A DIRECT PROOF OF THE QUANTUM VERSION OF MONK'S FORMULA

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1. Introduction

The quantum version of Monk's formula of Fomin, Gelfand, and Postnikov [6] gives an explicit rule for multiplying by a codimension one Schubert class in the (small) quantum cohomology ring of a flag variety SL_n/B. The proof given in [6] relies on a formula of Ciocan-Fontanine [3] for the quantum classes of certain special Schubert varieties given by cyclic permutations, which is obtained using degeneracy loci formulas on hyper-quot schemes. In the present paper we give a direct geometric proof of the quantum Monk's formula which relies only on classical Schubert calculus and the definition of Gromov-Witten invariants. In particular, no compactifications of moduli spaces are required. Our proof uses an adaption of the ideas from [1] where we give a similar proof of the quantum Pieri formula for Grassmann varieties.

Since the quantum cohomology ring of a flag variety is generated by the codimension one Schubert classes, the quantum Monk's formula uniquely determines this ring as well as the associated Gromov-Witten invariants. Thus, if associativity of quantum cohomology is granted [16, 12, 9], we obtain a completely elementary understanding of this ring.

The presentation of the quantum cohomology ring of a flag variety due to Givental, Kim, and Ciocan-Fontanine [10, 11, 3] and Ciocan-Fontanine's formula for special quantum Schubert classes [3] are easy consequences of the quantum Monk's formula. In fact, the quantum Monk's formula implies that Ciocan-Fontanine's classes satisfy the same recursive relations as those defining the quantum elementary symmetric polynomials (cf. [15, Lemma 4.2]). These results in turn are the only facts required in the combinatorial proof of the quantum Giambelli formula for flag varieties given in [6]. Alternatively, the quantum Schubert polynomials constructed in [6] can easily be computed by using only the quantum Monk's formula (cf. [6, §8] and [13, (4.16)]). The quantum Pieri formula of Ciocan-Fontanine [4] can also be derived combinatorially from the quantum Monk's formula [15, 7] [1], or it can be proved by an enhancement of the methods of the present paper [2]. For a survey of combinatorial approaches to quantum cohomology of flag varieties we refer the reader to [5].

In Section 2 we fix notation regarding Schubert varieties in partial flag varieties and prove a result which relates the Schubert varieties in different partial flag varieties. In Section 3 we give some tools for handling rational curves in flag

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[1] Ciocan-Fontanine's result is more general and covers all partial flag varieties SL_n/P.

varieties. The proof of the quantum Monk's formula is finally given in Section 4 after a short introduction of the quantum ring of a flag variety.

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2. Schubert varieties in partial flag varieties

Our notation for Schubert varieties is based on [8]. Set $E = \mathbb{C}^n$ and let $\mathrm{F}\ell(E) = \{V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset E \mid \dim V_i = i\}$ denote the variety of full flags in E. Given a fixed flag $F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset E$ and a permutation $w \in S_n$ there is a Schubert variety

$$\Omega_w(F_\bullet) = \{V_\bullet \in \mathrm{F}\ell(E) \mid \dim(V_p \cap F_q) \geq p - r_w(p, n-q) \ \forall p, q\}$$

where $r_w(p,q) = \#\{i \leq p \mid w(i) \leq q\}$. The codimension of this variety is equal to the length $\ell(w)$ of w. Notice that the rank conditions on V_p are redundant unless w has a descent at position p, i.e. $w(p) > w(p+1)$.

Given a sequence of integers $a = (a_1 \leq a_2 \leq \cdots \leq a_k)$ with $a_1 \geq 0$ and $a_k \leq n$, we have the partial flag variety $\mathrm{F}\ell(a; E) = \{V_{a_1} \subset \cdots \subset V_{a_k} \subset E \mid \dim V_{a_i} = a_i\}$. Although all such varieties can be obtained from strictly increasing sequences a, it will be convenient to allow weakly increasing sequences in the notation. Similarly it is useful to set $a_0 = 0$ and $a_{k+1} = n$. Let $S_n(a) \subset S_n$ denote the set of permutations whose descent positions are contained in the set $\{a_1, a_2, \dots, a_k\}$. The Schubert varieties in $\mathrm{F}\ell(a; E)$ are indexed by these permutations; the Schubert variety corresponding to $w \in S_n(a)$ is given by

$$\Omega_w^{(a)}(F_\bullet) = \{V_\bullet \in \mathrm{F}\ell(a; E) \mid \dim(V_{a_i} \cap F_q) \geq a_i - r_w(a_i, n-q) \ \forall i, q\}\,.$$

Let $\rho_a : \mathrm{F}\ell(E) \to \mathrm{F}\ell(a; E)$ be the projection which maps a full flag $V_\bullet$ to the sub-flag $V_{a_1} \subset \cdots \subset V_{a_k}$. Then for any $w \in S_n(a)$ we have $\rho_a^{-1}(\Omega_w^{(a)}(F_\bullet)) = \Omega_w(F_\bullet)$. On the other hand, if $w \in S_n$ is any permutation then $\rho_a(\Omega_w(F_\bullet)) = \Omega_{\widetilde{w}}^{(a)}(F_\bullet)$ where $\widetilde{w} \in S_n(a)$ is the permutation obtained from w by rearranging the elements $w(a_i + 1), w(a_i + 2), \dots, w(a_{i+1})$ in increasing order for each $0 \leq i \leq k$. In other words, $\widetilde{w}$ is the shortest representative for w modulo the subgroup $W_a \subset S_n$ generated by the simple reflections $s_i = (i, i+1)$ for $i \notin \{a_1, \dots, a_k\}$. For example, if $n = 6$, $a = (2, 5)$, and $w = 6\,2\,3\,1\,5\,4$ then $\widetilde{w} = 2\,6\,1\,3\,5\,4$.

Now let $b = (b_1 \leq b_2 \leq \cdots \leq b_k)$ be another sequence with the same length as a, such that $b_i \leq a_i$ for each i. Given a permutation $w \in S_n(a)$ we will need a description of the set $\{K_\bullet \in \mathrm{F}\ell(b; E) \mid \exists\ V_\bullet \in \Omega_w^{(a)}(F_\bullet) : K_{b_i} \subset V_{a_i} \ \forall i\}$.

We construct a permutation $\overline{w} \in S_n(b)$ from w as follows. Set $w^{(0)} = w$. Then for each $1 \leq i \leq k$ we let $w^{(i)}$ be the permutation obtained from $w^{(i-1)}$ by rearranging the elements $w^{(i-1)}(b_i + 1), \dots, w^{(i-1)}(a_{i+1})$ in increasing order. Finally we set $\overline{w} = w^{(k)}$. For example, if $n = 6$, $a = (2, 5)$, $b = (1, 2)$, and $w = 2\,6\,3\,4\,5\,1$ then $w^{(1)} = 2\,3\,4\,5\,6\,1$ and $\overline{w} = 2\,3\,1\,4\,5\,6$.

Lemma 1. *The set* $\{K_\bullet \in \mathrm{F}\ell(b; E) \mid \exists\ V_\bullet \in \Omega_w^{(a)}(F_\bullet) : K_{b_i} \subset V_{a_i} \ \forall i\}$ *is equal to the Schubert variety* $\Omega_{\overline{w}}^{(b)}(F_\bullet)$ *in* $\mathrm{F}\ell(b; E)$.

Proof. We prove that the subset Ω_i of $\mathrm{F}\ell_i = \mathrm{F}\ell(b_1, \dots, b_i, a_{i+1}, \dots, a_k; E)$ defined by $\Omega_i = \{K_\bullet \mid \exists\ V_\bullet \in \Omega_w^{(a)}(F_\bullet) : K_{b_j} \subset V_{a_j}$ for $j \leq i$ and $K_{a_j} = V_{a_j}$ for $j > i\}$ is equal to the Schubert variety in $\mathrm{F}\ell_i$ given by the permutation $w^{(i)}$. This is true when $i = 0$. Let $\rho_j : \mathrm{F}\ell(E) \to \mathrm{F}\ell_j$ denote the projection. Then it is easy to check that $\Omega_{i+1} = \rho_{i+1}(\rho_i^{-1}(\Omega_i))$, so the lemma follows from the above remarks about images and inverse images of projections ρ_a. □

Lemma 1 has a dual version which we will also need. Let a and c be weakly increasing sequences of integers between 0 and n, each of length k, such that $a_i \leq c_i$ for each $1 \leq i \leq k$. Given $w \in S_n(a)$ we define a permutation $\widehat{w} \in S_n(c)$ as follows. Set $w^{(k+1)} = w$. For each $i = k, k-1, \ldots, 1$ we then let $w^{(i)}$ be the permutation obtained from $w^{(i+1)}$ by rearranging the elements $w^{(i+1)}(a_{i-1}+1), \ldots, w^{(i+1)}(c_i)$ in increasing order. Finally we set $\widehat{w} = w^{(1)}$.

Lemma 2. *The set* $\{W_\bullet \in \mathrm{F}\ell(c;E) \mid \exists\ V_\bullet \in \Omega_w^{(a)}(F_\bullet) : V_{a_i} \subset W_{c_i}\ \forall i\}$ *is equal to the Schubert variety* $\Omega_{\widehat{w}}^{(c)}(F_\bullet)$ *in* $\mathrm{F}\ell(c;E)$.

Notice that the definitions of the permutations $\overline{w}$ and $\widehat{w}$ imply that $\ell(\overline{w}) \geq \ell(w) - \sum_{i=1}^{k}(a_i - b_i)(a_{i+1} - a_i)$ and $\ell(\widehat{w}) \geq \ell(w) - \sum_{i=1}^{k}(c_i - a_i)(a_i - a_{i-1})$. In particular, if $a = (1, 2, \ldots, n-1)$ so that $\mathrm{F}\ell(a;E) = \mathrm{F}\ell(E)$ then $\ell(\overline{w}) \geq \ell(w) - \sum_{i=1}^{n-1}(i - b_i)$ and $\ell(\widehat{w}) \geq \ell(w) - \sum_{i=1}^{n-1}(c_i - i)$.

3. Rational curves in partial flag varieties

By a rational curve in $\mathrm{F}\ell(a;E)$ we will mean the image C of a regular function $\mathbb{P}^1 \to \mathrm{F}\ell(a;E)$. (We will tolerate that a rational curve can be a point according to this definition.) Given a rational curve $C \subset \mathrm{F}\ell(a;E)$ we let $C_i = \rho_{a_i}(C) \subset \mathrm{Gr}(a_i, E)$ be the image of C by the projection $\rho_{a_i} : \mathrm{F}\ell(a;E) \to \mathrm{Gr}(a_i, E)$. This curve C_i then has a *kernel* and a *span* [1]. The kernel is the largest subspace of E contained in all the a_i-dimensional subspaces of E corresponding to points of C_i. We let b_i be the dimension of this kernel and denote the kernel itself by K_{b_i}. Similarly, the span of C_i is the smallest subspace of E containing all the subspaces given by points of C_i. We let c_i be the dimension of this span and denote the span by W_{c_i}. These subspaces define partial flags $K_\bullet \in \mathrm{F}\ell(b;E)$ and $W_\bullet \in \mathrm{F}\ell(c;E)$ where $b = (b_1 \ldots, b_k)$ and $c = (c_1, \ldots, c_k)$, which we will call the kernel and span of C.

Proposition 1. *Let* $C \subset \mathrm{F}\ell(a;E)$ *be a rational curve with kernel* $K_\bullet \in \mathrm{F}\ell(b;E)$ *and span* $W_\bullet \in Fl(c;E)$ *and let* $w \in S_n(a)$. *If* $C \cap \Omega_w^{(a)}(F_\bullet) \neq \emptyset$ *then* $K_\bullet \in \Omega_{\overline{w}}^{(b)}(F_\bullet)$ *and* $W_\bullet \in \Omega_{\widehat{w}}^{(c)}(F_\bullet)$.

Proof. If $V_\bullet \in C \cap \Omega_w^{(a)}(F_\bullet)$ then we have $K_{b_i} \subset V_{a_i} \subset W_{c_i}$ for all i. The proposition therefore follows from Lemma 1 and Lemma 2. □

Now let $a = (a_1 < a_2 < \cdots < a_k)$ be a strictly increasing sequence of integers with $1 \leq a_i \leq n-1$. Define the *multidegree* of a rational curve $C \subset \mathrm{F}\ell(a;E)$ to be the sequence $d = (d_1, \ldots, d_k)$ where d_i is the number of points in the intersection $C \cap \Omega_{s_{a_i}}(F_\bullet)$ for any flag $F_\bullet$ in general position. Notice that d_i is greater than or equal to the degree of the image $C_i \subset \mathrm{Gr}(a_i;E)$. If $K_\bullet \in \mathrm{F}\ell(b;E)$ is the kernel and $W_\bullet \in \mathrm{F}\ell(c;E)$ the span of C, it therefore follows from [1, Lemma 1] that $b_i \geq a_i - d_i$ and $c_i \leq a_i + d_i$ for all $1 \leq i \leq k$.

Next we shall need a fact about rational curves in the full flag variety $\mathrm{F}\ell(E)$. For integers $1 \leq i < j \leq n$, let $d_{ij} = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$ denote the multidegree consisting of $i-1$ zeros followed by $j-i$ ones followed by $n-j$ zeros, i.e. $(d_{ij})_p = 1$ for $i \leq p < j$ and $(d_{ij})_p = 0$ otherwise. We set $a = (1, 2, \ldots, n-1)$ and $b = a - d_{ij} = (b_1, \ldots, b_{n-1})$ where $b_p = p - (d_{ij})_p$.

Proposition 2. *Let $K_\bullet \in \mathrm{F}\ell(b;E)$ and let $W \subset E$ be a subspace of dimension $i+1$ such that $K_{j-2} \cap W = K_{i-1}$ and $K_{j-2} + W = K_j$. Then there exists a unique rational curve $C \subset \mathrm{F}\ell(E)$ of multidegree d_{ij} such that $K_\bullet$ is the kernel of C and W is the span of $C_i \subset \mathrm{Gr}(i,E)$.*

Proof. The only curve satisfying the conditions of the proposition is the set of flags

$$V_\bullet = (K_1 \subset \cdots \subset K_{i-1} \subset L \subset K_i + L \subset \cdots \subset K_{j-2} + L \subset K_j \subset \cdots \subset K_{n-1})$$

for all i-dimensional subspaces L such that $K_{i-1} \subset L \subset W$. □

It is easy to show that the rational curves $C \subset \mathrm{F}\ell(E)$ of multidegree d_{ij} are in fact in 1-1 correspondence with the pairs $(K_\bullet, W)$ of the proposition, but we shall not need this fact.

4. Quantum cohomology of flag varieties

For each permutation $w \in S_n$ we let Ω_w denote the class of $\Omega_w(F_\bullet)$ in the cohomology ring $H^* \mathrm{F}\ell(E) = H^*(\mathrm{F}\ell(E);\mathbb{Z})$. The Schubert classes Ω_w form a basis for this ring. If $d = (d_1, \dots, d_{n-1})$ is a multidegree we set $|d| = \sum d_i$. Given three permutations $u, v, w \in S_n$ such that $\ell(u) + \ell(v) + \ell(w) = \binom{n}{2} + 2|d|$, the Gromov-Witten invariant $\langle \Omega_u, \Omega_v, \Omega_w \rangle_d$ is defined as the number of rational curves of multidegree d in $\mathrm{F}\ell(E)$ which meet each of the Schubert varieties $\Omega_u(F_\bullet)$, $\Omega_v(G_\bullet)$, and $\Omega_w(H_\bullet)$ for general fixed flags $F_\bullet, G_\bullet, H_\bullet$ in E. If $\ell(u)+\ell(v)+\ell(w) \neq \binom{n}{2}+2|d|$ then $\langle \Omega_u, \Omega_v, \Omega_w \rangle_d = 0$.

Let $q_1, \dots, q_{n-1}$ be independent variables, and write $\mathbb{Z}[q] = \mathbb{Z}[q_1, \dots, q_{n-1}]$. The quantum cohomology ring $QH^* \mathrm{F}\ell(E)$ is a $\mathbb{Z}[q]$-algebra which is isomorphic to $H^* \mathrm{F}\ell(E) \otimes \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$. In this ring we have quantum Schubert classes $\sigma_w = \Omega_w \otimes 1$. Multiplication in $QH^* \mathrm{F}\ell(E)$ is defined by

$$\sigma_u \cdot \sigma_v = \sum_{w,d} \langle \Omega_u, \Omega_v, \Omega_{w^\vee} \rangle_d \, q^d \, \sigma_w$$

where the sum is over all permutations $w \in S_n$ and multidegrees $d = (d_1, \dots, d_{n-1})$; here we set $q^d = \prod q_i^{d_i}$ and we let $w^\vee \in S_n$ denote the permutation of the dual Schubert class to Ω_w, i.e. $w^\vee = w_0 w$ where w_0 is the longest permutation in S_n. It is a non-trivial fact that this defines an associative ring [16, 12, 9].

For $1 \leq i < j \leq n$ we let $t_{ij} = (i,j) \in S_n$ denote the transposition which interchanges i and j. We furthermore set $q_{ij} = q^{d_{ij}} = q_i \, q_{i+1} \dots q_{j-1}$. Our goal is to prove the following quantum version of the Monk's formula from [6].

Theorem 1. *For $w \in S_n$ and $1 \leq r < n$ we have*

$$\sigma_{s_r} \cdot \sigma_w = \sum \sigma_{w\, t_{kl}} + \sum q_{ij} \, \sigma_{w\, t_{ij}}$$

where the first sum is over all transpositions t_{kl} such that $k \leq r < l$ and $\ell(w\, t_{kl}) = \ell(w) + 1$, and the second sum is over all transpositions t_{ij} such that $i \leq r < j$ and $\ell(w\, t_{ij}) = \ell(w) - \ell(t_{ij}) = \ell(w) - 2(j-i) + 1$.

Proof. The first sum is dictated by the classical Monk's formula [14]. The second sum is equivalent to the following statement. If $d = (d_1, \dots, d_{n-1})$ is a non-zero multidegree and $u, w \in S_n$ are permutations such that $\ell(u)+\ell(w)+\ell(s_r) = \binom{n}{2}+2|d|$ then the Gromov-Witten invariant $\langle \Omega_u, \Omega_w, \Omega_{s_r} \rangle_d$ is equal to one if $d = d_{ij}$ for some i, j such that $i \leq r < j$ and $u^{-1} w_0 w = t_{ij}$; otherwise $\langle \Omega_u, \Omega_w, \Omega_{s_r} \rangle_d = 0$.

Suppose $\langle \Omega_u, \Omega_w, \Omega_{s_r} \rangle_d \neq 0$ and let C be a rational curve of multidegree d which meets three Schubert varieties $\Omega_u(F_\bullet)$, $\Omega_w(G_\bullet)$, and $\Omega_{s_r}(H_\bullet)$ in general position. Let $K_\bullet \in \mathrm{F}\ell(b;E)$ be the kernel of C and set $a = (1,2,\dots,n-1)$. Then $b_p \geq a_p - d_p$ for all $1 \leq p \leq n-1$. By Proposition 1 we have $K_\bullet \in \Omega^{(b)}_{\overline{u}}(F_\bullet) \cap \Omega^{(b)}_{\overline{w}}(G_\bullet) \cap \Omega^{(b)}_{\overline{s_r}}(H_\bullet)$. Since the flags are general this implies that $\ell(\overline{u}) + \ell(\overline{w}) + \ell(\overline{s_r}) \leq \dim \mathrm{F}\ell(b;E)$. On the other hand the inequalities $\ell(\overline{u}) \geq \ell(u) - \sum(p-b_p)$, $\ell(\overline{w}) \geq \ell(w) - \sum(p-b_p)$, $\ell(\overline{s_r}) \geq 0$, and $\sum(p-b_p) \leq |d|$ imply that $\ell(\overline{u}) + \ell(\overline{w}) + \ell(\overline{s_r}) \geq \binom{n}{2} - 1$. Since this is the maximal possible dimension of $\mathrm{F}\ell(b;E)$ we conclude that all inequalities are satisfied with equality.

This first implies that $b = a - d = (1-d_1, 2-d_2, \dots, n-1-d_{n-1})$. Furthermore, since $\dim \mathrm{F}\ell(b;E) = \binom{n}{2} - 1$ we deduce that $d = d_{ij}$ for some $1 \leq i < j \leq n$. Thus $\mathrm{F}\ell(b;E) = \mathrm{F}\ell(1,\dots,j-2,j,\dots,n-1;E)$ is the variety of partial flags with subspaces of all dimensions other than $j-1$. Since $\ell(\overline{s_r}) = 0$ it follows that $i \leq r < j$. The fact that $\ell(\overline{u}) = \ell(u) - |d|$ implies that $\overline{u} = u\, s_i s_{i+1} \cdots s_{j-1}$ by the definition of $\overline{u}$. Similarly we have $\overline{w} = w\, s_i s_{i+1} \cdots s_{j-1}$. Now since $\ell(\overline{u}) + \ell(\overline{w}) = \dim \mathrm{F}\ell(b;E)$ and $\Omega^{(b)}_{\overline{u}}(F_\bullet) \cap \Omega^{(b)}_{\overline{w}}(G_\bullet) \neq \emptyset$ we conclude that $\overline{u}$ and $\overline{w}$ are dual with respect to $\mathrm{F}\ell(b;E)$, i.e. $\overline{u}^{-1} w_0 \overline{w} = s_{j-1}$ or equivalently $u^{-1} w_0 w = t_{ij}$ as required.

It remains to be proved that if $d = d_{ij}$ and $u^{-1} w_0 w = t_{ij}$ for some $i \leq r < j$ then there exists a unique rational curve of multidegree d which meets the three given Schubert varieties. Set $\overline{u} = u\, s_i s_{i+1} \cdots s_{j-1}$ and $\overline{w} = w\, s_i s_{i+1} \cdots s_{j-1}$. Since $\ell(u\, t_{ij}) = \ell(w_0 w) = \binom{n}{2} - \ell(w) = \ell(u) - \ell(t_{ij})$ it follows that $\ell(\overline{u}) = \ell(u) - |d|$ and similarly $\ell(\overline{w}) = \ell(w) - |d|$. Thus $\ell(\overline{u}) + \ell(\overline{w}) = \dim \mathrm{F}\ell(b;E)$ where $b = a - d$. Since $\overline{u}^{-1} w_0 \overline{w} = s_{j-1}$ we conclude that there is a unique partial flag $K_\bullet \in \Omega^{(b)}_{\overline{u}}(F_\bullet) \cap \Omega^{(b)}_{\overline{w}}(G_\bullet)$. Similarly, if we set $\widehat{u} = u\, s_{j-1} s_{j-2} \cdots s_i$ and $\widehat{w} = w\, s_{j-1} s_{j-2} \cdots s_i$ then there exists a unique partial flag $W_\bullet \in \Omega^{(c)}_{\widehat{u}}(F_\bullet) \cap \Omega^{(c)}_{\widehat{w}}(G_\bullet)$ where $c = a + d$.

In fact, we can say precisely what these partial flags look like. For $1 \leq p \leq n$ we set $L_p = F_{n+1-p} \cap G_p$. Since the flags $F_\bullet$ and $G_\bullet$ are general, these spaces have dimension one, and $E = L_1 \oplus \cdots \oplus L_n$. Now $K_p = L_{\overline{u}(1)} \oplus \cdots \oplus L_{\overline{u}(p)}$ for each $p \neq j-1$ and $W_p = L_{\widehat{u}(1)} \oplus \cdots \oplus L_{\widehat{u}(p)}$ for $p \neq i$. Otherwise stated we have $K_p = W_p = L_{u(1)} \oplus \cdots \oplus L_{u(p)}$ for $1 \leq p \leq i-1$ and for $j \leq p < n$. For $i-1 \leq p \leq j-2$ we have $K_p = K_{i-1} \oplus L_{u(i+1)} \oplus \cdots \oplus L_{u(p+1)}$ while $W_{p+2} = K_p \oplus U$ where $U = L_{u(i)} \oplus L_{u(j)}$. In particular we get $W_{i+1} \cap K_{j-2} = K_{i-1}$ and $W_{i+1} + K_{j-2} = K_j$ so by Proposition 2 there is exactly one rational curve of multidegree d with kernel $K_\bullet$ and span $W_\bullet$. This curve consists of all flags

$$V_\bullet = (\, K_1 \subset \cdots \subset K_{i-1} \subset K_{i-1} \oplus L \subset \cdots \subset K_{j-2} \oplus L \subset K_j \subset \cdots \subset K_{n-1} \,)$$

where $L \subset U$ is a one dimensional subspace. When $L = L_{u(i)}$ we have $V_\bullet \in \Omega_u(F_\bullet)$, while $V_\bullet \in \Omega_w(G_\bullet)$ when $L = L_{u(j)}$. Finally, $V_\bullet$ belongs to $\Omega_{s_r}(H_\bullet)$ if and only if $V_r \cap H_{n-r} \neq 0$. Now take any non-zero element $x \in W_{r+1} \cap H_{n-r}$ and let x' be the U-component of x in $W_{r+1} = K_{r-1} \oplus U$. Taking $L = \mathbb{C}x'$ then gives a point $V_\bullet \in \Omega_{s_r}(H_\bullet)$. This completes the proof. □

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