{P}_X$ and $[X^{n-1}]$ is a Seidel class, we obtain $\phi(i,j) < \phi(i+1,j)$ for $i+1 < j < i+n$. For $i < j < i+n-1$ we have

$$\begin{aligned} \phi(i,j) &= [X^{n-1}]^{j-n} \star \tau(\alpha_{i+n-j}) = [X^{n-1}]^{j+1-n} \star \tau(\alpha_{i+n-j-1}') \\ &< [X^{n-1}]^{j+1-n} \star \tau(\alpha_{i+n-j-1}) = \phi(i,j+1)\,. \end{aligned}$$

This implies that ϕ is order-preserving. Assume that $\widehat{\alpha}' \lessdot \widehat{\alpha}$ is a covering in $\widehat{\mathcal{P}}_X$. We must show that $\phi^{-1}(\widehat{\alpha}') < \phi^{-1}(\widehat{\alpha})$. Since ϕ is surjective and quantum multiplication by $[X^{n-1}]$ is an order automorphism of $\widehat{\mathcal{P}}_X$, we may assume that $\widehat{\alpha} = \tau(\alpha_i)$ for some i. Lemma 4.5 then shows that $\widehat{\alpha}' = \tau(\alpha')$ for some $\alpha' \in \mathcal{P}_X$. We deduce that $\widehat{\alpha}' = \tau(\alpha_i')$ or $\widehat{\alpha}' = \tau(\alpha_{i-1})$. In either case we obtain $\phi^{-1}(\widehat{\alpha}') < \phi^{-1}(\widehat{\alpha})$. This proves that ϕ is an order isomorphism. Finally, using that $\phi(0,n-1) = 1$ and $\phi(n-1,n) = [1.P_X]$, the last claim in part (c) follows from Proposition 4.4(c).

The identity $[X^{n-1}] \star \phi(i,j) = \phi(i+1,j+1)$ follows from the definition of ϕ. Quantum multiplication by $[1.P_X]$ corresponds to an order automorphism of $\overline{\mathcal{P}}_X$ that commutes with multiplication by $[X^{n-1}]$, and any such order automorphism of $\overline{\mathcal{P}}_X$ is a translation along a northwest-to-southeast line, possibly combined with a reflection in such a line. Using that $[1.P_X] \star \phi(0,n-1) = \phi(n-1,n)$, we deduce that multiplication by $[1.P_X]$ corresponds to the automorphism $(i,j) \mapsto (j,i+n)$ of $\overline{\mathcal{P}}_X$, which proves part (d). ☐

We may identify $\widehat{\mathcal{P}}_X$ with the set of boxes $\overline{\mathcal{P}}_X$. Given a shape $\lambda \subset \widehat{\mathcal{P}}_X$ and $d \in \mathbb{Z}$, it follows from Proposition 6.1 that the shifted shape $\lambda[d] = q^d \star \lambda$ is obtained by moving λ by $2d$ diagonal steps in southeast direction.

Remark 6.2. It is natural to extend the notation $\lambda[d]$ to half-integer shifts by setting $\lambda[k/2] = [X^{n-1}]^k \star \lambda$. We then have $(\mathcal{O}^{n-1})^k \star \mathcal{O}^\lambda = \mathcal{O}^{\lambda[k/2]}$ in $\mathrm{QK}(X)_q$.

6.2. **Pieri formula.** The Pieri formula for the K-theory ring $K(X)$ proved in [BR12] expresses the structure constants of Pieri products as signed counts of KOG-tableaux, defined as follows.

Definition 6.3 (KOG-tableau, [BR12])**.** Given a skew shape $\theta \subset \mathcal{P}_X$, a *KOG-tableau* of shape θ is a labeling of the boxes of θ with positive integers, such that (i) each row of θ is strictly increasing from left to right; (ii) each column of θ is strictly increasing from top to bottom; and (iii) each box of θ is either smaller than or equal to all boxes south-west of it, or it is greater than or equal to all boxes south-west of it. The *content* of a KOG-tableau is the set of integers contained in its boxes. Let $\mathcal{B}(\theta, p)$ denote $(-1)^{|\theta|-p}$ times the number of KOG-tableaux of shape θ with content $\{1, 2, \ldots, p\}$.

The skew shape θ is called a *rim* if no box in θ is strictly south and strictly east of another box in θ. If θ is not a rim, then there are no KOG-tableau of shape θ, hence $\mathcal{B}(\theta, p) = 0$ for all p.

Theorem 6.4. *Let $X = \mathrm{OG}(n, 2n)$, let $\lambda \subset \widehat{\mathcal{P}}_X$ be any quantum shape, and let $1 \leq p \leq n-1$. Then*

$$\mathcal{O}^p \star \mathcal{O}^\lambda = \sum_\nu \mathcal{B}(\nu/\lambda, p)\, \mathcal{O}^\nu$$

holds in $\mathrm{QK}(X)_q$, where the sum is over all quantum shapes $\nu \subset \widehat{\mathcal{P}}_X$ containing λ.

Proof. Choose k maximal such that $\phi(k, k+n-1) \in \lambda$. By Corollary 3.7 and Proposition 6.1 we have $(\mathcal{O}^{n-1})^{-k} \star \mathcal{O}^\lambda = \mathcal{O}^\mu$, where $\mu \subset \mathcal{P}_X$ is a classical shape with $\mu_1 \leq n-2$. Corollary 4.3 then implies that $d_{\max}(p, \mu) = 0$, so [BCMP22, Cor. 8.3] shows that $\mathcal{O}^p \star \mathcal{O}^\mu$ agrees with the classical product $\mathcal{O}^p \cdot \mathcal{O}^\mu$ in $K(X)$. Notice that, if $\nu \supset \mu$ is any quantum shape such that ν/μ is a rim, then ν is a classical shape. It therefore follows from [BR12, Cor. 4.8] that

$$\mathcal{O}^p \star \mathcal{O}^\mu = \sum_\nu \mathcal{B}(\nu/\mu, p)\, \mathcal{O}^\nu$$

holds in $\mathrm{QK}(X)$, where the sum is over all shapes $\nu \subset \widehat{\mathcal{P}}_X$ containing μ. Since quantum multiplication by $(\mathcal{O}^{n-1})^k$ defines a module automorphism of $\mathrm{QK}(X)$ and defines an order automorphism of $\widehat{\mathcal{P}}_X$, this identity is equivalent to the theorem. $\square$

Corollary 6.5. *Let $\lambda \subset \mathcal{P}_X$ be any shape and let $1 \leq p \leq n-1$. Then*

$$\mathcal{O}^p \star \mathcal{O}^\lambda \;=\; \sum_\mu \mathcal{B}(\mu/\lambda)\, \mathcal{O}^\mu + q \sum_\nu \mathcal{B}(\nu[1]/\lambda)\, \mathcal{O}^\nu$$

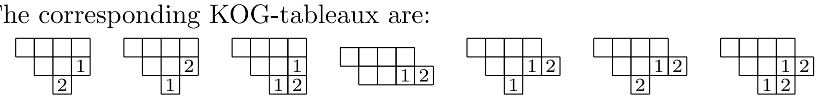
holds in $\mathrm{QK}(X)$, where the first sum is over all shapes $\mu \subset \mathcal{P}_X$ containing λ, and the second sum is over all shapes $\nu \subset \mathcal{P}_X$ for which $\nu[1]$ contains λ.

Proof. This is a direct translation of Theorem 6.4, using that $\mathcal{O}^{\nu[1]} = q\, \mathcal{O}^\nu$. $\square$

Example 6.6. Let $X = \mathrm{OG}(5, 10)$. Then the following holds in $\mathrm{QK}(X)$.

$$\mathcal{O}^2 \star \mathcal{O}^{(4,2)} \;=\; 2\, \mathcal{O}^{(4,3,1)} - \mathcal{O}^{(4,3,2)} + q - 2\, q\, \mathcal{O}^1 + q\, \mathcal{O}^2\,.$$

The corresponding KOG-tableaux are:



7. Pieri formula for Lagrangian Grassmannians

7.1. **Quantum shapes.** Let $X = \mathrm{LG}(n, 2n)$ be the Lagrangian Grassmannian of maximal isotropic subspaces of $\mathbb{C}^{2n}$ endowed with a symplectic bilinear form. The quantum cohomology ring $\mathrm{QH}(X)$ was computed in [KT03], and a Pieri formula for the ordinary K-theory ring $K(X)$ was obtained in [BR12].

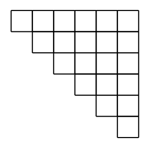
The Lagrangian Grassmannian X is cominuscule, but not minuscule, of type C_n. We identify the simple roots of type C_n with the vectors

$$\Delta = \{e_n - e_{n-1}, \dots, e_3 - e_2, e_2 - e_1, 2e_1\},$$

where $\gamma = 2e_1$ is the cominuscule simple root defining X. We then obtain

$$\mathcal{P}_X = \{e_i + e_j \mid 1 \le i \le j \le n\},$$

where the partial order is given by $e_{i'} + e_{j'} \le e_i + e_j$ if and only if $i' \le i$ and $j' \le j$. We represent $\mathcal{P}_X$ as a staircase shape with n rows, where $e_i + e_j$ corresponds to the box in row i and column j:

$\mathcal{P}_{\mathrm{LG}(6,12)} =$ 

Each shape $\lambda \subset \mathcal{P}_X$ can be identified with a strict partition

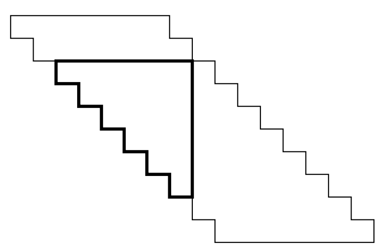
$$\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$$

with $\lambda_1 \le n$, where λ_i is the number of boxes in the i-th row of λ. If $\lambda \subset \mathcal{P}_X$ consists of a single row of boxes, then λ will also be identified with the integer $p = |\lambda|$. The special Schubert classes in $K(X)$ are the classes $\mathcal{O}^p$ for $1 \le p \le n$.

Define the set

$$\overline{\mathcal{P}}_X = \{(i,j) \in \mathbb{Z}^2 \mid i \le j \le i + n\},$$

and give $\overline{\mathcal{P}}_X$ the northwest-to-southeast order $(i', j') \le (i, j)$ if and only if $i' \le i$ and $j' \le j$. We represent $\overline{\mathcal{P}}_X$ as an infinite set of boxes (i,j) in the plane, where the row number i increases from north to south, and the column number j increases from east to west. Each row in $\overline{\mathcal{P}}_X$ contains $n+1$ boxes. The set $\mathcal{P}_X$ will be identified with the subset $\{(i,j) \in \mathbb{Z}^2 \mid 1 \le i \le j \le n\} \subset \overline{\mathcal{P}}_X$.

$\overline{\mathcal{P}}_{\mathrm{LG}(6,12)} =$ 

Recall that $z_d \in W^X$ is defined by $X_{z_d} = \Gamma_d(1.P_X)$ for $d \ge 0$.

Proposition 7.1. *Let $X = \mathrm{LG}(n, 2n)$.*

(a) *The group of Seidel classes in $\mathrm{QH}(X)_q^\times$ is generated by $[1.P_X]$ and q.*

(b) *We have $[1.P_X]^2 = q^n$ in $\mathrm{QH}(X)$.*

(c) *The map $\phi : \overline{\mathcal{P}}_X \to \widehat{\mathcal{P}}_X$ defined by $\phi(i,j) = q^{j-n}\,[X^{z_{i+n-j}}]$ is an order isomorphism which identifies $\mathcal{P}_X$ with the set $\{(i,j) \in \mathbb{Z}^2 \mid 1 \le i \le j \le n\}$.*

(d) *The action of Seidel classes on* $\widehat{\mathcal{P}}_X$ *is determined by* $q \star \phi(i,j) = \phi(i+1, j+1)$ *and* $[1.P_X] \star \phi(i,j) = \phi(j, i+n)$.

Proof. Since the root system of type C_n has only one cominuscule root, we have $W^{\text{comin}} = \{1, w_0^X\}$. It follows that $[1.P_X]^2$ is a power of q in $\mathrm{QH}(X)$, and since $\dim(X) = \binom{n+1}{2}$ and $\deg(q) = n+1$, we must have $[1.P_X]^2 = q^n$. Parts (a) and (b) follow from this.

We have $\mathcal{P}_X \smallsetminus I(z_1^\vee) = \{e_1 + e_n, \dots, e_{n-1} + e_n, 2e_n\}$. Since $e_i + e_n$ is the unique maximal box of $I(z_i)$, it follows from Proposition 4.4 that the map ϕ is bijective. Notice that for $a, b \in [0, n]$ and $d \in \mathbb{Z}$, $[X^{z_a}] \leq q^d [X^{z_b}]$ holds in $\widehat{\mathcal{P}}_X$ if and only if $d \geq 0$ and $\Gamma_d(X_{z_b}) \cap X^{z_a} \neq \emptyset$, which is equivalent to $d \geq 0$ and $a \leq b + d$, see [BCMP22, Lemma 7.9]. It follows that $\phi(i', j') \leq \phi(i,j)$ holds in $\widehat{\mathcal{P}}_X$ if and only if $(i', j') \leq (i, j)$ holds in $\overline{\mathcal{P}}_X$. This shows that ϕ is an order isomorphism. The last claim in part (c) follows from Proposition 4.4(c), noting that $\phi(0, n) = 1$ and $\phi(n,n) = [1.P_X]$.

The identity $q \star \phi(i,j) = \phi(i+1, j+1)$ follows from the definition of ϕ. Quantum multiplication by $[1.P_X]$ corresponds to an order automorphism of $\overline{\mathcal{P}}_X$ that commutes with multiplication by q, and any such order automorphism of $\overline{\mathcal{P}}_X$ is a translation along a northwest-to-southeast line, possibly combined with a reflection in such a line. Using that $[1.P_X] \star \phi(0, n) = \phi(n, n)$, we deduce the formula $[1.P_X] \star \phi(i,j) = \phi(j, i+n)$, proving part (d). $\square$

We may identify $\widehat{\mathcal{P}}_X$ with the set of boxes $\overline{\mathcal{P}}_X$. Given a shape $\lambda \subset \widehat{\mathcal{P}}_X$ and $d \in \mathbb{Z}$, it follows from Proposition 7.1 that the shifted shape $\lambda[d] = q^d \star \lambda$ is obtained by moving λ by d diagonal steps in southeast direction.

7.2. **Pieri formula.** The Pieri formula for the K-theory ring $K(X)$ proved in [BR12] expresses the structure constants of Pieri products as signed counts of KLG-tableaux, defined as follows.

Definition 7.2 (KLG-tableau, [BR12])**.** Let $\theta \subset \mathcal{P}_X$ be a rim. A *KLG-tableau* of shape θ is a labeling of the boxes of θ with elements from the ordered set

$$\{1' < 1 < 2' < 2 < \cdots\}$$

such that (i) each row of θ is strictly increasing from left to right; (ii) each column of θ is strictly increasing from top to bottom; (iii) each box containing an unprimed integer is larger than or equal to all boxes southwest of it; (iv) each box containing a primed integer is smaller than or equal to all boxes southwest of it; (v) no SW diagonal box contains a primed integer. The *content* of a KLG-tableau is the set of integers i such that some box contains i or i'. Define $\mathcal{C}(\theta, p)$ to be $(-1)^{|\theta| - p}$ times the number of KLG-tableaux of shape θ with content $\{1, 2, \dots, p\}$. If $\theta \subset \mathcal{P}_X$ is a skew shape that is not a rim, then set $\mathcal{C}(\theta, p) = 0$.

In contrast to the case of maximal orthogonal Grassmannians, we need to adjust the definition of KLG-tableau with extra conditions in the quantum case.

Definition 7.3 (QKLG-tableau)**.** Let T be a KLG-tableau whose shape is a rim contained in $\widehat{\mathcal{P}}_X$. A box of T is called *unrepeated* if its label is distinct from all other labels when ignoring primes. A box of T is a *quantum box* if it belongs to the NE diagonal of $\widehat{\mathcal{P}}_X$ or is connected to an unrepeated quantum box. A box of T is *terminal* if it is not on the SW diagonal of $\widehat{\mathcal{P}}_X$ and is not connected to a box to the left or below it. We say that T is a *QKLG-tableau* if (vi) every primed non-terminal quantum box is unrepeated, and (vii) every terminal quantum box is primed. For any rim θ contained in $\widehat{\mathcal{P}}_X$, we let $\mathcal{N}(\theta, p)$ denote $(-1)^{|\theta|-p}$ times the number of QKLG-tableaux of shape θ with content $\{1, 2, \dots, p\}$. If $\theta \subset \widehat{\mathcal{P}}_X$ is a skew shape that is not a rim, then set $\mathcal{N}(\theta, p) = 0$.

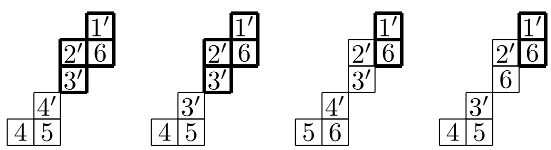
The integers $\mathcal{N}(\theta, p)$ can also be defined recursively, see Definition 10.5.

Theorem 7.4. *Let $X = \mathrm{LG}(n, 2n)$, let $\lambda \subset \widehat{\mathcal{P}}_X$ be any quantum shape, and let $1 \le p \le n$. Then*
$$\mathcal{O}^p \star \mathcal{O}^\lambda = \sum_\nu \mathcal{N}(\nu/\lambda, p)\, \mathcal{O}^\nu$$
holds in $\mathrm{QK}(X)_q$, where the sum is over all quantum shapes $\nu \subset \widehat{\mathcal{P}}_X$ containing λ.

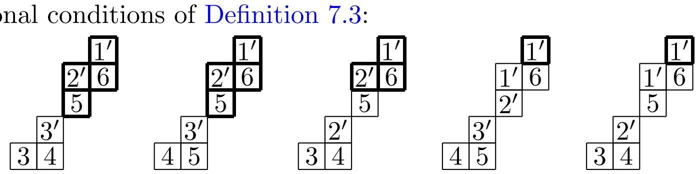
The proof of Theorem 7.4 is given in the three remaining sections of this paper.

Corollary 7.5. *Let $\lambda \subset \mathcal{P}_X$ be any shape and let $1 \le p \le n$. Then*
$$\mathcal{O}^p \star \mathcal{O}^\lambda = \sum_\mu \mathcal{C}(\mu/\lambda, p)\, \mathcal{O}^\mu + q \sum_\nu \mathcal{N}(\nu[1]/\lambda, p)\, \mathcal{O}^\nu$$
holds in $\mathrm{QK}(X)$, where the first sum is over all shapes $\mu \subset \mathcal{P}_X$ containing λ, and the second sum is over all shapes $\nu \subset \mathcal{P}_X$ for which $\nu[1]$ contains λ.

Example 7.6. Let $X = \mathrm{LG}(7, 14)$ and set $\lambda = (7, 5, 4, 2)$ and $\nu = (7, 5, 3, 2)$. Then $\nu[1]/\lambda$ meets both the SW diagonal and the NE diagonal of $\widehat{\mathcal{P}}_X$. The coefficient of $q\,\mathcal{O}^\nu$ in $\mathcal{O}^6 \star \mathcal{O}^\lambda$ is -4, due to the following list of QKLG-tableaux of shape $\nu[1]/\lambda$ with content $\{1, 2, 3, 4, 5, 6\}$:



Quantum boxes are indicated with a think border. There are five additional KLG-tableaux of shape $\nu[1]/\lambda$ with content $\{1, 2, 3, 4, 5, 6\}$ which do not satisfy the additional conditions of Definition 7.3:



The first two violate condition (vii) and the last three violate condition (vi).

8. Perpendicular incidences of symplectic Richardson varieties

Let Y_P^Q be a Richardson variety in the symplectic Grassmannian $Y = \mathrm{SG}(m, 2n)$. Each point $L \in \mathbb{P}^{2n-1}$ defines the subvariety $Y_P^Q \cap L^\perp = \{V \in Y_P^Q \mid V \subset L^\perp\}$. Let $\mathbb{P}_P^Q \subset \mathbb{P}^{2n-1}$ be the subset of points L for which $Y_P^Q \cap L^\perp$ is not empty. In this section we show that $\mathbb{P}_P^Q$ is a complete intersection defined by explicitly given equations. We also show that $Y_P^Q \cap L^\perp$ is rational for all points L in a dense open subset of $\mathbb{P}_P^Q$. This will be used in Section 9 to compute the Gromov-Witten invariants required to prove our Pieri formula for the quantum K-theory of Lagrangian Grassmannians.

8.1. **Symplectic Grassmannians.** Let $\{e_1, \dots, e_{2n}\}$ denote the standard basis of $\mathbb{C}^{2n}$. Define the symplectic vector space $E = \mathbb{C}^{2n}$, where the skew-symmetric bilinear form on E is given by $(e_i, e_j) = \delta_{i+j,2n+1}$ for $1 \leq i \leq j \leq 2n$. Given $0 \leq m \leq n$, let $Y = \mathrm{SG}(m, E) = \mathrm{SG}(m, 2n)$ denote the symplectic Grassmannian of m-dimensional isotropic subspaces of E,

$$Y = \mathrm{SG}(m, E) = \{V \subset E \mid \dim(V) = m \text{ and } (V, V) = 0\}\,.$$

This space has a transitive action by the symplectic group $G = \mathrm{Sp}(E)$. Let $B \subset G$ be the Borel subgroup of upper triangular matrices, let $B^- \subset G$ be the opposite Borel subgroup of lower triangular matrices, and let $T = B \cap B^-$ be the maximal torus of symplectic diagonal matrices.

For $a, b \in \mathbb{Z}$, let $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ denote the corresponding integer interval. Given any subset $P \subset [1, 2n]$, we let $E_P = \mathrm{Span}_{\mathbb{C}}\{e_p \mid p \in P\}$ be the span of the basis elements corresponding to P. A *Schubert symbol* for $\mathrm{SG}(m, 2n)$ is a subset $P \subset [1, 2n]$ of cardinality m, such that $p' + p'' \neq 2n + 1$ for all $p', p'' \in P$. The subspace E_P is a point of $\mathrm{SG}(m, 2n)$ if and only if P is a Schubert symbol, and the T-fixed points of $\mathrm{SG}(m, E)$ are exactly the points E_P for which P is a Schubert symbol for Y. Each Schubert symbol P defines the Schubert varieties

$$Y_P = \overline{B.E_P} \quad \text{ and } \quad Y^P = \overline{B^-.E_P} \subset Y\,.$$

These varieties can also be defined by (see [BKT15, §4.1])

$$\begin{aligned} Y_P &= \{V \in Y \mid \dim(V \cap E_{[1,b]}) \geq \#(P \cap [1, b]) \ \forall\, b \in [1, 2n]\} \quad \text{ and} \\ Y^P &= \{V \in Y \mid \dim(V \cap E_{[a,2n]}) \geq \#(P \cap [a, 2n]) \ \forall\, a \in [1, 2n]\}\,. \end{aligned}$$

Given Schubert symbols P and Q for Y, we will denote the elements of these sets by $P = \{p_1 < p_2 < \cdots < p_m\}$ and $Q = \{q_1 < q_2 < \cdots < q_m\}$. The *Bruhat order* on Schubert symbols is defined by $Q \leq P$ if and only if $q_i \leq p_i$ for $1 \leq i \leq m$. With this notation we have

$$Q \leq P \quad \Leftrightarrow \quad E_Q \in Y_P \quad \Leftrightarrow \quad Y_Q \subset Y_P \quad \Leftrightarrow \quad Y_P \cap Y^Q \neq \emptyset\,.$$

Define the length $\ell(P)$ to be

$$\ell(P) \;=\; \sum_{i=1}^{m} (p_i - i) \;-\; \#\{i < j : p_i + p_j > 2n + 1\}\,.$$

We then have $\dim(Y_P) = \mathrm{codim}(Y^P, Y) = \ell(P)$. Notice also that Y^P is a translate of $Y_{P^\vee}$, where $P^\vee = \{2n + 1 - p \mid p \in P\}$ is the Poincare dual Schubert symbol.

8.2. **Richardson varieties.** Two Schubert symbols P and Q for $Y = \mathrm{SG}(m, 2n)$ such that $Q \leq P$ define the Richardson variety

$$Y_P^Q = Y_P \cap Y^Q\,.$$

This variety is known to be rational [Ric92]. Using that $\dim(Y_P^Q) = \ell(P) - \ell(Q)$, we obtain

$$\dim(Y_P^Q) \;=\; \sum_{i=1}^{m} (p_i - q_i) \;-\; \#\{i < j : q_i + q_j < 2n + 1 < p_i + p_j\}\,. \tag{4}$$

For any point $V \in Y_P^Q$ we have $V \subset E_{[q_1, p_m]}$ and $V \cap E_{[q_i, p_i]} \neq 0$ for $1 \leq i \leq m$; this holds because $\dim(V \cap E_{[1,p_i]}) \geq i$, $\dim(V \cap E_{[q_i,2n]}) \geq m + 1 - i$, and $\dim(V) = m$.

Let $Y = \mathrm{SG}(m, E)$ and $Y' = \mathrm{SG}(m-1, E)$, and define the 2-step symplectic flag variety

$$Z = \mathrm{SF}(m-1, m; E) = \{(V', V) \in Y' \times Y \mid V' \subset V\}\,.$$

Let $a : Z \to Y$ and $b : Z \to Y'$ be the projections. The T-fixed points in Z have the form $(E_{P'}, E_P)$, where P' and P are Schubert symbols for Y' and Y, respectively, such that $P' \subset P$. The corresponding Schubert varieties are denoted

$$Z_{P',P} = \overline{B.(E_{P'}, E_P)} \quad \text{and} \quad Z^{P',P} = \overline{B^-.(E_{P'}, E_P)}\,.$$

A Richardson variety in Z is denoted by $Z_{P',P}^{Q',Q} = Z_{P',P} \cap Z^{Q',Q}$. Recall our standing notation $P = \{p_1 < \cdots < p_m\}$ and $Q = \{q_1 < \cdots < q_m\}$ for Schubert symbols for $\mathrm{SG}(m, 2n)$.

Proposition 8.1. *Let $Q \leq P$ be Schubert symbols for $Y = \mathrm{SG}(m, 2n)$, and let $1 \leq k \leq m$. Set $Q' = Q \smallsetminus \{q_k\}$ and $P' = P \smallsetminus \{p_k\}$. Then the restricted map $a : Z_{P',P}^{Q',Q} \to Y_P^Q$ is birational. In addition, the restricted map $b : Z_{P',P}^{Q',Q} \to {Y'}_{P'}^{Q'}$ is surjective if and only if $\dim({Y'}_{P'}^{Q'}) \leq \dim(Y_P^Q)$.*

We will prove Proposition 8.1 after introducing some additional notation and results. We will identify the Weyl group of $\mathrm{Sp}(2n)$ with the group of permutations

$$W = \{w \in S_{2n} \mid w(i) + w(2n+1-i) = 2n+1 \text{ for } i \in [1, 2n]\}\,.$$

This group is generated by the simple reflections $s_1, \ldots, s_n \in W$ defined by

$$s_i(i) = i+1\,, \quad s_i(i+1) = i\,, \quad \text{and} \quad s_i(j) = j \ \text{ for } j \in [1, n] \smallsetminus \{i, i+1\}\,.$$

The simple reflection s_n corresponds to the unique long simple root of the root system of type C_n. The parabolic subgroup $P_Y \subset G$ defining $Y = \mathrm{SG}(m, 2n)$ corresponds to the subgroup $W_Y \subset W$ generated by s_i for $i \neq m$. Let $W^Y \subset W$ be the subset of minimal representatives of the cosets in W/W_Y. Then W^Y is in bijective correspondence with the Schubert symbols of Y. The Schubert symbol $P = \{p_1 < p_2 < \cdots < p_m\}$ corresponds to the permutation $w \in W^Y$ defined by

$$w(j) = p_j \ \text{ for } \ 1 \leq j \leq m\,, \quad \text{and} \quad w(m+1) < w(m+2) < \cdots < w(n) \leq n\,.$$

This correspondence preserves the Bruhat order.

The permutation $\widehat{w} \in W^Z$ corresponding of a T-fixed point $(E_{P'}, E_P)$ of $Z = \mathrm{SF}(m-1, m; 2n)$, with $P' = P \smallsetminus \{p_k\}$, is defined by

$$\widehat{w}(j) = \begin{cases} p_j & \text{if } 1 \leq j < k, \\ p_{j+1} & \text{if } k \leq j < m, \\ p_k & \text{if } j = m, \end{cases}$$

and $\widehat{w}(m+1) < \widehat{w}(m+2) < \cdots < \widehat{w}(n) \leq n$. Equivalently, if $w \in W^Y$ corresponds to P, then

$$\widehat{w} = w s_k s_{k+1} \ldots s_{m-1}\,.$$

Let $w' \in W^{Y'}$ be the permutation corresponding to P'. Then w' is obtained from $\widehat{w}$ by first replacing the value $\widehat{w}(m)$ with $\min(p_k, 2n+1-p_k)$, and then rearranging the values $\widehat{w}(m), \ldots, \widehat{w}(n)$ in increasing order. Since $\widehat{w}(m+1) < \cdots < \widehat{w}(n) \leq n$, we can write $w' = \widehat{w} y$, where y is the product of the first $\ell(\widehat{w}) - \ell(w')$ simple reflections in the product

$$s_m s_{m+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_{m+1} s_m\,. \tag{5}$$

Let $F = \mathrm{Sp}(2n)/B$ be the variety of complete symplectic flags, and let $M = \mathrm{Sp}(2n)/P_M$ be any flag variety of $G = \mathrm{Sp}(2n)$. For $\tau \leq \sigma$ in W, let $\Pi_\sigma^\tau(M) \subset M$ denote the *projected Richardson variety* obtained as the image of F_σ^τ under the projection $F \to M$. Recall from [BCMP22, §2] that the M-Bruhat order $\leq_M$ on W can be defined by

$$\tau \leq_M \sigma \quad \Longleftrightarrow \quad \tau \leq \sigma \;\text{ and }\; \sigma_M \leq_L \tau_M \,,$$

where $\sigma = \sigma^M \sigma_M$ and $\tau = \tau^M \tau_M$ are the parabolic factorizations with respect to W_M, and $\leq_L$ is the left weak Bruhat order on W. We need the following properties of projected Richardson varieties from [KLS14] (see also [BCMP22, §3]).

Proposition 8.2 ([KLS14])**.** *Let $\tau \leq \sigma$ in W. The projected Richardson variety $\Pi_\sigma^\tau(M)$ satisfies the following properties.*

(a) *We have $\Pi_\sigma^\tau(M) \subset M_\sigma^\tau$.*

(b) *If $\sigma \in W^M$, then equality $\Pi_\sigma^\tau(M) = M_\sigma^\tau$ holds if and only if $\tau \in W^M$.*

(c) *The projection $F_\sigma^\tau \to \Pi_\sigma^\tau(M)$ is birational if and only if $\tau \leq_M \sigma$.*

(d) *For any simple reflection $s_i \in W_M$ with $\sigma s_i < \sigma$, we have $\Pi_\sigma^\tau(M) = \Pi_{\sigma s_i}^{\min(\tau, \tau s_i)}$.*

Here $\min(\tau, \tau s_i)$ denotes the smaller element among τ and τs_i in the Bruhat order on W.

Proof of Proposition 8.1. Let $u \in W^Y$ correspond to P and let $v \in W^Y$ correspond to Q. Then $Y_P^Q = Y_u^v$ and $Z_{P',P}^{Q',Q} = Z_{\widehat{u}}^{\widehat{v}}$, where $\widehat{u} = ux$ and $\widehat{v} = vx$, with $x = s_k s_{k+1} \cdots s_{m-1}$. Since $\widehat{u}, \widehat{v} \in W^Z$, we have $Z_{\widehat{u}}^{\widehat{v}} = \Pi_{\widehat{u}}^{\widehat{v}}(Z)$ by Proposition 8.2(b). Using that $\widehat{u} = ux$ and $\widehat{v} = vx$ are parabolic factorizations with respect to W_Y, we obtain $\widehat{v} \leq_Y \widehat{u}$, so Proposition 8.2(d,b,c) shows that $\Pi_{\widehat{u}}^{\widehat{v}}(Y) = \Pi_u^v(Y) = Y_u^v$ and $a : Z_{\widehat{u}}^{\widehat{v}} \to Y_u^v$ is birational. This proves the first claim.

Since $Z_{\widehat{u}}^{\widehat{v}} = \Pi_{\widehat{u}}^{\widehat{v}}(Z)$, we have $b(Z_{P',P}^{Q',Q}) = \Pi_{\widehat{u}}^{\widehat{v}}(Y')$. Let $u', v' \in W^{Y'}$ be the elements corresponding to P' and Q'. Then $u' = \widehat{u}y$ and $v' = \widehat{v}z$, where y is the product of the first $\ell(\widehat{u}) - \ell(u')$ simple reflections in (5), and z is the product of the first $\ell(\widehat{v}) - \ell(v')$ simple reflections. Using Proposition 8.2(d), we obtain

$$\Pi_{\widehat{u}}^{\widehat{v}}(Y') = \Pi_{\widehat{u}y}^{\widehat{v}\min(y,z)}(Y') \,.$$

By Proposition 8.2(b), this variety is equal to ${Y'}_{u'}^{v'}$ if and only if $z \leq y$, which is equivalent to $\ell(u') - \ell(v') \leq \ell(\widehat{u}) - \ell(\widehat{v})$. The second claim follows from this. □

8.3. **Matrix representations of Richardson varieties.** We need a parametrization of an open subset of Y_P^Q by matrices with perpendicular rows, which is based on a combinatorial diagram used in [BKT09, Rav15]. Let M_P^Q be the variety of all $m \times (2n)$-matrices $A = (a_{i,j})$, with $a_{i,j} \in \mathbb{C}$, such that for $1 \leq i \leq m$ we have $a_{i,q_i} = 1$, $a_{i,p_i} \neq 0$, and $a_{i,j} = 0$ for $j \notin [q_i, p_i]$, and such that each pair of rows of A are perpendicular as vectors in E, that is,

$$\sum_{t=1}^{n} (a_{i,t}\, a_{j,2n+1-t} - a_{i,2n+1-t}\, a_{j,t}) = 0 \tag{6}$$

for $1 \leq i < j \leq m$. Notice that this equation is vacuous unless

$$q_i + q_j < 2n + 1 < p_i + p_j \,.$$

We will say that rows i and j in M_P^Q are *correlated* if $i \neq j$ and this inequality holds. We will show in Theorem 8.5 that M_P^Q is isomorphic to a dense open subset of the Richardson variety Y_P^Q. In particular, M_P^Q is non-empty and irreducible. Identity (4) states that $\dim(Y_P^Q)$ is equal to the number of entries $a_{i,j}$ that are not explicitly assigned to a constant value, minus the number of pairs of correlated rows in M_P^Q.

Example 8.3. Let $Y = \mathrm{SG}(4,12)$, $Q = \{2,3,8,9\}$, and $P = \{5,7,10,12\}$. Then M_P^Q is the variety of all matrices of the form

$$A = \begin{bmatrix} 0 & 1 & a_{1,3} & a_{1,4} & a_{1,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_{3,9} & a_{3,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_{4,10} & a_{4,11} & a_{4,12} \end{bmatrix},$$

such that $a_{1,5} \neq 0$, $a_{2,7} \neq 0$, $a_{3,10} \neq 0$, $a_{4,12} \neq 0$, and the rows of A are pairwise perpendicular. The variety M_P^Q has 12 unassigned entries and 4 pairs of correlated rows, so $\dim(Y_P^Q) = 8$.

Remark 8.4. Let $Q \leq P$ be Schubert symbols for $Y = \mathrm{SG}(m,2n)$ and $1 \leq k \leq m$. Set $Q' = Q \smallsetminus \{q_k\}$ and $P' = P \smallsetminus \{p_k\}$. Then $Q' \leq P'$ are Schubert symbols for $Y' = \mathrm{SG}(m-1,2n)$ and we have

$$\dim Y_P^Q - \dim {Y'}_{P'}^{Q'} = (p_k - q_k) - \#\{j \in [1,m] \mid j \neq k \text{ and } q_j + q_k < 2n+1 < p_j + p_k\}\,.$$

This is the number of unassigned entries in row k of M_P^Q, minus the number of rows correlated to row k.

Let $\mathring{Y}_P^Q \subset Y_P^Q$ be the open subvariety defined by

$$\mathring{Y}_P^Q = \{V \in Y_P^Q \mid \forall\, 1 \leq i \leq m : V \cap E_{[q_i+1,p_i]} = V \cap E_{[q_i,p_i-1]} = 0\}\,.$$

The following result confirms a conjecture of Ravikumar [Rav13, Conj. 6.5.3].

Theorem 8.5. *The variety $\mathring{Y}_P^Q$ is a dense open subset of Y_P^Q. Moreover, the map $M_P^Q \to \mathring{Y}_P^Q$ sending a matrix to its row span is an isomorphism of varieties.*

Proof. Since Y_P^Q is irreducible and the subsets

$$U_i^L = \{V \in Y_P^Q \mid V \cap E_{[q_i+1,p_i]} = 0\} \quad \text{and}$$
$$U_i^R = \{V \in Y_P^Q \mid V \cap E_{[q_i,p_i-1]} = 0\}$$

are open in Y_P^Q, the first claim will follow if we can show that U_i^L and U_i^R are non-empty for all $1 \leq i \leq m$. By replacing Y_P^Q with $Y_{Q^\vee}^{P^\vee}$, if required, we may assume that $q_1 + p_m \geq 2n+1$. The sets U_m^L and U_1^R are non-empty since $E_Q \in U_m^L$ and $E_P \in U_1^R$. Set $\Omega = \{V \in Y \mid V \subset E_{[1,p_m-1]}\}$. Then $Y_P \cap \Omega$ is a B-stable proper closed subset of Y_P, so $Y_P \cap \Omega$ is a union of Schubert varieties $Y_{\widehat{P}}$ that are properly contained in Y_P. It follows that $Y_P^Q \cap \Omega$ is a union of Richardson varieties $Y_{\widehat{P}}^Q$ that are properly contained in Y_P^Q. Therefore, $U_m^L \smallsetminus \Omega$ is a dense open subset of Y_P^Q.

Set $Y' = \mathrm{SG}(m-1,2n)$, $Q' = \{q_1 < \cdots < q_{m-1}\}$, and $P' = \{p_1 < \cdots < p_{m-1}\}$. By induction we may assume $\mathring{Y'}_{P'}^{Q'} \neq \emptyset$. By Remark 8.4, the condition $q_1 + p_m \geq 2n+1$ implies that $\dim({Y'}_{P'}^{Q'}) \leq \dim(Y_P^Q)$. In fact, if row i of M_P^Q is correlated to row m, then $2n+1-p_m \leq q_1 \leq q_i < 2n+1-q_m$, so row m is correlated to at most $p_m - q_m$ rows. Using Proposition 8.1, we can therefore choose a point $(V',V) \in Z^{Q',Q}_{P',P}$ such that $V' \in \mathring{Y'}^{Q'}_{P'}$ and $V \in U^L_m \smallsetminus \Omega$. Since $V' \subset E_{[1,p_m-1]}$ and $V \not\subset E_{[1,p_m-1]}$, we have $V' = V \cap E_{[1,p_m-1]}$. The condition $V' \in \mathring{Y'}^{Q'}_{P'}$ therefore implies that $V \in U^L_i \cap U^R_i$ for $1 \leq i \leq m-1$. Set $L = V \cap E_{[q_m,p_m]}$. Since $V' \subset E_{[1,p_{m-1}]}$, we obtain $V' \cap L \subset V' \cap E_{[q_{m-1}+1,p_{m-1}]} = 0$, hence $V = V' \oplus L$ and $\dim(L) = 1$. Since $V' \subset E_{[1,p_m-1]}$ and $V \not\subset E_{[1,p_m-1]}$, it follows that $L \not\subset E_{[1,p_m-1]}$. We deduce that $V \cap E_{[q_m,p_m-1]} = L \cap E_{[q_m,p_m-1]} = 0$, so that $V \in U^R_m$. We conclude that $V \in \mathring{Y}^Q_P$, so this set is a dense open subset of Y^Q_P.

It is clear from the definitions that $A \mapsto \operatorname{Span}(A)$ is a well defined morphism of varieties $M^Q_P \to \mathring{Y}^Q_P$. On the other hand, given $V \in \mathring{Y}^Q_P$, each space $L_i = V \cap E_{[q_i,p_i]}$ is one-dimensional, for $1 \leq i \leq m$. In addition, if we write $L_i = \mathbb{C}u_i$ with $u_i \in E$, then the q_i-th and p_i-th coordinates of u_i are non-zero. By rescaling u_i, we may assume that the q_i-th coordinate is 1. Let A be the $m \times (2n)$ matrix whose i-th row is u_i. Then $A \in M^Q_P$ and $\operatorname{Span}(A) = V$. This completes the proof. □

Let $Q \leq P$ be Schubert symbols for $Y = \mathrm{SG}(m,2n)$, let $1 \leq k \leq m$, and let $A = (a_{i,j}) \in M^Q_P$. Define the *submatrix of constraints* on row k in A to be the matrix $A[k]$ with entries $a_{i,j}$ for which $i \neq k$, $q_i + q_k < 2n+1 < p_i + p_k$, and $2n+1-p_k \leq j \leq 2n+1-q_k$. This matrix has one row for each row correlated to the k-th row of M^Q_P. For example, if A is the matrix of Example 8.3, then the submatrix of constraints on row 2 is the matrix

$$A[2] = \begin{bmatrix} 0 & 0 & 1 & a_{3,9} & a_{3,10} \\ 0 & 0 & 0 & 1 & a_{4,10} \end{bmatrix}.$$

The constraints (6) on row k in A imposed by the other rows depend only on the entries of $A[k]$. We will say that the vector $v = (v_{q_k}, \dots, v_{p_k}) \in \mathbb{C}^{p_k-q_k+1}$ is *perpendicular* to $A[k]$ if the entries of v satisfy the quadratic equations (6) imposed on the k-th row in A, that is,

$$\sum_{t=1}^{n} (a_{i,t}\, v_{2n+1-t} - a_{i,2n+1-t}\, v_t) = 0$$

for all $i \neq k$ with $q_i + q_k < 2n+1 < p_i + p_k$, where we set $v_t = 0$ for $t \notin [q_k, p_k]$.

Set $Q' = Q \smallsetminus \{q_k\}$, $P' = P \smallsetminus \{p_k\}$, and $Y' = \mathrm{SG}(m-1,2n)$. Motivated by Proposition 8.1 and Theorem 8.5, we will say that the k-th row of M^Q_P is *solvable* if $\dim(Y'^{Q'}_{P'}) \leq \dim(Y^Q_P)$. By Remark 8.4, this means that there are at most $p_k - q_k$ constraints on the k-th row of M^Q_P. The k-th row of M^Q_P is *movable* if $\dim(Y'^{Q'}_{P'}) < \dim(Y^Q_P)$, that is, there are fewer than $p_k - q_k$ constraints on the k-th row. If the k-th row of M^Q_P is movable, then for each matrix $A \in M^Q_P$, we can vary the k-th row of A in a positive dimensional parameter space while fixing the remaining rows.

Corollary 8.6. *Let $Q \leq P$ be Schubert symbols for $\mathrm{SG}(m,2n)$, and assume that the k-th row of M^Q_P is solvable. Then M^Q_P contains a dense open subset of points A for which the submatrix $A[k]$ of constraints on row k has linearly independent rows.*

Proof. Set $Q' = Q \smallsetminus \{q_k\}$ and $P' = P \smallsetminus \{p_k\}$. Given $A \in M^Q_P$, let $A' \in M^{Q'}_{P'}$ denote the result of removing the k-th row from A. It follows from Proposition 8.1 and

Theorem 8.5 that $A \mapsto A'$ defines a dominant morphism $M_P^Q \to M_{P'}^{Q'}$. This implies that, for all points A in a dense open subset of M_P^Q, the fiber over A' in M_P^Q is non-empty of dimension $\dim(M_P^Q) - \dim(M_{P'}^{Q'})$. This fiber can be identified with the set of vectors $v = (1, v_{q_k+1}, \ldots, v_{p_k})$, with $v_{p_k} \neq 0$, that are perpendicular to $A[k]$. We deduce that the rows of $A[k]$ are linearly independent by Remark 8.4. $\square$

8.4. **Perpendicular incidence varieties.** Let $Y = \mathrm{SG}(m, 2n)$ and define the *perpendicular incidence variety*

$$S = \{(V, L) \in Y \times \mathbb{P}(E) \mid V \subset L^\perp\}.$$

Let $f : S \to \mathbb{P}(E)$ and $g : S \to Y$ be the projections. Given Schubert symbols $Q \leq P$ for Y, we set $S_P^Q = g^{-1}(Y_P^Q)$. Since g is locally trivial with fibers $g^{-1}(V) = \mathbb{P}(V^\perp)$ by [BCMP13, Prop. 2.3], it follows that S_P^Q is irreducible with $\dim(S_P^Q) = \dim(Y_P^Q) + 2n - m - 1$.

Following [BKT09, Rav15], we define a *cut* of M_P^Q to be an integer $c \in [0, 2n]$ such that $p_i \leq c$ or $c < q_i$ holds for each $i \in [1, m]$. This implies that no row of M_P^Q contains non-zero entries in both column c and column $c+1$. A *lone star* is an integer $s \in [1, 2n]$ such that $q_i = p_i = s$ for some $i \in [1, m]$. This implies that $s - 1$ and s are cuts of M_P^Q. The integer c is a *double-cut* of M_P^Q if both c and $2n - c$ are cuts. A *component* of M_P^Q is a pair of integers (a, b), with $0 \leq a < b \leq n$, such that (i) a is a double-cut, (ii) b is a double-cut or $b = n$, and (iii) no double-cut belongs to $[a+1, b-1]$. We will say that row i of M_P^Q is contained in the component (a, b) if $a < q_i \leq p_i \leq b$, or $2n - b < q_i \leq p_i \leq 2n - a$, or $b = n$ and $a < q_i \leq p_i \leq 2n - a$. Each row of M_P^Q belongs to a unique component, and two rows can be correlated only if they belong to the same component. Any component (a, b) contains at most $b - a$ rows. The component (a, b) is called a *quadratic component* if b is a double-cut, $b - a \geq 2$, and (a, b) contains $b - a$ rows.

Let $\mathbb{P}_P^Q \subset \mathbb{P}(E)$ denote the closed subvariety defined by the linear equations $x_{2n+1-s} = 0$ for all lone stars s of M_P^Q, as well as the quadratic equations

$$x_{a+1} x_{2n-a} + \cdots + x_b x_{2n+1-b} = 0$$

given by all quadratic components (a, b) of M_P^Q. Using that the quadratic equations involve pairwise disjoint sets of variables, it follows that $\mathbb{P}_P^Q$ is an irreducible complete intersection in $\mathbb{P}(E)$ with rational singularities.

Example 8.7. Let $Y = \mathrm{SG}(8, 20)$ and define $Q = \{1, 2, 4, 6, 9, 11, 16, 18\}$ and $P = \{2, 3, 7, 8, 11, 12, 16, 20\}$. The shape of non-zero entries in M_P^Q is given by the diagram:

$$\left[\begin{array}{ccc|ccccc|cccc|ccccc|ccc}
\star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \star & \star & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \star & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \star & \star
\end{array}\right]$$

Here we ignore that the lone star in column 16 forces the entry in column 5 to vanish. The double-cuts of M_P^Q are indicated with vertical lines. The components of M_P^Q are $(0,3)$, $(3,8)$, and $(8,10)$, and we have

$$\mathbb{P}_P^Q = Z(x_5\,,\, x_1x_{20}+x_2x_{19}+x_3x_{18}) \subset \mathbb{P}(E)\,.$$

Our main result about perpendicular incidences is the following theorem, which will be proved at the end of this section.

Theorem 8.8. *Let $Q \le P$ be Schubert symbols for $Y = \mathrm{SG}(m,2n)$. Then f restricts to a surjective morphism $f : S_P^Q \to \mathbb{P}_P^Q$ with rational general fibers.*

The analogue of Theorem 8.8 with $S \subset Y \times \mathbb{P}(E)$ defined by the condition $L \subset V$ has been established in [BKT09, BR12, Rav15]. When $Y = \mathrm{LG}(n,2n)$ is a Lagrangian Grassmannian, the conditions $V \subset L^\perp$ and $L \subset V$ are equivalent, so this case of Theorem 8.8 is equivalent to [BR12, Lemma 5.2]. One new complication in our case is that S is not a flag variety, so the map $f : S_P^Q \to \mathbb{P}_P^Q$ is not a projection from a Richardson variety, as studied in e.g. [BC12, KLS14, BCMP22].

Lemma 8.9. *Let $Q \le P$ be Schubert symbols for $\mathrm{SG}(m,2n)$ and let $1 \le k \le m$. If $q_k \le n < p_k$, then row k of M_P^Q is movable.*

Proof. Assume that row j is correlated to row k. If $j < k$, then $2n+1-p_k < p_j < p_k$, which holds for at most $p_k - n - 1$ rows j. If $j > k$, then $q_k < q_j < 2n+1-q_k$, which holds for at most $n - q_k$ rows j. It follows that row k is correlated to at most $p_k - q_k - 1$ rows. ☐

Proposition 8.10. *Let $Q \le P$ be Schubert symbols for $Y = \mathrm{SG}(m,2n)$, and let (a,b) be a component of M_P^Q with $b-a \ge 2$. Then (a,b) is a quadratic component if and only if no row contained in (a,b) is movable. In this case all rows contained in (a,b) are solvable, and M_P^Q has no cuts c with $a < c < b$ or $2n-b < c < 2n-a$.*

Proof. Since two rows of M_P^Q can be correlated only if they belong to the same component, we may assume that $(a,b) = (0,n)$ is the only component of M_P^Q. By Lemma 8.9 we may further assume that n is a cut. By replacing M_P^Q with $M_{Q^\vee}^{P^\vee}$, if necessary, we may also assume that $p_m = 2n$. Set $r = p_m - q_m \ge 1$. If row m of M_P^Q is not movable, then since $1 \notin P$ and $r+1 = 2n+1-q_m \notin Q$, we must have $q_i = i < p_i$ for $1 \le i \le r$. The same conclusion holds if $(0,n)$ is a quadratic component of M_P^Q, since in this case we have $x \in Q$ or $2n+1-x \in Q$ for all $x \in [1,n]$. Set $Q' = (Q \smallsetminus \{r, q_m\}) \cup \{r+1, q_m+1\}$, so that the shape of $M_P^{Q'}$ is obtained from the shape of M_P^Q by removing the leftmost entry from rows r and m. Then M_P^Q and $M_P^{Q'}$ have the same pairs of correlated rows, except that rows r and m are correlated in M_P^Q but not in $M_P^{Q'}$. It follows that any row is movable in M_P^Q if and only if it is movable in $M_P^{Q'}$, and the same holds with movable replaced by solvable. The component $(0,n)$ is quadratic if and only if $m = n$. Since $M_P^{Q'}$ has no empty components, $m = n$ holds if and only if all components of $M_P^{Q'}$ are quadratic or lone stars. By induction on $\sum_{i=1}^m (p_i - q_i)$, this holds if and only if $M_P^{Q'}$ has no movable rows, which proves the first claim. Assuming that $(0,n)$ is a quadratic component, it also follows by induction that all rows of M_P^Q are solvable. Noting that all double-cuts of $M_P^{Q'}$ belong to the set $\{0, r, n\}$, it follows by induction that all cuts of $M_P^{Q'}$ belong to $\{0, r, n, 2n-r, 2n\}$. The last claim follows from this since r and $2n-r$ are not cuts of M_P^Q. ☐

Corollary 8.11. *Let $Q \leq P$ be Schubert symbols, and assume that row k in M_P^Q is movable. Then $\mathbb{P}_P^Q = \mathbb{P}_{Q'}^{P'}$, where $Q' = Q \smallsetminus \{q_k\}$ and $P' = P \smallsetminus \{p_k\}$.*

Proof. This holds because a movable row cannot be a lone star and cannot belong to a quadratic component by Proposition 8.10. $\square$

Given Schubert symbols $Q \leq P$ for $Y = \mathrm{SG}(m, 2n)$, define the variety

$$\widehat{M}_P^Q = \{(A, u) \in M_P^Q \times E \mid A \perp u\}\,,$$

where $A \perp u$ indicates that u is perpendicular to all rows of A. The variety $\widehat{M}_P^Q$ is irreducible with $\dim(\widehat{M}_P^Q) = \dim(M_P^Q) + 2n - m$.

Proposition 8.12. *Let $Q \leq P$ be Schubert symbols for $Y = \mathrm{SG}(m, 2n)$ and assume that the k-th row of M_P^Q is movable. Set $Q' = Q \smallsetminus \{q_k\}$, $P' = P \smallsetminus \{p_k\}$, and $r = \dim(M_P^Q) - \dim(M_{P'}^{Q'}) > 0$. Let $\pi : \widehat{M}_P^Q \to \widehat{M}_{P'}^{Q'}$ be the projection that forgets row k in its first argument. There exists a morphism $\phi : \widehat{M}_P^Q \to \mathbb{C}^{r-1}$, given by projection to $r - 1$ of the entries of the k-th row of M_P^Q, such that the morphism $\pi \times \phi : \widehat{M}_P^Q \to \widehat{M}_{P'}^{Q'} \times \mathbb{C}^{r-1}$ is birational.*

Proof. By Corollary 8.6 we can choose $A \in M_P^Q$ such that the submatrix $A[k]$ of constraints on row k has linearly independent rows. The number of rows in $A[k]$ is equal to $p_k - q_k - r$ by Remark 8.4. We can therefore choose a vector

$$(u_{2n+1-p_k}, \ldots, u_{2n+1-q_k}) \in \mathbb{C}^{p_k - q_k + 1}$$

which is perpendicular to the k-th row of A and not in the row span of $A[k]$. Using that $a_{i,q_i} = 1$ and $a_{i,p_i} \neq 0$ for each row i, we can extend this vector to $u \in E$, such that u is perpendicular to all rows of A. Let $A' \in M_{P'}^{Q'}$ be the result of removing the k-th row from A. Then the fiber of $\pi : \widehat{M}_P^Q \to \widehat{M}_{P'}^{Q'}$ over (A', u) contains (A, u), so it is not empty. This fiber can be identified with the set of vectors $(1, v_{q_k+1}, \ldots, v_{p_k})$, with $v_{p_k} \neq 0$, which are perpendicular to both $A[k]$ and $(u_{2n+1-p_k}, \ldots, u_{2n+1-q_k})$. Therefore the fiber has dimension $r - 1 = \dim(\widehat{M}_P^Q) - \dim(\widehat{M}_{P'}^{Q'})$. Since $\widehat{M}_{P'}^{Q'}$ is irreducible, this implies that $\pi : \widehat{M}_P^Q \to \widehat{M}_{P'}^{Q'}$ is dominant. It also follows that (A, u) is determined by (A', u) together with some collection of $r - 1$ entries $a_{k,j}$ from the k-th row of A. Since this holds whenever a particular minor in (A', u) is non-zero, we deduce that (A, u) is determined by (A', u) and the same $r - 1$ entries from row k, for all points (A, u) in a dense open subset of $\widehat{M}_P^Q$. The result follows from this. $\square$

Assume that $c \in [1, n-1]$ is a double-cut of M_P^Q, and set $Q' = Q \cap [c+1, 2n-c]$, $P' = P \cap [c+1, 2n-c]$, $Q'' = Q \smallsetminus Q'$, and $P'' = P \smallsetminus P'$. Set $m' = \#P'$, $Y' = \mathrm{SG}(m', 2n)$, $m'' = \#P''$, $Y'' = \mathrm{SG}(m'', 2n)$, and let $S' \subset Y' \times \mathbb{P}(E)$ and $S'' \subset Y'' \times \mathbb{P}(E)$ be the corresponding perpendicular incidence varieties, with projections $f' : S' \to \mathbb{P}(E)$ and $f'' : S'' \to \mathbb{P}(E)$. Since we have $\mathbb{P}_P^Q = \mathbb{P}_{P'}^{Q'} \cap \mathbb{P}_{P''}^{Q''}$, the following lemma shows that Theorem 8.8 can be proved under the assumption that M_P^Q has only one component $(0, n)$.

Lemma 8.13. *The map $(V', V'') \mapsto V' \oplus V''$ is an isomorphism ${Y'}_{P'}^{Q'} \times {Y''}_{P''}^{Q''} \cong Y_P^Q$, and we have $f(S_P^Q) = f'({S'}_{P'}^{Q'}) \cap f''({S''}_{P''}^{Q''})$. For all points $L \in f(S_P^Q)$, the fiber of*

$f : S_P^Q \to f(S_P^Q)$ over L is the product of the fibers of $f' : {S'}_{P'}^{Q'} \to f'({S'}_{P'}^{Q'})$ and $f'' : {S''}_{P''}^{Q''} \to f''({S''}_{P''}^{Q''})$ over L.

Proof. Set $E' = E_{[c+1,2n-c]}$ and $E'' = E_{[1,c]\cup[2n-c+1,2n]}$. Using that $V' \subset E'$ holds for all $V' \in {Y'}_{P'}^{Q'}$, and $V'' \subset E''$ holds for all $V'' \in {Y''}_{P''}^{Q''}$, it follows that the map ${Y'}_{P'}^{Q'} \times {Y''}_{P''}^{Q''} \to Y_P^Q$ is well defined. The inverse map $V \mapsto (V \cap E', V \cap E'')$ is well defined because $\dim(V \cap E') = m'$, $\dim(V \cap E_{[1,c]}) = \#(P \cap [1,c])$, and $\dim(V \cap E_{[2n-c+1,2n]}) = \#(P \cap [2n-c+1, 2n])$ holds for all $V \in Y_P^Q$. This proves the first claim. The remaining claims follow because $V' \oplus V'' \subset L^\perp$ is equivalent to $V' \subset L^\perp$ and $V'' \subset L^\perp$. □

Proof of Theorem 8.8. We may assume that $(0, n)$ is the only component of M_P^Q by Lemma 8.13. If M_P^Q has no movable rows, then Proposition 8.10 implies that $m = n$, so the claim follows from [BR12, Lemma 5.2]. Otherwise M_P^Q has at least one movable row, say row k. Set $Y' = \mathrm{SG}(m-1, 2n)$, $Q' = Q \smallsetminus \{q_k\}$, $P' = P \smallsetminus \{p_k\}$, and $r = \dim(M_P^Q) - \dim(M_{P'}^{Q'})$. Let $S \subset Y \times \mathbb{P}(E)$ and $S' \subset Y' \times \mathbb{P}(E)$ be the perpendicular incidence varieties, with projections $f : S \to \mathbb{P}(E)$ and $f' : S' \to \mathbb{P}(E)$. It follows from Proposition 8.12 that $f(S_P^Q) = f'({S'}_{P'}^{Q'})$, and for all points L in a dense open subset of $f(S_P^Q)$, the fiber $f^{-1}(L) \cap S_P^Q$ is birational to $({f'}^{-1}(L) \cap {S'}_{P'}^{Q'}) \times \mathbb{C}^{r-1}$. The result therefore follows by induction on m. □

9. Gromov-Witten invariants of Pieri type

9.1. **Incidences of projected Richardson varieties.** Let $X = \mathrm{LG}(n, 2n)$ be a Lagrangian Grassmannian and $Y = \mathrm{SG}(m, 2n)$ a symplectic Grassmannian. Set $Z = \mathrm{SF}(m, n; 2n)$ and let $p : Z \to X$ and $q : Z \to Y$ be the projections. We also set $\widehat{X} = \mathrm{SF}(1, n; 2n)$, with projections $\eta : \widehat{X} \to \mathbb{P}^{2n-1}$ and $\pi : \widehat{X} \to X$. Our computation of Gromov-Witten invariants of X is based on the following result.

Theorem 9.1. *Let $Q \le P$ be Schubert symbols for $Y = \mathrm{SG}(m, 2n)$, and let $X_P^Q = p(q^{-1}(Y_P^Q))$ be the corresponding projected Richardson variety in $X = \mathrm{LG}(n, 2n)$. Then η restricts to a cohomologically trivial morphism $\eta : \pi^{-1}(X_P^Q) \to \mathbb{P}_P^Q$.*

Proof. Define $\widehat{Z} = Z \times_X \widehat{X} = \{(K, V, L) \in Y \times X \times \mathbb{P}^{2n-1} \mid K \subset V \supset L\}$ and $S = \{(K, L) \in Y \times \mathbb{P}^{2n-1} \mid K \subset L^\perp\}$. Consider the following commutative diagram, where all morphisms are the natural projections.

$$\begin{tikzcd}
S \arrow[drrr, "f"] \arrow[dddr, "g"'] & & & \\
& \widehat{Z} \arrow[ul, "\sigma"'] \arrow[r, "\widehat{p}"'] \arrow[d, "\tau"] & \widehat{X} \arrow[r, "\eta"'] \arrow[d, "\pi"] & \mathbb{P}^{2n-1} \\
& Z \arrow[r, "p"'] \arrow[d, "q"] & X & \\
& Y & &
\end{tikzcd}$$

Since the morphisms of this diagram are equivariant for the action of $\mathrm{Sp}(2n)$, and all targets other than S are flag varieties of $\mathrm{Sp}(2n)$, it follows that all morphisms other than σ are locally trivial fibrations with non-singular fibers [BCMP13, Prop. 2.3]. Let Z_P^Q, $\widehat{Z}_P^Q$, and S_P^Q be the inverse images of Y_P^Q in Z, $\widehat{Z}$, and S, respectively, and set $\widehat{X}_P^Q = \pi^{-1}(X_P^Q)$. Since Y_P^Q and X_P^Q have rational singularities [Bri02, BC12, KLS14], it follows that Z_P^Q, $\widehat{Z}_P^Q$, S_P^Q, and $\widehat{X}_P^Q$ have rational singularities as well.

All fibers of σ are rational. In fact, for $(K,L) \in S$ we have $\sigma^{-1}(K,L) = \mathrm{LG}(m', (K+L)^\perp/(K+L))$, where $m' = n - \dim(K+L)$. Since $\widehat{Z}_P^Q = \sigma^{-1}(S_P^Q)$, this implies that $\sigma : \widehat{Z}_P^Q \to S_P^Q$ is cohomologically trivial [BM11, Thm. 3.1]. Since $f : S_P^Q \to f(S_P^Q)$ is cohomologically trivial by Theorem 8.8, it follows that $\eta\widehat{p} = f\sigma : \widehat{Z}_P^Q \to f(S_P^Q)$ is cohomologically trivial [BCMP18b, Lemma 2.4].

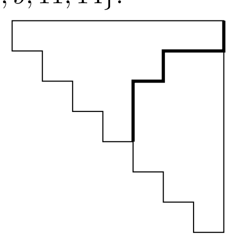
Using that the outer rectangle and the right square of the following diagram are fiber squares, it follows that $\widehat{p} : \widehat{Z}_P^Q \to \widehat{X}_P^Q$ is the base extension of $p : Z_P^Q \to X_P^Q$ along π.

$$\begin{tikzcd}
\widehat{Z}_P^Q \arrow[r, "\widehat{p}"'] \arrow[d, "\tau"] & \widehat{X}_P^Q \arrow[r, "\subset"'] \arrow[d, "\pi"] & \widehat{X} \arrow[d, "\pi"] \\
Z_P^Q \arrow[r, "p"'] & X_P^Q \arrow[r, "\subset"'] & X
\end{tikzcd}$$

This implies that $\widehat{p} : \widehat{Z}_P^Q \to \widehat{X}_P^Q$ is cohomologically trivial, for example because its general fibers are Richardson varieties by [BCMP22, Cor. 2.11]. It follows that $\eta : \widehat{X}_P^Q \to f(S_P^Q)$ is cohomologically trivial. In particular, $\eta(\widehat{X}_P^Q) = f(S_P^Q)$ is a complete intersection of the required form. This completes the proof. $\square$

9.2. **Gromov-Witten invariants of Pieri type.** The Schubert varieties in $X = \mathrm{LG}(n,2n)$ are indexed by shapes $\lambda \subset \mathcal{P}_X$. The Schubert symbol P corresponding to $\lambda \subset \mathcal{P}_X$ is obtained as follows. The border of λ forms a path from the upper-right corner of $\mathcal{P}_X$ to the diagonal. Number the steps of this path from 1 to n, starting from the upper-right corner. Then P consists of the integers i for which the i-th step is horizontal, and the integers $2n+1-i$ for which the i-th step is vertical. By observing that the map from shapes to Schubert symbols is compatible with the Bruhat order, this description of the Schubert varieties in X follows from e.g. [BS16, Lemma 2.9].

Example 9.2. Let $X = \mathrm{LG}(7,14)$ and $\lambda = (7,4,2,1)$. Then λ corresponds to the Schubert symbol $P = \{2,3,5,8,9,11,14\}$.



Recall that a classical shape $\lambda \subset \mathcal{P}_X$ is identified with the quantum shape $I([X^\lambda]) = \lambda \cup I(1)$ in $\widehat{\mathcal{P}}_X$, and $\lambda[d]$ is the result of shifting this shape by d diagonal steps for each $d \in \mathbb{Z}$.

Let $\lambda, \mu \subset \mathcal{P}_X$ be shapes and $d \geq 0$ a degree. Then $\Gamma_d(X_\lambda, X^\mu) \neq \emptyset$ if and only if $\mu \subset \lambda[d]$. When this holds, we let $\lambda[d]/\mu$ be the skew shape in $\widehat{\mathcal{P}}_X$ of boxes in $\lambda[d]$ that are not contained in μ. Let $R(\lambda[d]/\mu)$ denote the size of a maximal rim in this skew shape, and let $N(\lambda[d]/\mu)$ be the number of connected components of $\lambda[d]/\mu$ that are disjoint from both of the diagonals in $\widehat{\mathcal{P}}_X$. The following result interprets Theorem 9.1 when the projected Richardson variety in X is a curve neighborhood $\Gamma_d(X_\lambda, X^\mu)$.

Corollary 9.3. *Let $X = \mathrm{LG}(n, 2n)$ and let $\lambda, \mu \subset \mathcal{P}_X$ be shapes such that $\Gamma_d(X_\lambda, X^\mu) \neq \emptyset$. Set $\theta = \lambda[d]/\mu$. If $R(\theta) = n+1$, then $\eta(\pi^{-1}(\Gamma_d(X_\lambda, X^\mu))) = \mathbb{P}^{2n-1}$. Otherwise $\eta(\pi^{-1}(\Gamma_d(X_\lambda, X^\mu)))$ is a complete intersection in $\mathbb{P}^{2n-1}$ of dimension $n + R(\theta) - 1$, defined by $N(\theta)$ quadratic equations and $n - R(\theta) - N(\theta)$ linear equations. Moreover, the restricted map $\eta : \pi^{-1}(\Gamma_d(X_\lambda, X^\mu)) \to \eta(\pi^{-1}(\Gamma_d(X_\lambda, X^\mu)))$ is cohomologically trivial.*

Proof. Write $X_\lambda = X_P$ and $X^\mu = X^Q$, where $P = \{p_1 < \cdots < p_n\}$ and $Q = \{q_1 < \cdots < q_n\}$ are the Schubert symbols corresponding to λ and μ. Then $q(p^{-1}(X_\lambda)) = Y_{P'}$ and $q(p^{-1}(X^\mu)) = Y^{Q'}$, where $P' = \{p_{d+1}, \ldots, p_n\}$ and $Q' = \{q_1, \ldots, q_{n-d}\}$, so we have $\Gamma_d(X_\lambda, X^\mu) = X_{P'}^{Q'}$. Theorem 9.1 shows that

$$\eta : \pi^{-1}(\Gamma_d(X_\lambda, X^\mu)) \to \mathbb{P}_{P'}^{Q'}$$

is cohomologically trivial. It remains to show that $\mathbb{P}_{P'}^{Q'}$ is a complete intersection defined by the expected equations. If $R(\theta) = n+1$, then we can make d and λ smaller and μ larger until we obtain $R(\theta) = n$ and $N(\theta) = 0$. This will make $\Gamma_d(X_\lambda, X^\mu)$ smaller, while the corollary still asserts that $\eta(\pi^{-1}(\Gamma_d(X_\lambda, X^\mu))) = \mathbb{P}^{2n-1}$. We may therefore assume that $R(\theta) \leq n$, which implies that the borders of μ and $\lambda[d]$ meet somewhere. In particular, μ has at least d vertical steps, and $\lambda[d]$ has at least d horizontal steps.

Let $\ell(\mu)$ be the number of vertical steps of μ. Then μ has $n - \ell(\mu)$ horizontal steps. Notice that, if $1 \leq k \leq n - \ell(\mu)$, then q_k is the step number of the k-th horizontal step of μ, while if $n - \ell(\mu) < k \leq n$, then $2n+1-q_k$ is the step number of the $(n+1-k)$-th vertical step of μ. Since the starting point of μ is d boxes north-west of the starting point of λ, and the endpoint of μ is north-west of the endpoint of $\lambda[d]$, we have $\ell(\mu) \leq \ell(\lambda) + d$. The condition $R(\theta) \leq n$ implies that $\ell(\mu) \geq d$ and $\ell(\lambda) \leq n - d$.

Write $P' = \{p'_1, \ldots, p'_{n-d}\}$ and $Q' = \{q'_1, \ldots, q'_{n-d}\}$, where $p'_i = p_{i+d}$ and $q'_i = q_i$. It follows from the construction of P and Q from λ and μ that the rows in $M_{P'}^{Q'}$ are in bijection with some of the steps of $\lambda[d]$, and also with some of the steps of μ. We will explain how to obtain the resulting bijection between steps of $\lambda[d]$ and μ, and how to obtain the rows of $M_{P'}^{Q'}$ from the corresponding pairs of steps in $\lambda[d]$ and μ. This will include drawing *connectors* between the paired steps of $\lambda[d]$ and μ, see Example 9.4.

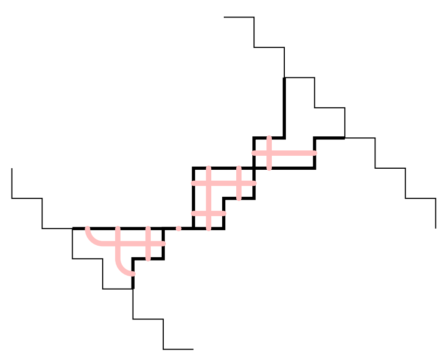
Consider row k of $M_{P'}^{Q'}$. Assume first that $d + k \leq n - \ell(\lambda)$. Then $k \leq n - \ell(\mu)$, q'_k is the step number of the k-th horizontal step of μ, and p'_k is the step number of the $(d+k)$-th horizontal step of $\lambda[d]$. These steps of μ and $\lambda[d]$ are in the same column, and $p'_k - q'_k$ is the distance (number of boxes) between the two steps. We draw a vertical line segment (connector) from the k-th horizontal step of μ to the $(d+k)$-th horizontal step of $\lambda[d]$.

Assume next that $k > n - \ell(\mu)$. Then $d + k > n - \ell(\lambda)$, $2n+1-q'_k$ is the step number of the $(n+1-k)$-th vertical step of μ, and $2n+1-p'_k$ is the step number of the $(n-d+1-k)$-th vertical step of $\lambda[d]$. These steps of μ and $\lambda[d]$ are in the same row, and $p'_k - q'_k$ is the distance between the two steps. We draw a horizontal line segment (connector) from the $(n+1-k)$-th vertical step of μ to the $(n-d+1-k)$-th vertical step of $\lambda[d]$.

We finally assume that $d+k > n-\ell(\lambda)$ and $k \leq n-\ell(\mu)$. Then q'_k is the step number of the k-th horizontal step of μ, and $2n+1-p'_k$ is the step number of the $(n-d+1-k)$-th vertical step of $\lambda[d]$. In this case, if we draw a vertical line segment going down from the horizontal step of μ, and a horizontal line segment going to the left from the vertical step of $\lambda[d]$, then these line segments meet in a diagonal box of $\widehat{\mathcal{P}}_X$. In this case the connector representing row k of $M_{P'}^{Q'}$ is obtained by connecting the two line segments, and $p'_k - q'_k$ is the number of boxes this connector passes through.

It follows from this description that the lone stars of $M_{P'}^{Q'}$ correspond to steps shared by μ and $\lambda[d]$, and there are exactly $n - R(\theta) - N(\theta)$ such steps. It also follows that, if μ and $\lambda[d]$ meet after c steps, then c is a double-cut of $M_{P'}^{Q'}$. The only other cuts of $M_{P'}^{Q'}$ are the integers in the set $[0, q'_1 - 1] \cup [p'_{n-d}, 2n]$. We deduce that any component of θ that is disjoint from both diagonals in $\widehat{\mathcal{P}}_X$ produces a quadratic component of $M_{P'}^{Q'}$. If a component of θ meets the SW diagonal of $\widehat{\mathcal{P}}_X$, then the corresponding component of $M_{P'}^{Q'}$ contains a row that crosses the middle, so this component is not quadratic. Finally, if a component of θ intersects the NE diagonal of $\widehat{\mathcal{P}}_X$, then the corresponding component (a,b) of $M_{P'}^{Q'}$ has fewer than $b-a$ rows, so it is not quadratic. It follows that $M_{P'}^{Q'}$ has exactly $N(\theta)$ quadratic components. □

Example 9.4. Let $X = \mathrm{LG}(12,24)$, $\mu = (12,11,9,6,5)$, and $\lambda = (11,8,6,3,1)$, and $d=2$. Then $\theta = \lambda[d]/\mu$ is the skew shape between the two thick black paths in the following picture. The connectors of θ are colored pink. We have $R(\theta) = 10$ and $N(\theta) = 1$.



The shapes μ and λ correspond to $Q = \{3,5,6,9,10,11,12,17,18,21,23,24\}$ and $P = \{1,3,4,6,8,9,11,13,15,18,20,23\}$. We obtain $\Gamma_d(X_\lambda, X^\mu) = X_{P'}^{Q'}$, where Q' and P' are determined by the shape of $M_{P'}^{Q'}$:

$$\left[\begin{array}{c|c|cc|cccc|c|cccccc|c|cccc|ccc|c}
\cdot & \cdot & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \star & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \star & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \star & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \star & \star & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \star & \star & \star & \cdot
\end{array}\right]$$

This diagram has $12 - R(\theta) - N(\theta) = 1$ lone stars, and $N(\theta) = 1$ quadratic components. The unique quadratic component is $(4, 8)$. The rows of $M_{P'}^{Q'}$ correspond to the connectors in θ, see the proof of Corollary 9.3. Rows 6, 7, and 10 are movable.

Consider a complete intersection $Y \subset \mathbb{P}^{a+b}$ of dimension b, defined by a quadratic equations. The K-theory class of Y is $[\mathcal{O}_Y] = (2H - H^2)^a$, where $H \in K(\mathbb{P}^{a+b})$ is the hyperplane class. It follows that the sheaf Euler characteristic of Y is given by $\chi(\mathcal{O}_Y) = h(a, b)$, where $h : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ is defined by [BR12, §4]

$$h(a, b) \;=\; \sum_{j=0}^{b} (-1)^j \, 2^{a-j} \binom{a}{j} . \tag{7}$$

Here we set $\binom{a}{j} = 0$ unless $0 \le j \le a$. Notice that for $b \ge a$ we have $h(a, b) = (2 - 1)^a = 1$, and $h(a, b) = 0$ for $b < 0$. We record for later the identity

$$h(a + 1, b) + h(a, b - 1) = 2\, h(a, b) \,, \tag{8}$$

which follows from the binomial formula. The following result is the quantum generalization of [BR12, Prop. 5.3].

Corollary 9.5. *The K-theoretic Gromov-Witten invariants of $X = \mathrm{LG}(n, 2n)$ of Pieri type are given by $I_d(\mathcal{O}_\lambda, \mathcal{O}^\mu, \mathcal{O}^p) = h(N(\theta), R(\theta) - p)$, with $\theta = \lambda[d]/\mu$.*

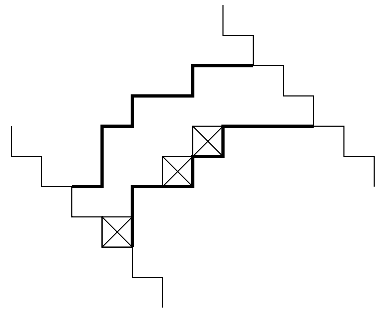
Proof. Let $L \subset \mathbb{P}^{2n-1}$ be the B^--stable linear subspace of dimension $n - p$. Then $\pi : \eta^{-1}(L) \to X^p$ is a birational isomorphism, so $\mathcal{O}^p = \pi_*(\eta^*([\mathcal{O}_L]))$. Using [BCMP18b, Thm. 4.1], the projection formula, and Corollary 9.3, we obtain

$$\begin{aligned}
I_d(\mathcal{O}_\lambda, \mathcal{O}^\mu, \mathcal{O}^p) \;&=\; \chi_X\left([\mathcal{O}_{\Gamma_d(X_\lambda, X^\mu)}] \cdot \pi_* \eta^* [\mathcal{O}_L]\right) \\
&=\; \chi_{\mathbb{P}^{2n-1}}\left([\mathcal{O}_{\eta(\pi^{-1}(\Gamma_d(X_\lambda, X^\mu)))}] \cdot [\mathcal{O}_L]\right) .
\end{aligned}$$

If $R(\theta) \le n$, then this is the sheaf Euler characteristic of a complete intersection of dimension $R(\theta) - p$ defined by $N(\theta)$ quadratic equations as well as linear equations in $\mathbb{P}^{2n-1}$, which proves the result. Finally, if $R(\theta) = n + 1$, then $I_d(\mathcal{O}_\lambda, \mathcal{O}^\mu, \mathcal{O}^p) = h(N(\theta), R(\theta) - p) = 1$, so the corollary also holds in this case. $\square$

9.3. **Quantum multiplication by special Schubert classes.** We finish this section by proving some preliminary formulas for quantum products with special Schubert classes. We start with the undeformed product $\mathcal{O}^p \odot \mathcal{O}^\mu$, see Section 2.5 or [BCMP18a, §2.5].

Given a skew shape $\theta \subset \widehat{\mathcal{P}}_X$, let $\theta^\circ \subset \theta$ be the skew shape obtained by removing all maximal boxes from θ that do not belong to the north-east diagonal of $\widehat{\mathcal{P}}_X$.



For $p \in \mathbb{Z}$ we then define

$$\mathcal{H}(\theta, p) \;=\; \sum_{\theta^\circ \subset \varphi \subset \theta} (-1)^{|\theta|-|\varphi|}\, h(N(\varphi), R(\varphi) - p)\,, \tag{9}$$

the sum over all subsets φ of θ that contain θ°.

Proposition 9.6. *For any shape $\mu \subset \widehat{\mathcal{P}}_X$ and $1 \leq p \leq n$, we have*

$$\mathcal{O}^p \odot \mathcal{O}^\mu \;=\; \sum_\nu \mathcal{H}(\nu/\mu, p)\, \mathcal{O}^\nu$$

in $\mathrm{QK}(X)_q$, where the sum is over all shapes $\nu \subset \widehat{\mathcal{P}}_X$ containing μ.

Proof. Given a shape $\nu \subset \mathcal{P}_X$ we let $\mathcal{I}_\nu \in K(X)$ denote the dual element of $\mathcal{O}^\nu$, defined by $\chi_X(\mathcal{I}_\nu \cdot \mathcal{O}^\lambda) = \delta_{\nu,\lambda}$ for all shapes $\lambda \subset \mathcal{P}_X$. We have [BR12, Lemma 3.5]

$$\mathcal{I}_\nu = \sum_{\nu/\kappa \text{ rook strip}} (-1)^{|\nu/\kappa|}\, \mathcal{O}_\kappa\,,$$

where the sum is over all shapes $\kappa \subset \nu$ such that ν/κ is a rook strip, that is, ν/κ has at most one box in each row and column. Assume that $\mu \subset \mathcal{P}_X$ is a classical shape. By Corollary 9.5 and equation (9) we have

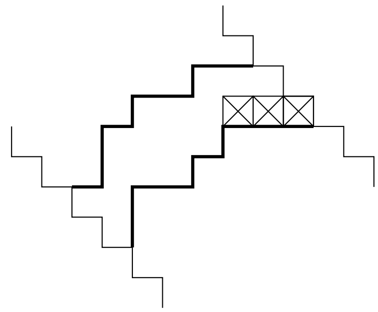
$$\begin{aligned} I_d(\mathcal{O}^p, \mathcal{O}^\mu, \mathcal{I}_\nu) &= \sum_{\nu/\kappa \text{ rook strip}} (-1)^{|\nu/\kappa|}\, I_d(\mathcal{O}^p, \mathcal{O}^\mu, \mathcal{O}_\kappa) \\ &= \sum_{\nu/\kappa \text{ rook strip}} (-1)^{|\nu/\kappa|}\, h\big(N(\kappa[d]/\mu), R(\kappa[d]/\mu) - p\big) \\ &= \mathcal{H}(\nu[d]/\mu, p)\,, \end{aligned}$$

where the sums are over all shapes $\kappa \subset \mathcal{P}_X$ such that $\mu \subset \kappa[d] \subset \nu[d]$ and ν/κ is a rook strip. By the definition of the undeformed product [BCMP18a, §2.5], we obtain

$$\mathcal{O}^p \odot \mathcal{O}^\mu = \sum_{\nu,d} I_d(\mathcal{O}^p, \mathcal{O}^\mu, \mathcal{I}_\nu)\, q^d\, \mathcal{O}^\nu = \sum_{\nu,d} \mathcal{H}(\nu[d]/\mu, p)\, \mathcal{O}^{\nu[d]}\,,$$

with the sum over $\nu \subset \mathcal{P}_X$ and $d \geq 0$ such that $\mu \subset \nu[d]$. The proposition is equivalent to this identity. □

We next consider the associative quantum product $\mathcal{O}^p \star \mathcal{O}^\mu$. Given a skew shape $\theta \subset \widehat{\mathcal{P}}_X$, let $\theta^- \subset \theta$ be the skew shape obtained by removing the maximal box on the north-east diagonal, if any, as well as any boxes in the same row that do not have a box immediately below them in θ.



For $p \in \mathbb{Z}$ we then define

$$\widehat{\mathcal{N}}(\theta,p) \;=\; \mathcal{H}(\theta,p) - \sum_{\theta^- \subset \varphi \subsetneq \theta} \mathcal{H}(\varphi,p)\,, \tag{10}$$

the sum over all proper lower order ideals φ in θ that contain θ^-.

We will prove in Corollary 10.11 that $\widehat{\mathcal{N}}(\theta,p) = \mathcal{N}(\theta,p)$ holds for all skew shapes $\theta \subset \widehat{\mathcal{P}}_X$ and $p \in \mathbb{Z}$, that is, $\widehat{\mathcal{N}}(\theta,p)$ is equal to $(-1)^{|\theta|-p}$ times the number of QKLG-tableaux of shape θ with content $\{1,2,\dots,p\}$. Theorem 7.4 is therefore equivalent to the following statement.

Proposition 9.7. *For any shape $\mu \subset \widehat{\mathcal{P}}_X$ and $1 \le p \le n$, we have*

$$\mathcal{O}^p \star \mathcal{O}^\mu \;=\; \sum_\nu \widehat{\mathcal{N}}(\nu/\mu,p)\,\mathcal{O}^\nu$$

in $\mathrm{QK}(X)_q$, where the sum is over all shapes $\nu \subset \widehat{\mathcal{P}}_X$ containing μ.

Proof. For any shape $\lambda \subset \widehat{\mathcal{P}}_X$, set $\lambda^+ = \lambda \cup I(q^{d+1})$, where $d \in \mathbb{Z}$ is maximal with $I(q^d) \subset \lambda$. In other words, $\lambda^+ \subset \widehat{\mathcal{P}}_X$ is the smallest shape that contains λ and contains one more box than λ on the north-east diagonal of $\widehat{\mathcal{P}}_X$. We then have $q\,\psi(\mathcal{O}^\lambda) = \mathcal{O}^{\lambda^+}$, where ψ is the line neighborhood operator from Section 2.5. It therefore follows from Proposition 9.6 that the coefficient of $\mathcal{O}^\nu$ in the product

$$\mathcal{O}^p \star \mathcal{O}^\mu \;=\; \mathcal{O}^p \odot \mathcal{O}^\mu - q\,\psi(\mathcal{O}^p \odot \mathcal{O}^\mu) \tag{11}$$

is equal to

$$\mathcal{H}(\nu/\mu,p) - \sum_{\lambda:\,\mu\subset\lambda \text{ and } \lambda^+=\nu} \mathcal{H}(\lambda/\mu,p) \;=\; \widehat{\mathcal{N}}(\nu/\mu,p)\,,$$

as required. ☐

Remark 9.8. The constants $\widehat{\mathcal{N}}(\theta,p)$ have alternating signs by Corollary 10.11, but the constants $\mathcal{H}(\theta,p)$ do not have easily predictable signs.

10. Combinatorial Identities

In this section we complete the proof of Theorem 7.4. Let $X = \mathrm{LG}(n,2n)$ be a Lagrangian Grassmannian. Any shape $\lambda \subset \mathcal{P}_X$ and integer $1 \le p \le n$ define three products

$$\mathcal{O}^p \cdot \mathcal{O}^\lambda = \sum_\nu \mathcal{C}(\nu/\lambda, p)\, \mathcal{O}^\nu \in K(X)\,,$$
$$\mathcal{O}^p \odot \mathcal{O}^\lambda = \sum_\nu \mathcal{H}(\nu/\lambda, p)\, \mathcal{O}^\nu \in \mathrm{QK}(X)\,, \quad \text{and}$$
$$\mathcal{O}^p \star \mathcal{O}^\lambda = \sum_\nu \widehat{\mathcal{N}}(\nu/\lambda, p)\, \mathcal{O}^\nu \in \mathrm{QK}(X)\,.$$

The first sum is over all shapes $\nu \subset \mathcal{P}_X$ containing λ, and the two last sums are over all quantum shapes $\nu \subset \widehat{\mathcal{P}}_X$ containing λ. The constants $\mathcal{H}(\theta, p)$ and $\widehat{\mathcal{N}}(\theta, p)$ are defined whenever θ is a skew shape in $\widehat{\mathcal{P}}_X$, and these constants depend on where θ is located in $\widehat{\mathcal{P}}_X$, including whether θ meets the two diagonals in $\widehat{\mathcal{P}}_X$. The constants $\mathcal{H}(\theta, p)$ and $\widehat{\mathcal{N}}(\theta, p)$ are therefore bound to our chosen Lagrangian Grassmannian $X = \mathrm{LG}(n, 2n)$. On the other hand, the constant $\mathcal{C}(\theta, p)$ does not depend on any NE diagonal, and its definition extends naturally to any (finite) skew shape θ in the partially ordered set $\mathcal{P}_X^\infty = \bigcup_m \widehat{\mathcal{P}}_{\mathrm{LG}(m,2m)}$, which is unbounded in north-east direction. This is equivalent to considering $\mathcal{C}(\theta, p)$ as a structure constant of $\varprojlim K(\mathrm{LG}(m, 2m))$. Notice that $\mathcal{C}(\theta, p) = \mathcal{H}(\theta, p) = \widehat{\mathcal{N}}(\theta, p)$ holds whenever $\theta \subset \widehat{\mathcal{P}}_X$ is disjoint from the NE diagonal.

Theorem 7.4 states that each quantum structure constant $\widehat{\mathcal{N}}(\theta, p)$ is equal to the (signed) number $\mathcal{N}(\theta, p)$ of QKLG-tableaux. We prove this by showing that $\widehat{\mathcal{N}}(\theta, p)$ and $\mathcal{N}(\theta, p)$ are determined by the same recursive identities. These identities simultaneously provide an alternative definition of these constants. We also prove an analogous recursive definition of the undeformed structure constants $\mathcal{H}(\theta, p)$ when θ contains at most one box on the NE diagonal of $\widehat{\mathcal{P}}_X$. Our recursive definitions refer to (quantum or undeformed) structure constants computed in the quantum K-theory of smaller Lagrangian Grassmannians $X' = \mathrm{LG}(n', 2n')$. For this reason we will introduce additional notation to make it easier to refer to the constants $\mathcal{H}(\theta, p)$ and $\widehat{\mathcal{N}}(\theta, p)$ when θ is regarded as a skew shape in $\widehat{\mathcal{P}}_{X'}$. We summarize this notation here and give precise definitions below. We will regard any skew shape θ as a subset of $\mathcal{P}_X^\infty$. Suppose θ is contained in a specific set $\widehat{\mathcal{P}}_{X'}$, and we wish to refer to the constants $\mathcal{H}(\theta, p)$ and $\widehat{\mathcal{N}}(\theta, p)$ computed in $\mathrm{QK}(X')$. If θ is disjoint from the NE diagonal of $\widehat{\mathcal{P}}_{X'}$, then we can use the structure constant $\mathcal{C}(\theta, p)$ of the ordinary K-theory ring $K(X)$. On the other hand, if θ meets the NE diagonal of $\widehat{\mathcal{P}}_{X'}$, then the values of $\mathcal{H}(\theta, p)$ and $\widehat{\mathcal{N}}(\theta, p)$ computed in $\mathrm{QK}(X')$ will be denoted $\mathcal{H}_q(\theta, p)$ and $\mathcal{N}_q(\theta, p)$. Equivalently, given any skew shape $\theta \subset \mathcal{P}_X^\infty$, we can define $\mathcal{H}_q(\theta, p)$ and $\mathcal{N}_q(\theta, p)$ as the values of $\mathcal{H}(\theta, p)$ and $\widehat{\mathcal{N}}(\theta, p)$ computed in $\mathrm{QK}(X')$, where $X' = \mathrm{LG}(n', 2n')$ is the smallest Lagrangian Grassmannian for which $\theta \subset \widehat{\mathcal{P}}_{X'}$.

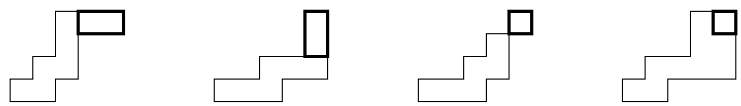
Define $\mathcal{P}_X^\infty = \{(i,j) \in \mathbb{Z}^2 \mid i \leq j\}$, and equip this set with the partial order defined by $(i', j') \leq (i'', j'')$ if and only if $i' \leq i''$ and $j' \leq j''$. We will consider $\mathcal{P}_X$ and $\widehat{\mathcal{P}}_X$ as subsets of $\mathcal{P}_X^\infty$. Define a *skew shape* in $\mathcal{P}_X^\infty$ to be any finite subset obtained as the difference between two lower order ideals. Given a skew shape $\theta \subset \mathcal{P}_X^\infty$, let $R(\theta)$ denote the size of a maximal rim contained in θ, and let $N'(\theta)$ be the number of components of θ that are disjoint from the SW diagonal. Let θ' denote the skew shape obtained by removing all south-east corners from θ. Given an integer $p \in \mathbb{Z}$, it was proved in [BR12] that the constant $\mathcal{C}(\theta, p)$ from Definition 7.2

is given by

$$\mathcal{C}(\theta, p) = \sum_{\theta' \subset \varphi \subset \theta} (-1)^{|\theta|-|\varphi|}\, h(N'(\varphi), R(\varphi) - p)\,,$$

where the function $h : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ is defined by (7).

Let $\theta \subset \mathcal{P}_X^\infty$ be a non-empty skew shape. Then θ contains a unique *north-east box* Q. A skew shape in $\mathcal{P}_X^\infty$ will be called a *line* if its boxes are contained in a single row or a single column. The *north-east arm* of θ is the largest line ψ that can be obtained by intersecting θ with a square whose upper-right box is Q.



We will say that the north-east arm ψ is a *row* if θ contains no box immediately below Q, and ψ is a *column* if θ contains no box immediately to the left of Q. Notice that ψ can be both a row and a column (if it is a disconnected single box), and it can be neither a row nor a column (only if θ is not a rim). We let $\widehat{\theta} = \theta \smallsetminus \psi$ denote the complement of the north-east arm. This set $\widehat{\theta}$ is a skew shape if and only if ψ is a row or a column. If ψ is not connected to $\widehat{\theta}$, then ψ is not a row if and only if ψ is a column with at least two boxes, and ψ is not a column if and only if ψ is a row with at least two boxes. We set $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

Proposition 10.1 ([BR12]). *Let $\theta \subset \mathcal{P}_X^\infty$ be any skew shape and let $p \in \mathbb{Z}$. If θ is not a rim, then $\mathcal{C}(\theta, p) = 0$, and $\mathcal{C}(\emptyset, p) = \chi(p \leq 0)$. If θ is a non-empty rim with north-east arm $\psi = \theta \smallsetminus \widehat{\theta}$ of size a, then $\mathcal{C}(\theta, p)$ is determined by the following rules.*

(i) *If $\widehat{\theta} = \emptyset$ and θ meets the SW diagonal, then $\mathcal{C}(\theta, p) = \delta_{p,|\theta|}$ if θ is a row, and $\mathcal{C}(\theta, p) = \delta_{p,|\theta|} - \delta_{p,|\theta|-1}$ if θ is not a row.*

(ii) *If $\widehat{\theta} = \emptyset$ and θ is disjoint from the SW diagonal, then $\mathcal{C}(\theta, p) = 2\,\delta_{p,|\theta|} - \chi(p \geq 1)\,\delta_{p,|\theta|-1}$.*

(iii) *If $\widehat{\theta} \neq \emptyset$ and ψ is connected to $\widehat{\theta}$, then $\mathcal{C}(\theta, p) = \mathcal{C}(\widehat{\theta}, p - a) - \mathcal{C}(\widehat{\theta}, p - a + 1)$.*

(iv) *If $\widehat{\theta} \neq \emptyset$ and ψ is not connected to $\widehat{\theta}$, then $\mathcal{C}(\theta, p) = 2\,\mathcal{C}(\widehat{\theta}, p-a) - 2\,\mathcal{C}(\widehat{\theta}, p-a+1)$ if $a = 1$, and $\mathcal{C}(\theta, p) = 2\,\mathcal{C}(\widehat{\theta}, p - a) - 3\,\mathcal{C}(\widehat{\theta}, p - a + 1) + \mathcal{C}(\widehat{\theta}, p - a + 2)$ if $a \geq 2$.*

Given a non-empty skew shape $\theta \subset \mathcal{P}_X^\infty$ with north-east box Q, let $N'_q(\theta) = \max(N'(\theta) - 1, 0)$ be the number of components of θ that do not meet the SW diagonal and do not contain Q, and let $\theta'_q = \theta' \cup Q$ be the result of removing all south-east corners except Q (in case Q is a south-east corner). For $p \in \mathbb{Z}$ we define

$$\mathcal{H}_q(\theta, p) = \sum_{\theta'_q \subset \varphi \subset \theta} (-1)^{|\theta|-|\varphi|}\, h(N'_q(\varphi), R(\varphi) - p)\,. \tag{12}$$

Remark 10.2. Assume that $\theta \subset \widehat{\mathcal{P}}_X$ is a skew shape containing at most one box from the NE diagonal of $\widehat{\mathcal{P}}_X$, for example a rim. Then the constant $\mathcal{H}(\theta, p)$ defined by equation (9) is given by

$$\mathcal{H}(\theta, p) = \begin{cases} \mathcal{C}(\theta, p) & \text{if } \theta \text{ is disjoint from the NE diagonal,} \\ \mathcal{H}_q(\theta, p) & \text{if } \theta \text{ contains one box on the NE diagonal.} \end{cases}$$

If $\theta \subset \widehat{\mathcal{P}}_X$ contains two or more boxes from the NE diagonal, then θ is not a skew shape in $\mathcal{P}_X^\infty$ and $\mathcal{H}_q(\theta, p)$ is not defined. Our next result together with Proposition 10.1 provides a recursive definition of the constants $\mathcal{H}_q(\theta, p)$.

Proposition 10.3. *Let $\theta \subset \mathcal{P}_X^\infty$ be any non-empty skew shape and let $p \in \mathbb{Z}$. If θ is not a rim, then $\mathcal{H}_q(\theta, p) = 0$. If θ is a rim with north-east arm $\psi = \theta \smallsetminus \widehat{\theta}$ of size a, then $\mathcal{H}_q(\theta, p)$ is determined by the following rules.*

(i$''$) *If $\widehat{\theta} = \emptyset$, then $\mathcal{H}_q(\theta, p) = \chi(p \le |\theta|)$ if θ is a row, and $\mathcal{H}_q(\theta, p) = \delta_{p,|\theta|}$ if θ is not a row.*

(iii$''$) *If $\widehat{\theta} \neq \emptyset$ and ψ is connected to $\widehat{\theta}$, then $\mathcal{H}_q(\theta, p) = \mathcal{H}_q(\widehat{\theta}, p - a)$ if ψ is a row or $p \ge |\theta|$, and $\mathcal{H}_q(\theta, p) = \mathcal{C}(\widehat{\theta}, p - a) - \mathcal{H}_q(\widehat{\theta}, p - a) + \mathcal{H}_q(\widehat{\theta}, p - a + 1)$ if ψ is a column and $p < |\theta|$.*

(iv$''$) *If $\widehat{\theta} \neq \emptyset$ and ψ is not connected to $\widehat{\theta}$, then $\mathcal{H}_q(\theta, p) = \mathcal{C}(\widehat{\theta}, p - a)$ if ψ is a row, and $\mathcal{H}_q(\theta, p) = \mathcal{C}(\widehat{\theta}, p - a) - \mathcal{C}(\widehat{\theta}, p - a + 1)$ if ψ is not a row.*

Proof. If θ is not a rim, then let $B \in \theta$ be a south-east corner such that θ contains a box strictly north and strictly west of B. For any skew shape φ with $\theta'_q \subset \varphi \subset \theta \smallsetminus B$ we have $h(N'_q(\varphi), R(\varphi) - p) = h(N'_q(\varphi \cup B), R(\varphi \cup B) - p)$, which implies that $\mathcal{H}_q(\theta, p) = 0$. We can therefore assume that θ is a non-empty rim. If $\theta = \psi$ is a row, then $\mathcal{H}_q(\theta, p) = h(0, |\theta| - p) = \chi(p \le |\theta|)$. If $\theta = \psi$ is not a row, and B is the bottom box of θ, then $\mathcal{H}_q(\theta, p) = h(0, |\theta| - p) - h(0, |\theta \smallsetminus B| - p) = \delta_{p,|\theta|}$.

Assume that $\widehat{\theta} \neq \emptyset$ and ψ is a row connected to $\widehat{\theta}$. Then the skew shapes occurring in (12) have the form $\varphi \cup \psi$, where $\widehat{\theta}' = (\widehat{\theta})' \subset \varphi \subset \widehat{\theta}$. Since $h(N'_q(\varphi \cup \psi), |\varphi \cup \psi| - p) = h(N'_q(\varphi), |\varphi| - p + a)$, we obtain $\mathcal{H}_q(\theta, p) = \mathcal{H}_q(\widehat{\theta}, p - a)$.

Assume that $\widehat{\theta} \neq \emptyset$ and ψ is a column connected to $\widehat{\theta}$. If $p \ge |\theta|$, then since $h(N'_q(\varphi), |\varphi| - p)$ is non-zero only when $p \le |\varphi|$, we obtain

$$\mathcal{H}_q(\theta, p) = h(N'_q(\theta), |\theta| - p) = h(N'_q(\widehat{\theta}), |\widehat{\theta}| - p + a) = \mathcal{H}_q(\widehat{\theta}, p - a)\,.$$

Assume that $p < |\theta|$ and let B be the north-east box of $\widehat{\theta}$. Then

$$\mathcal{H}_q(\theta, p) - \mathcal{C}(\widehat{\theta}, p - a) + \mathcal{H}_q(\widehat{\theta}, p - a) - \mathcal{H}_q(\widehat{\theta}, p - a + 1)$$

is equal to the sum over all skew shapes φ, with $\widehat{\theta}' \subset \varphi \subset \widehat{\theta} \smallsetminus B$, of $(-1)^{|\widehat{\theta}| - |\varphi|}$ times

$$\begin{aligned} &h(N'_q(\varphi \cup \psi), |\varphi \cup \psi| - p) - h(N'_q(\varphi \cup B \cup \psi), |\varphi \cup B \cup \psi| - p) \\ &- h(N'(\varphi), |\varphi| - p + a) + h(N'(\varphi \cup B), |\varphi \cup B| - p + a) \\ &- h(N'_q(\varphi \cup B), |\varphi \cup B| - p + a) + h(N'_q(\varphi \cup B), |\varphi \cup B| - p + a - 1)\,. \end{aligned} \tag{13}$$

Using that

$$N'_q(\varphi \cup \psi) = N'(\varphi \cup B) = N'(\varphi) \quad \text{and} \quad N'_q(\varphi \cup B \cup \psi) = N'_q(\varphi \cup B) = N'_q(\varphi)\,,$$

it follows that (13) is equal to

$$h(N'(\varphi), |\varphi| - p + a + 1) + h(N'_q(\varphi), |\varphi| - p + a) - 2\, h(N'_q(\varphi), |\varphi| - p + a + 1)\,.$$

If $N'(\varphi) > 0$, then this expression is zero by identity (8). Otherwise we have $N'(\varphi) = N'_q(\varphi) = 0$, which implies $\varphi = \widehat{\theta} \smallsetminus B$, so the expression is zero because $|\varphi| - p + a \ge 0$.

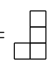
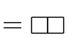
We finally assume that $\widehat{\theta} \neq \emptyset$ and ψ is not connected to $\widehat{\theta}$. If ψ is a row, then

$$\begin{aligned}
\mathcal{H}_q(\theta,p) &= \sum_{\widehat{\theta}' \subset \varphi \subset \widehat{\theta}} (-1)^{|\theta|-|\varphi\cup\psi|}\, h(N'_q(\varphi\cup\psi), |\varphi\cup\psi|-p) \\
&= \sum_{\widehat{\theta}' \subset \varphi \subset \widehat{\theta}} (-1)^{|\widehat{\theta}|-|\varphi|}\, h(N'(\varphi), |\varphi|-p+a) \;=\; \mathcal{C}(\widehat{\theta}, p-a)\,.
\end{aligned}$$

If ψ is not a row, B is the bottom box of ψ, and $\psi' = \psi \smallsetminus B$, then

$$\begin{aligned}
&\mathcal{H}_q(\theta,p) \\
&= \sum_{\widehat{\theta}' \subset \varphi \subset \widehat{\theta}} (-1)^{|\theta|-|\varphi\cup\psi|} \big(h(N'_q(\varphi\cup\psi), |\varphi\cup\psi|-p) - h(N'_q(\varphi\cup\psi'), |\varphi\cup\psi'|-p) \big) \\
&= \sum_{\widehat{\theta}' \subset \varphi \subset \widehat{\theta}} (-1)^{|\widehat{\theta}|-|\varphi|} \left(h(N'(\varphi), |\varphi|-p+a) - h(N'(\varphi), |\varphi|-p+a-1) \right) \\
&= \mathcal{C}(\widehat{\theta}, p-a) - \mathcal{C}(\widehat{\theta}, p-a+1)\,.
\end{aligned}$$

This completes the proof. $\square$

Example 10.4. For any skew shape $\theta =$  $\subset \mathcal{P}_X^\infty$ and $p \leq 2$, we obtain $\widehat{\theta} =$  and

$$\mathcal{H}_q(\theta,p) = \mathcal{C}(\widehat{\theta}, p-2) - \mathcal{H}_q(\widehat{\theta}, p-2) + \mathcal{H}_q(\widehat{\theta}, p-1) = 0 - 1 + 1 = 0\,.$$

This illustrates that negative values of p must be allowed in Proposition 10.3 to obtain correct recursive identities without including additional special cases.

Definition 10.5. Given a non-empty skew shape $\theta \subset \mathcal{P}_X^\infty$ and $p \in \mathbb{Z}$, define an integer $\mathcal{N}_q(\theta,p)$ as follows. If θ is not a rim, then $\mathcal{N}_q(\theta,p) = 0$. If θ is a rim with north-east arm $\psi = \theta \smallsetminus \widehat{\theta}$ of size a, then $\mathcal{N}_q(\theta,p)$ is determined by the following rules.

(i′) If $\widehat{\theta} = \emptyset$ and θ meets the SW diagonal, then $\mathcal{N}_q(\theta,p) = \delta_{p,|\theta|}$.

(ii′) If $\widehat{\theta} = \emptyset$ and θ is disjoint from the SW diagonal, then $\mathcal{N}_q(\theta,p) = \delta_{p,|\theta|}$ if θ is a column, and $\mathcal{N}_q(\theta,p) = \delta_{p,|\theta|} - \delta_{p,|\theta|-1}$ if θ is not a column.

(iii′) If $\widehat{\theta} \neq \emptyset$ and ψ is connected to $\widehat{\theta}$, then $\mathcal{N}_q(\theta,p) = \mathcal{N}_q(\widehat{\theta}, p-a)$ if ψ is a column, and $\mathcal{N}_q(\theta,p) = \mathcal{N}_q(\widehat{\theta}, p-a) - \mathcal{C}(\widehat{\theta}, p-a+1)$ if ψ is a row.

(iv′) If $\widehat{\theta} \neq \emptyset$ and ψ is not connected to $\widehat{\theta}$, then $\mathcal{N}_q(\theta,p) = \mathcal{C}(\widehat{\theta}, p-a) - \mathcal{C}(\widehat{\theta}, p-a+1)$ if ψ is a column, and $\mathcal{N}_q(\theta,p) = \mathcal{C}(\widehat{\theta}, p-a) - 2\,\mathcal{C}(\widehat{\theta}, p-a+1) + \mathcal{C}(\widehat{\theta}, p-a+2)$ if ψ is not a column.

Recall from Definition 7.3 that $|\mathcal{N}(\theta,p)|$ is the number of QKLG-tableaux of shape $\theta \subset \widehat{\mathcal{P}}_X$ with content $\{1, 2, \ldots, p\}$.

Lemma 10.6. *Let $\theta \subset \widehat{\mathcal{P}}_X$ be a rim meeting the NE diagonal of $\widehat{\mathcal{P}}_X$ and let $p \in \mathbb{Z}$. Then $\mathcal{N}_q(\theta,p) = \mathcal{N}(\theta,p)$.*

Proof. Let $\psi = \theta \smallsetminus \widehat{\theta}$ be the north-east arm and set $a = |\psi|$. The pictures in this proof will be drawn for the case $a = 4$. Assume first that $\widehat{\theta} = \emptyset$. If θ meets the SW diagonal of $\widehat{\mathcal{P}}_X$, then there exists only one QKLG-tableau of shape θ, which is one of the following cases:

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&a\\\hline\end{array} \qquad \text{or} \qquad \begin{array}{|c|}\hline 1'\\\hline 2'\\\hline 3'\\\hline a\\\hline\end{array}$$

If θ is disjoint from the SW diagonal, then there is a unique QKLG-tableau of shape θ when θ is a column or a single box, and exactly two QKLG-tableaux of shape θ when θ is a row with at least two boxes:

$$\begin{array}{|c|}\hline 1'\\\hline 2'\\\hline 3'\\\hline a'\\\hline\end{array} \qquad \text{or} \qquad \begin{array}{|c|c|c|c|}\hline 1'&2&3&a\\\hline\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|}\hline 1'&1&2&b\\\hline\end{array}\,, \quad \text{where } b = a-1.$$

This accounts for cases (i′) and (ii′) of Definition 10.5.

Assume next that $\widehat{\theta} \neq \emptyset$ and that ψ is connected to $\widehat{\theta}$. Then any QKLG-tableau of shape θ and content $\{1, \ldots, p\}$ must assign the following labels to the boxes of ψ (with $b_i = p - a + i$):

$$\begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{1'}\\\cline{2-2} & \multicolumn{1}{|c|}{2'}\\\cline{2-2} & \multicolumn{1}{|c|}{3'}\\\cline{2-2} & \multicolumn{1}{|c|}{a'}\\\hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{p}\\\hline\end{array} \qquad \text{or} \qquad \begin{array}{|c|c|c|c|c|}\hline 1'&b_1&b_2&b_3&p\\\hline \multicolumn{1}{|c|}{} \\\cline{1-1}\end{array}$$

The pictures also show two of the boxes from $\widehat{\theta}$. If ψ is a column, then a' must be an unrepeated quantum box, so the labels of $\widehat{\theta}$ can be any QKLG-tableau with content $\{a+1, a+2, \ldots, p\}$ (with p considered on the NE diagonal). If ψ is a row, then the labels of $\widehat{\theta}$ must have content either $\{1, 2, \ldots, p-a\}$ or $\{1, 2, \ldots, p-a+1\}$. In the first case b_1 is an unrepeated quantum box, so $1'$ is also a quantum box, and the labels of $\widehat{\theta}$ can be any QKLG-tableau with content $\{1, 2, \ldots, p-a\}$ (with $1'$ considered on the NE diagonal). In the second case b_1 is repeated, $1'$ is not a quantum box, so the labels of $\widehat{\theta}$ can be any KLG-tableau with content $\{1, 2, \ldots, p-a+1\}$. This accounts for case (iii′) of Definition 10.5.

Finally, assume that $\widehat{\theta} \neq \emptyset$ and ψ is not connected to $\widehat{\theta}$. Then any QKLG-tableau of shape θ and content $\{1, \ldots, p\}$ must assign the following labels to the boxes of ψ (with $b_i = p - a + i$):

$$\begin{array}{|c|}\hline 1'\\\hline 2'\\\hline 3'\\\hline a'\\\hline\end{array} \qquad \text{or} \qquad \begin{array}{|c|c|c|c|}\hline 1'&b_2&b_3&p\\\hline\end{array}$$

If ψ is a column or a single box, then the labels of $\widehat{\theta}$ must form a KLG-tableau with content $\{a+1, a+2, \ldots, p\}$ or $\{a, a+1, \ldots, p\}$. If ψ is a row with at least two boxes, then the labels of $\widehat{\theta}$ must form a KLG-tableau with content $\{2, 3, \ldots, p-a+1\}$, $\{1, 2, \ldots, p-a+1\}$, $\{2, 3, \ldots, p-a+2\}$, or $\{1, 3, \ldots, p-a+2\}$. This accounts for case (iv′) of Definition 10.5. $\square$

Lemma 10.7. *Let $\theta \subset \mathcal{P}_X^\infty$ be a non-empty rim and let $p \in \mathbb{Z}$.*

(a) *For $p \leq 0$ we have $\mathcal{C}(\theta,p) = \mathcal{N}_q(\theta,p) = 0$, and*

$$\mathcal{H}_q(\theta,p) = \begin{cases} 1 & \text{if } \theta \text{ is a single row of boxes,} \\ 0 & \text{otherwise.} \end{cases}$$

(b) *We have $\mathcal{C}(\theta,|\theta|) = 2^{N'(\theta)}$ and $\mathcal{N}_q(\theta,|\theta|) = \mathcal{H}_q(\theta,|\theta|) = 2^{N'_q(\theta)}$.*

(c) *For $p > |\theta|$ we have $\mathcal{C}(\theta,p) = \mathcal{N}_q(\theta,p) = \mathcal{H}_q(\theta,p) = 0$.*

Proof. These identities follow from the recursive definitions by induction on $|\theta|$. □

Lemma 10.8. *Let $\theta \subset \mathcal{P}_X^\infty$ be a non-empty rim, such that the north-east arm $\psi = \theta \smallsetminus \widehat{\theta}$ is not a disconnected single box, and let $p < |\theta|$. Then,*

$$2\mathcal{N}_q(\theta,p) - \mathcal{N}_q(\theta,p+1) \;=\; \begin{cases} \mathcal{C}(\theta,p) - \mathcal{C}(\theta,p+1) & \text{if } \psi \text{ is a row,} \\ \mathcal{C}(\theta,p) & \text{if } \psi \text{ is a column.} \end{cases}$$

Proof. Assume that $\widehat{\theta} = \emptyset$. If ψ meets the SW diagonal or is a column, then both sides of the identity are equal to $-\delta_{p+1,|\theta|}$, and otherwise both sides are equal to $-3\,\delta_{p+1,|\theta|} + \delta_{p+2,|\theta|}$.

Assume next that $\widehat{\theta} \neq \emptyset$ and ψ is connected to $\widehat{\theta}$. Set $a = |\psi|$. If ψ is a row, then

$$\begin{aligned} &2\,\mathcal{N}_q(\theta,p) - \mathcal{N}_q(\theta,p+1) - \mathcal{C}(\theta,p) + \mathcal{C}(\theta,p+1) \\ &\qquad = \; 2\,\mathcal{N}_q(\widehat{\theta},p-a) - \mathcal{N}_q(\widehat{\theta},p-a+1) - \mathcal{C}(\widehat{\theta},p-a)\,, \end{aligned}$$

which vanishes by induction on $|\theta|$, since the north-east arm of $\widehat{\theta}$ is a column. If ψ is a column, then

$$\begin{aligned} &2\,\mathcal{N}_q(\theta,p) - \mathcal{N}_q(\theta,p+1) - \mathcal{C}(\theta,p) \\ &\qquad = \; 2\,\mathcal{N}_q(\widehat{\theta},p-a) - \mathcal{N}_q(\widehat{\theta},p-a+1) - \mathcal{C}(\widehat{\theta},p-a) + \mathcal{C}(\widehat{\theta},p-a+1) \end{aligned}$$

which vanishes by induction on $|\theta|$, since the north-east arm of $\widehat{\theta}$ is a row.

Finally we assume that $\widehat{\theta} \neq \emptyset$ and ψ is not connected to $\widehat{\theta}$. If ψ is a column, then both sides are equal to

$$2\,\mathcal{C}(\widehat{\theta},p-a) - 3\,\mathcal{C}(\widehat{\theta},p-a+1) + \mathcal{C}(\widehat{\theta},p-a+2)\,,$$

and if ψ is a row, then both sides are equal to

$$2\,\mathcal{C}(\widehat{\theta},p-a) - 5\,\mathcal{C}(\widehat{\theta},p-a+1) + 4\,\mathcal{C}(\widehat{\theta},p-a+2) - \mathcal{C}(\widehat{\theta},p-a+3)\,.$$

The identity follows from this. □

Lemma 10.9. *Let $\theta \subset \mathcal{P}_X^\infty$ be a non-empty rim and let $p < |\theta|$. Then,*

$$\mathcal{H}_q(\theta,p) - \mathcal{H}_q(\theta,p+1) \;=\; \mathcal{C}(\theta,p) - \mathcal{N}_q(\theta,p)\,.$$

Proof. Let $\psi = \theta \smallsetminus \widehat{\theta}$ be the north-east arm of θ and set $a = |\psi|$. Assume first that $\widehat{\theta} = \emptyset$. If ψ is a row, then both sides of the identity are zero, and otherwise both sides are equal to $-\delta_{p+1,|\theta|}$.

Assume next that $\widehat{\theta} \neq \emptyset$ and ψ is connected to $\widehat{\theta}$. If ψ is a row, then it follows by induction on $|\theta|$ that

$$\begin{aligned} \mathcal{H}_q(\theta,p) - \mathcal{H}_q(\theta,p+1) &= \mathcal{H}_q(\widehat{\theta},p-a) - \mathcal{H}_q(\widehat{\theta},p-a+1) \\ &= \mathcal{C}(\widehat{\theta},p-a) - \mathcal{N}_q(\widehat{\theta},p-a) = \mathcal{C}(\theta,p) - \mathcal{N}_q(\theta,p)\,. \end{aligned}$$

If ψ is a column and $p \leq |\theta|-2$, then the recursive definitions and induction on $|\theta|$ yield

$$\begin{aligned}
&\mathcal{H}_q(\theta,p) - \mathcal{H}_q(\theta,p+1) - \mathcal{C}(\theta,p) + \mathcal{N}_q(\theta,p) \\
&\quad = -\mathcal{H}_q(\widehat{\theta},p-a) + 2\,\mathcal{H}_q(\widehat{\theta},p-a+1) - \mathcal{H}_q(\widehat{\theta},p-a+2) + \mathcal{N}_q(\widehat{\theta},p-a) \\
&\quad = 2\,\mathcal{N}_q(\widehat{\theta},p-a) - \mathcal{N}_q(\widehat{\theta},p-a+1) - \mathcal{C}(\widehat{\theta},p-a) + \mathcal{C}(\widehat{\theta},p-a+1)\,.
\end{aligned}$$

This expression is equal to zero by Lemma 10.8, as the north-east arm of $\widehat{\theta}$ is a row. If ψ is a column and $p = |\theta|-1$, then the recursive definitions and induction on $|\theta|$ gives

$$\begin{aligned}
&\mathcal{H}_q(\theta,p) - \mathcal{H}_q(\theta,p+1) - \mathcal{C}(\theta,p) + \mathcal{N}_q(\theta,p) \\
&\quad = \mathcal{N}_q(\widehat{\theta},p-a) - \mathcal{H}_q(\widehat{\theta},p-a) + \mathcal{C}(\widehat{\theta},p-a+1) \\
&\quad = 2\,\mathcal{N}_q(\widehat{\theta},p-a) - \mathcal{H}_q(\widehat{\theta},p-a+1) - \mathcal{C}(\widehat{\theta},p-a) + \mathcal{C}(\widehat{\theta},p-a+1)\,.
\end{aligned}$$

This expression is equal to zero by Lemma 10.7(b) and Lemma 10.8, as $p-a+1 = |\widehat{\theta}|$ and the north-east arm of $\widehat{\theta}$ is a row.

Finally assume that $\widehat{\theta} \neq \emptyset$ and ψ is not connected to $\widehat{\theta}$. If ψ is a row, then both sides of the identity are equal to $\mathcal{C}(\widehat{\theta},p-a) - \mathcal{C}(\widehat{\theta},p-a+1)$, and otherwise both sides are equal to $\mathcal{C}(\widehat{\theta},p-a) - 2\,\mathcal{C}(\widehat{\theta},p-a+1) + \mathcal{C}(\widehat{\theta},p-a+2)$. This proves the identity. $\square$

Proposition 10.10. *Let $\theta \subset \mathcal{P}_X^\infty$ be a non-empty skew shape with north-east arm $\psi = \theta \smallsetminus \widehat{\theta}$, and let $p \in \mathbb{Z}$. Then,*

$$\mathcal{H}_q(\theta,p) - \mathcal{N}_q(\theta,p) = \begin{cases} \sum_{\widehat{\theta} \subset \varphi \subsetneq \theta} \mathcal{C}(\varphi,p) & \text{if } \psi \text{ is a row,} \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over all proper lower order ideals φ of θ that contain $\widehat{\theta}$.

Proof. We may assume that θ is a rim, since otherwise $\widehat{\theta}$ is also not a rim, and both sides of the identity vanish. Set $a = |\psi|$. Assume first that ψ is not a row. If $\widehat{\theta} = \emptyset$, then $\mathcal{H}_q(\theta,p) = \delta_{p,|\theta|} = \mathcal{N}_q(\theta,p)$. If $\widehat{\theta} \neq \emptyset$ and ψ is connected to $\widehat{\theta}$, then $\mathcal{H}_q(\theta,p) = \mathcal{N}_q(\theta,p)$ for $p \geq |\theta|$ by Lemma 10.7(b,c), and for $p < |\theta|$ we have

$$\mathcal{H}_q(\theta,p) - \mathcal{N}_q(\theta,p) = \mathcal{C}(\widehat{\theta},p-a) - \mathcal{H}_q(\widehat{\theta},p-a) + \mathcal{H}_q(\widehat{\theta},p-a+1) - \mathcal{N}_q(\widehat{\theta},p-a)\,,$$

which is equal to zero by Lemma 10.9. Finally, if $\widehat{\theta} \neq \emptyset$ and ψ is not connected to $\widehat{\theta}$, then $\mathcal{H}_q(\theta,p) = \mathcal{C}(\widehat{\theta},p-a) - \mathcal{C}(\widehat{\theta},p-a+1) = \mathcal{N}_q(\theta,p)$.

Assume that ψ is a row. For $0 \leq i \leq a-1$, we let φ_i be the union of $\widehat{\theta}$ with the leftmost i boxes of ψ. Then $\varphi_0, \varphi_1, \ldots, \varphi_{a-1}$ are the proper lower order ideals φ in θ that contain $\widehat{\theta}$. If $\widehat{\theta} = \emptyset$ and ψ meets the SW diagonal, then

$$\mathcal{H}_q(\theta,p) - \mathcal{N}_q(\theta,p) = \chi(p \leq |\theta|) - \delta_{p,|\theta|} = \chi(p \leq 0) + \sum_{i=1}^{a-1} \delta_{p,i} = \sum_{i=0}^{a-1} \mathcal{C}(\varphi_i,p)\,.$$

If θ is a single box not on the SW diagonal, then $\mathcal{H}_q(\theta,p) - \mathcal{N}_q(\theta,p) = \chi(p \leq 0) = \mathcal{C}(\varphi_0, p)$. If $\widehat{\theta} = \emptyset$, $|\theta| \geq 2$, and ψ does not meet the SW diagonal, then

$$\begin{aligned}\mathcal{H}_q(\theta,p) - \mathcal{N}_q(\theta,p) &= \chi(p < |\theta|) + \delta_{p,a-1} \\ &= \chi(p \leq 0) + 2\,\delta_{p,1} + \sum_{i=2}^{a-1}(2\,\delta_{p,i} - \delta_{p,i-1}) \;=\; \sum_{i=0}^{a-1}\mathcal{C}(\varphi_i, p)\,.\end{aligned}$$

If $\widehat{\theta} \neq \emptyset$ and ψ is connected to $\widehat{\theta}$, then since the north-east arm of $\widehat{\theta}$ is not a row, we obtain by induction on $|\theta|$ that

$$\begin{aligned}&\mathcal{H}_q(\theta,p) - \mathcal{N}_q(\theta,p) \;=\; \mathcal{H}_q(\widehat{\theta}, p-a) - \mathcal{N}_q(\widehat{\theta}, p-a) + \mathcal{C}(\widehat{\theta}, p-a+1) \\ &= \mathcal{C}(\widehat{\theta}, p-a+1) \;=\; \mathcal{C}(\widehat{\theta}, p) + \sum_{i=1}^{a-1}(\mathcal{C}(\widehat{\theta}, p-i) - \mathcal{C}(\widehat{\theta}, p-i+1)) \;=\; \sum_{i=0}^{a-1}\mathcal{C}(\varphi_i, p)\,.\end{aligned}$$

If $\widehat{\theta} \neq \emptyset$ and ψ is a single box that is not connected to $\widehat{\theta}$, then $\mathcal{H}_q(\theta,p) - \mathcal{N}_q(\theta,p) = \mathcal{C}(\widehat{\theta}, p)$ follows from the definitions. Finally, if $\widehat{\theta} \neq \emptyset$, ψ is not connected to $\widehat{\theta}$, and $a \geq 2$, we obtain

$$\begin{aligned}\mathcal{H}_q(\theta,p) - \mathcal{N}_q(\theta,p) \;&=\; 2\,\mathcal{C}(\widehat{\theta}, p-a+1) - \mathcal{C}(\widehat{\theta}, p-a+2) \\ =\; &\mathcal{C}(\widehat{\theta}, p) + \Big(2\,\mathcal{C}(\widehat{\theta}, p-1) - 2\,\mathcal{C}(\widehat{\theta}, p)\Big) \\ &\quad + \sum_{i=2}^{a-1}\Big(2\,\mathcal{C}(\widehat{\theta}, p-i) - 3\,\mathcal{C}(\widehat{\theta}, p-i+1) + \mathcal{C}(\widehat{\theta}, p-i+2)\Big) \\ =\; &\sum_{i=0}^{a-1}\mathcal{C}(\varphi_i, p)\,.\end{aligned}$$

The identity follows from these observations. □

We finally prove that the Pieri structure constants $\widehat{\mathcal{N}}(\theta,p)$ of $\mathrm{QK}(X)$ are signed counts of QKLG-tableaux.

Corollary 10.11. *Let $\theta \subset \widehat{\mathcal{P}}_X$ be a skew shape and $1 \leq p \leq n$. Then $\widehat{\mathcal{N}}(\theta,p) = \mathcal{N}(\theta,p)$.*

Proof. If θ is disjoint from the NE diagonal of $\widehat{\mathcal{P}}_X$, then $\widehat{\mathcal{N}}(\theta,p) = \mathcal{C}(\theta,p) = \mathcal{N}(\theta,p)$ by [BR12]. If θ contains two or more boxes from the NE diagonal, then $\mathcal{N}(\theta,p) = 0$ by definition (since θ is not a rim), and since $d_{\max}(p) = 1$, it follows from [BCMP22, Thm. 8.3] that $\widehat{\mathcal{N}}(\theta,p) = 0$. Assume that θ contains exactly one box from the NE diagonal of $\widehat{\mathcal{P}}_X$. Then θ^- equals $\widehat{\theta}$ if the north-east arm of θ is a row, and $\theta^- = \theta$ otherwise. Lemma 10.6 shows that $\mathcal{N}(\theta,p) = \mathcal{N}_q(\theta,p)$, and Proposition 10.10 and the definition (10) show that $\mathcal{N}_q(\theta,p) = \widehat{\mathcal{N}}(\theta,p)$, noting that the condition $\theta^- \subset \varphi \subsetneq \theta$ implies that $\mathcal{H}(\varphi,p) = \mathcal{C}(\varphi,p)$ by Remark 10.2. □

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