QUANTUM COHOMOLOGY OF GRASSMANNIANS

ANDERS SKOVSTED BUCH

1. INTRODUCTION

The purpose of this paper is to give simple proofs of the main theorems about the (small) quantum cohomology ring of a Grassmann variety. This first of all includes Bertram's quantum versions of the Pieri and Giambelli formulas [1]. Bertram's proofs of these theorems required the use of quot schemes. Our proof of the quantum Pieri formula uses no moduli spaces and only the definition of Gromov-Witten invariants. In fact we show that this formula is a consequence of the classical Pieri formula. We then show that the quantum Giambelli formula follows immediately from the quantum Pieri formula together with the classical Giambelli formula and associativity of quantum cohomology [17, 13].

We also give a short proof of the Grassmannian case of a formula of Fulton and Woodward for the minimal q-power which appears in a quantum product of two Schubert classes [10]. In addition we supply a proof of a simple version of the rim-hook algorithm [2] based on "mod n" arithmetic which is due to F. Sottile [19]. Finally we recover Siebert and Tian's presentation of the quantum cohomology of Grassmannians [18].

In this paper we only assume associativity of quantum cohomology and standard facts about the usual cohomology. The basic idea is that if a rational curve of degree d passes through a Schubert variety in the Grassmannian $\mathrm{Gr}(l, \mathbb{C}^n)$ of l-dimensional subspaces of $\mathbb{C}^n$, then the linear span of the points of this curve gives rise to a point in $\mathrm{Gr}(l + d, \mathbb{C}^n)$ which must lie in a related Schubert variety. Remarkably, this simple idea can in many cases be used to conclude that no curves pass through three Schubert varieties in general position because the intersection of the related Schubert varieties in $\mathrm{Gr}(l + d, \mathbb{C}^n)$ is empty. In particular, the quantum Giambelli formula can be deduced by knowing that certain Gromov-Witten invariants are zero, and in each case this follows because the codimensions of the related Schubert varieties add up to more than the dimension of $\mathrm{Gr}(l + d, \mathbb{C}^n)$. The same idea can be used to prove the quantum Monk's formula for flag varieties SL_n/B [7, 4] and even the quantum Pieri formula for partial flag varieties SL_n/P [6, 5]. In a paper [3] with A. Kresch and H. Tamvakis this idea will furthermore be applied to obtain a similar treatment of the quantum cohomology of Lagrangian and orthogonal Grassmannians [14, 15].

We thank Sottile for triggering our search for simplifications to the theory of quantum cohomology for Grassmannians, when he explained his simplified rim-hook algorithm to us. We also thank W. Fulton for numerous helpful comments to our paper, some of which greatly enhanced the clarity of our proof of the quantum

Giambelli formula. Finally we thank Kresch for simplifying our proof of the key Lemma 1.

2. Preliminaries

Set $E = \mathbb{C}^n$, $X = \mathrm{Gr}(l, E)$, and $k = n - l$. Given a flag of subspaces $F_1 \subset F_2 \subset \cdots \subset F_n = E$ and a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0)$ with $\lambda_1 \leq k$, we define the Schubert variety

$$(1) \qquad \Omega_\lambda(F_\bullet) = \{V \in X \mid \dim(V \cap F_{k+i-\lambda_i}) \geq i \ \forall 1 \leq i \leq l\}.$$

The codimension of this variety is equal to the weight $|\lambda| = \sum \lambda_i$ of λ. We let Ω_λ denote the class of $\Omega_\lambda(F_\bullet)$ in the cohomology ring $H^*X = H^*(X; \mathbb{Z})$. The Schubert classes Ω_λ form a basis for this ring. The partitions indexing this basis are exactly those where the Young diagram fits in an $l \times k$ rectangle.

The cohomology ring has the presentation $H^*X = \mathbb{Z}[\Omega_1, \ldots, \Omega_k]/(Y_{l+1}, \ldots, Y_n)$ where $Y_p = \det(\Omega_{1+j-i})_{1 \leq i, j \leq p}$. Here we set $\Omega_i = 0$ for $i < 0$ or $i > k$ for convenience. In this presentation the class Ω_λ is given by the Giambelli formula

$$(2) \qquad \Omega_\lambda = \det(\Omega_{\lambda_i + j - i})_{1 \leq i, j \leq l}.$$

This is usually deduced (cf. [8]) from the Pieri formula, which states that

$$(3) \qquad \Omega_i \cdot \Omega_\lambda = \sum \Omega_\nu$$

where the sum is over all partitions ν which can be obtained by adding i boxes to the Young diagram of λ with no two in the same column.

Recall that the degree of a rational curve $f : \mathbb{P}^1 \to \mathbb{P}^N$ is equal to the number of points in the inverse image by f of a general hyperplane in $\mathbb{P}^N$. The degree of a map $f : \mathbb{P}^1 \to X$ is the degree of the composition of f with the Plücker embedding $X \subset \mathbb{P}(\bigwedge^l E)$ which maps a point $V \in X$ to $v_1 \wedge \cdots \wedge v_l$ for any basis $\{v_1, \ldots, v_l\}$ of V. Now let λ, μ, and ν be three partitions contained in an $l \times k$ rectangle, and let $d \geq 0$ be an integer such that $|\lambda| + |\mu| + |\nu| = lk + dn$. The Gromov-Witten invariant $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d$ is defined as the number of rational curves of degree d on X, which meet all of the Schubert varieties $\Omega_\lambda(F_\bullet)$, $\Omega_\mu(G_\bullet)$, and $\Omega_\nu(H_\bullet)$ for general flags $F_\bullet$, $G_\bullet$, $H_\bullet$, up to automorphisms of $\mathbb{P}^1$. A simple proof that this number is well defined is given in [1]. If $|\lambda| + |\mu| + |\nu| \neq lk + dn$ then $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d = 0$.

The (small) quantum cohomology ring $QH^*X = QH^*(X; \mathbb{Z})$ of X is a $\mathbb{Z}[q]$-algebra which is isomorphic to $H^*X \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$. In this ring we have Schubert classes $\sigma_\lambda = \Omega_\lambda \otimes 1$. The ring structure on QH^*X is defined by

$$(4) \qquad \sigma_\lambda \cdot \sigma_\mu = \sum_{\nu, \, d \geq 0} \langle \Omega_\lambda, \Omega_\mu, \Omega_{\nu^\vee} \rangle_d \, q^d \, \sigma_\nu$$

where $\nu^\vee = (k - \nu_l, k - \nu_{l-1}, \ldots, k - \nu_1)$ is the partition for the dual Schubert class of Ω_ν. It is a non-trivial fact that this defines an associative ring structure [17, 13] (see also [9]). Notice that the definition implies that QH^*X is a graded ring where σ_λ has degree $|\lambda|$ and q has degree n. Furthermore, the map $QH^*X/(q) \to H^*X$ which sends σ_λ to Ω_λ is an isomorphism of rings, so the quantum ring is a deformation of the usual cohomology ring.

Siebert and Tian have given a presentation of this ring which is similar to the above presentation of the cohomology ring [18]. We will comment on this presentation towards the end of this paper. For now we will start from the basic assumption that the quantum ring is a well defined associative ring, and aim towards proving

generalizations of the Pieri and Giambelli formulas stated above to reveal its structure.

3. The span and kernel of a curve

Our main new tool is the following definition. If Y is any subvariety of $X = \mathrm{Gr}(l, E)$ we define the *span* of Y to be the smallest subspace of E containing all the l-dimensional spaces given by points of Y. Similarly we define the *kernel* of Y to be the largest subspace of E contained in all the spaces given by points of Y.

Lemma 1. *Let C be a rational curve of degree d in X. Then the span of C has dimension at most $l + d$ and the kernel of C has dimension at least $l - d$.*

Proof. Let C be the image of a regular function $f : \mathbb{P}^1 \to X$ of degree d, and let $S \subset E \otimes \mathcal{O}_X$ be the tautological subbundle on X. Then $f^* S = \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^1}(-a_i)$ for integers $a_i \geq 0$ with sum d, and f is given by an inclusion $\bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^1}(-a_i) \subset E \otimes \mathcal{O}_{\mathbb{P}^1}$, i.e. a point $p \in \mathbb{P}^1$ is mapped to the fiber over p of the image of this bundle map. If $(s : t)$ are homogeneous coordinates on $\mathbb{P}^1$ then $\Gamma(\mathcal{O}_{\mathbb{P}^1}(a_i))$ has the basis $\{s^j t^{a_i - j}\}_{0 \leq j \leq a_i}$, so each map $\mathcal{O}_{\mathbb{P}^1}(-a_i) \to E \otimes \mathcal{O}_{\mathbb{P}^1}$ has the form

$$\sum_{j=0}^{a_i} \alpha_j \, s^{-j} t^{j - a_i} \;\mapsto\; \sum_{j=0}^{a_i} v_j^{(i)} \otimes \alpha_j$$

for vectors $v_j^{(i)} \in E$ (which will depend on the chosen identification of $f^* S$ with $\bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^1}(-a_i)$). The span of C must therefore be contained in the span of the set $\{v_j^{(i)}\}$ which has cardinality $\sum 1 + a_i = l + d$. On the other hand, at least $l - d$ of the integers a_i must be zero, and the kernel of C contains the span of the corresponding vectors $v_0^{(i)}$. $\qquad\square$

The above lemma can also be obtained by proving that any regular function $f : \mathbb{P}^1 \to X$ can be written as $f(t) = f_1(t) \wedge \cdots \wedge f_l(t)$ for maps $f_i : \mathbb{P}^1 \to \mathbb{P}(E)$. This is what we did until Kresch showed us the simpler argument given here.

If λ is a partition and d a non-negative integer, we let $\hat{\lambda}$ denote the partition obtained by removing the leftmost d columns from the Young diagram of λ, i.e. $\hat{\lambda}_i = \max(\lambda_i - d, 0)$.

Lemma 2. *Let $C \subset X$ be a rational curve of degree $d \leq k$ and let $W \subset E$ be an $l + d$ dimensional subspace containing the span of C. If λ is a partition such that $C \cap \Omega_\lambda(F_\bullet) \neq \emptyset$ then W belongs to the Schubert variety $\Omega_{\hat{\lambda}}(F_\bullet)$ in $\mathrm{Gr}(l + d, E)$.*

Proof. Let $V \in C \cap \Omega_\lambda(F_\bullet)$. Since $V \subset W$, the Schubert conditions on V imply that $\dim(W \cap F_{k+i-\lambda_i}) \geq i$ for all i, which says exactly that W belongs to $\Omega_{\hat{\lambda}}(F_\bullet)$. $\quad\square$

Similarly, if $K \subset E$ is an $l - d$ dimensional subspace contained in the kernel of C, then $K \in \Omega_{\bar{\lambda}}(F_\bullet) \subset \mathrm{Gr}(l - d, E)$, where $\bar{\lambda} = (\lambda_{d+1}, \ldots, \lambda_l)$ is the result of removing the top d rows of λ. Therefore the condition that a curve meets a given set of Schubert varieties in X implies that intersections of related Schubert varieties in $\mathrm{Gr}(l + d, E)$ and in $\mathrm{Gr}(l - d, E)$ are not empty, which is a statement about the usual cohomology of these spaces. As we shall see, this simple idea is sufficient to compute Gromov-Witten invariants in many important cases.

4. The quantum Pieri formula

We start with the following quantum version of the Pieri formula [1].

Theorem 1 (Bertram). *If λ is contained in an $l \times k$ rectangle and $p \leq k$ then*

$$\sigma_p \cdot \sigma_\lambda = \sum \sigma_\mu + q \sum \sigma_\nu$$

where the first sum is over all partitions μ such that $|\mu| = |\lambda| + p$ and $k \geq \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_l \geq \lambda_l$, and the second sum is over all partitions ν such that $|\nu| = |\lambda| + p - n$ and $\lambda_1 - 1 \geq \nu_1 \geq \lambda_2 - 1 \geq \nu_2 \geq \cdots \geq \lambda_l - 1 \geq \nu_l \geq 0$.

Recall that the length $\ell(\lambda)$ of a partition λ is the number of non-zero parts of λ. Notice that the second sum in the theorem is non-zero only if $\ell(\lambda) = l$.

Proof. The first sum is dictated by the classical Pieri formula. Notice that this classical case is equivalent to the following statement. If α and β are partitions such that $|\alpha| + |\beta| + p = lk$ then

$$\langle \Omega_\alpha, \Omega_\beta, \Omega_p \rangle_0 = \begin{cases} 1 & \text{if } \alpha_i + \beta_j \geq k \text{ for } i + j = l \text{ and } \alpha_i + \beta_j \leq k \text{ for } i + j = l + 1; \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose $|\alpha| + |\beta| + p = lk + dn$ for some $d \geq 1$ and let C be a rational curve of degree d in X which meets each of the varieties $\Omega_\alpha(F_\bullet)$, $\Omega_\beta(G_\bullet)$, and $\Omega_p(H_\bullet)$ for general flags $F_\bullet$, $G_\bullet$, $H_\bullet$. Notice that $dn \leq lk + k$ which implies that $d \leq k$. Let $W \subset E$ be a subspace of dimension $l + d$ which contains the span of C. Then $W \in \text{Gr}(l + d, E)$ lies in the intersection $\Omega_{\hat\alpha}(F_\bullet) \cap \Omega_{\hat\beta}(G_\bullet) \cap \Omega_{\hat p}(H_\bullet)$ where $\hat\alpha$ and $\hat\beta$ are the results of removing the leftmost d columns from α and β, and $\hat p = \max(p - d, 0)$. Since the flags $F_\bullet$, $G_\bullet$, $H_\bullet$ are general, this implies that $|\hat\alpha| + |\hat\beta| + \hat p \leq (l + d)(k - d)$. Since we also have

$$|\hat\alpha| + |\hat\beta| + \hat p \geq |\alpha| + |\beta| - 2ld + p - d = (l + d)(k - d) + d^2 - d$$

we deduce that $d = 1$ and $\ell(\alpha) = \ell(\beta) = l$. Using this, the quantum Pieri formula becomes equivalent to the statement that if $|\alpha| + |\beta| + p = lk + n$ then $\langle \Omega_\alpha, \Omega_\beta, \Omega_p \rangle_1 = 1$ if $\alpha_i + \beta_j \geq k + 1$ for $i + j = l + 1$ and $\alpha_i + \beta_j \leq k + 1$ for $i + j = l + 2$; otherwise $\langle \Omega_\alpha, \Omega_\beta, \Omega_p \rangle_1 = 0$. In other words, the quantum Pieri formula states that $\langle \Omega_\alpha, \Omega_\beta, \Omega_p \rangle_1 = \langle \Omega_{\hat\alpha}, \Omega_{\hat\beta}, \Omega_{\hat p} \rangle_0$ where the right hand side is a coefficient of the classical Pieri formula for $\text{Gr}(l + 1, E)$.

If $\langle \Omega_{\hat\alpha}, \Omega_{\hat\beta}, \Omega_{\hat p} \rangle_0$ is zero then the space W can't exist, so neither can C. On the other hand, if $\langle \Omega_{\hat\alpha}, \Omega_{\hat\beta}, \Omega_{\hat p} \rangle_0 = 1$ then there exists a unique subspace $W \subset E$ of dimension $l + 1$ which is contained in the intersection $\Omega_{\hat\alpha}(F_\bullet) \cap \Omega_{\hat\beta}(G_\bullet) \cap \Omega_{\hat p}(H_\bullet)$. Furthermore, since the flags are general, W must lie in the interior of each of these Schubert varieties. In particular, each of the spaces $V_1 = W \cap F_{n-\alpha_l}$ and $V_2 = W \cap G_{n-\beta_l}$ have dimension l. Notice also that $V_1 \in \Omega_\alpha(F_\bullet)$ and $V_2 \in \Omega_\beta(G_\bullet)$. Since $\Omega_\alpha(F_\bullet) \cap \Omega_\beta(G_\bullet) = \emptyset$ we deduce that $V_1 \neq V_2$, so $K = V_1 \cap V_2$ has dimension $l - 1$. We conclude that the only rational curve of degree 1 in X which meets the Schubert varieties for α, β, and p is the line $\mathbb{P}(W/K)$ of l-dimensional subspaces between K and W. $\square$

5. The quantum Giambelli formula

Now set $\sigma_i = 0$ for $i < 0$ and for $k < i < n$. The quantum Giambelli formula states that the classical formula (2) continues to be valid if all Schubert classes are replaced with the corresponding quantum Schubert classes [1].

Theorem 2 (Bertram)**.** *If λ is a partition contained in an $l \times k$ rectangle, then the Schubert class σ_λ in $QH^* \operatorname{Gr}(l, E)$ is given by $\sigma_\lambda = \det(\sigma_{\lambda_i + j - i})_{1 \le i, j \le l}$.*

Proof. We claim that if $0 \le i_j \le k$ for $1 \le j \le l$ then $\sigma_{i_1} \cdot \sigma_{i_2} \cdots \sigma_{i_l} = (\Omega_{i_1} \cdot \Omega_{i_2} \cdots \Omega_{i_l}) \otimes 1$, i.e. no q-terms show up when the first product is expanded in the quantum ring. Using induction, this can be established by proving that if $\ell(\mu) < l$ then the expansion of $\sigma_i \cdot \sigma_\mu$ involves no q-terms and no partitions of lengths greater than $\ell(\mu) + 1$. The claim therefore follows from Theorem 1. Since the determinant of the quantum Giambelli formula is a signed sum of products of the form $\sigma_{i_1} \cdot \sigma_{i_2} \cdots \sigma_{i_l}$, we conclude from the classical Giambelli formula (2) that $\det(\sigma_{\lambda_i + j - i}) = \det(\Omega_{\lambda_i + j - i}) \otimes 1 = \Omega_\lambda \otimes 1 = \sigma_\lambda$ as required. $\square$

Notice that this proof uses only that no q-terms are contained in a product $\sigma_i \cdot \sigma_\mu$ when $\ell(\mu) < l$ (in addition to the classical Giambelli and Pieri formulas). In fact, Theorem 1 could be replaced with the following lemma.

Lemma 3. *Let λ and μ be partitions contained in an $l \times k$ rectangle such that $\ell(\lambda) + \ell(\mu) \le l$. Then $\sigma_\lambda \cdot \sigma_\mu = (\Omega_\lambda \cdot \Omega_\mu) \otimes 1$.*

Proof. If $d \ge 1$ and ν is a partition such that $|\lambda| + |\mu| + |\nu| = lk + nd$, then any intersection $\Omega_{\hat\lambda}(F_\bullet) \cap \Omega_{\hat\mu}(G_\bullet) \cap \Omega_{\hat\nu}(H_\bullet)$ of general Schubert varieties in $\operatorname{Gr}(l + d, E)$ must be empty since $|\hat\lambda| + |\hat\mu| + |\hat\nu| \ge |\lambda| + |\mu| + |\nu| - 2dl = lk + dk - dl > (l + d)(k - d)$. This shows that $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d = 0$. $\square$

A dual version of this lemma states that if $\lambda_1 + \mu_1 \le k$ then $\sigma_\lambda \cdot \sigma_\mu = (\Omega_\lambda \cdot \Omega_\mu) \otimes 1$. This follows by replacing the span of a curve with its kernel in the above proof, or by using the duality isomorphism $QH^* \operatorname{Gr}(l, E) \cong QH^* \operatorname{Gr}(k, E^*)$.

6. The minimal power of q in a quantum product

As a further demonstration of the use of the span of a rational curve in X, we will give a short proof of a recent theorem of Fulton and Woodward [10] in the case of Grassmannians. It generalizes the fact that any product $\sigma_\lambda \cdot \sigma_\mu$ in $QH^* X$ is non-zero.

Given a partition λ and an integer $d \ge 0$, we let $\hat{\hat\lambda}$ denote the partition obtained by removing the top d rows and the leftmost d columns from λ. Recall that a product $\Omega_\lambda \cdot \Omega_\mu$ is non-zero in $H^* \operatorname{Gr}(l, E)$ if and only if $\lambda_i + \mu_j \le k$ for $i + j = l + 1$.

Theorem 3 (Fulton and Woodward)**.** *The smallest power of q which appears in a product $\sigma_\lambda \cdot \sigma_\mu$ in $QH^* X$ is equal to the smallest d for which $\Omega_{\hat{\hat\lambda}} \cdot \Omega_\mu \ne 0$ in $H^* X$.*

Proof. If $\sigma_\lambda \cdot \sigma_\mu$ contains q^d times a Schubert class then some curve of degree d meets each of the Schubert varieties $\Omega_\lambda(F_\bullet)$ and $\Omega_\mu(G_\bullet)$ where $F_\bullet$ and $G_\bullet$ are general flags. If $W \subset E$ has dimension $l + d$ and contains the span of this curve then W lies in the intersection $\Omega_{\hat\lambda}(F_\bullet) \cap \Omega_{\hat\mu}(G_\bullet)$ in $\operatorname{Gr}(l + d, E)$. In particular this intersection is not empty which implies that $\Omega_{\hat\lambda} \cdot \Omega_{\hat\mu} \ne 0$ in $H^* \operatorname{Gr}(l + d, E)$. Since this is equivalent to $\Omega_{\hat{\hat\lambda}} \cdot \Omega_\mu \ne 0$ in $H^* X$, this proves the inequality "$\ge$" of the theorem. (If $d > k$ then it is also true that $\Omega_{\hat{\hat\lambda}} \cdot \Omega_\mu \ne 0$.)

Now let d be the smallest number for which $\Omega_{\hat{\lambda}} \cdot \Omega_\mu \neq 0$. Notice that this implies that λ contains a $d \times d$ rectangle, i.e. $\lambda_d \geq d$. Set $\alpha = (k + d - \lambda_d, \ldots, k + d - \lambda_1)$ and let β be the partition given by $\beta_i = \max(d - \lambda_{l+1-i}, 0)$ for $1 \leq i \leq l$. If the Young diagram for λ is put in the upper-left corner of an l by $k + d$ rectangle, then α is the complement of λ in the top d rows of this rectangle, turned 180 degrees, and β is the complement of λ in the leftmost d columns, also turned.

It follows from the Littlewood-Richardson rule that the product $\sigma_\lambda \cdot \sigma_\beta$ contains the class $\sigma_{(d^l)+\hat{\lambda}} = \sigma_{(d^l)} \cdot \sigma_{\hat{\lambda}}$ and that $\sigma_{\hat{\lambda}} \cdot \sigma_\alpha$ contains $\sigma_{(k^d),\hat{\lambda}} = \sigma_{(k^d)} \cdot \sigma_{\hat{\hat{\lambda}}}$. Since the structure constants of QH^*X are all non-negative, this implies that $\sigma_\lambda \cdot \sigma_\beta \cdot \sigma_\alpha$ contains the product $\sigma_{(d^l)} \cdot \sigma_{(k^d)} \cdot \sigma_{\hat{\hat{\lambda}}} = q^d \sigma_{\hat{\hat{\lambda}}}$. Since $\Omega_{\hat{\hat{\lambda}}} \cdot \Omega_\mu \neq 0$ by assumption, we conclude that $\sigma_\lambda \cdot \sigma_\mu \cdot \sigma_\alpha \cdot \sigma_\beta$ contains q^d times some Schubert class. In particular, at least one term of the product $\sigma_\lambda \cdot \sigma_\mu$ must involve a power of q which is less than or equal to d. This proves the other inequality "$\leq$" in the theorem. Notice that the identities we have used in this argument follow easily from Theorem 1 and the dual version of Lemma 3, combined with the classical Pieri rule. $\qquad\square$

7. THE RIM-HOOK ALGORITHM

We will next recall how to carry out computations in the quantum cohomology ring of X. Let $c_i \in QH^*X$ denote the ith Chern class of the dual of the tautological subbundle $S \subset E \otimes \mathcal{O}_X$. This means that $c_i = \sigma_{(1^i)}$ for $0 \leq i \leq l$ and $c_i = 0$ for $i < 0$ and $i > l$. For $p \geq 1$ we then define $\sigma_p = \det(c_{1+j-i})_{1 \leq i,j \leq p}$ in QH^*X. Notice that for $p < n$ we have $\sigma_p = \Omega_p \otimes 1$, so this definition is compatible with our previous definition of σ_p. The definition implies that for every $m \geq 1$ we have

$$(5) \qquad\qquad \sum_{i=0}^{l} (-1)^i \, \sigma_{m-i} \, \sigma_{(1^i)} = 0 \, .$$

Lemma 4. *For any $p \geq k + 1$ we have $\sigma_p = (-1)^{l-1} q \, \sigma_{p-n}$.*

Proof. Both sides of the identity are zero for $k + 1 \leq p < n$. It follows from (5) with $m = n$ that $\sigma_n + (-1)^l \sigma_k \, \sigma_{(1^l)} = 0$, so $\sigma_n = (-1)^{l-1} \sigma_k \, \sigma_{(1^l)} = (-1)^{l-1} q$ by Theorem 1. For $p > n$ we finally obtain

$$\sigma_p = \sum_{i=1}^{l} (-1)^{i-1} \sigma_{p-i} \, \sigma_{(1^i)} = (-1)^{l-1} q \sum_{i=1}^{l} (-1)^{i-1} \sigma_{p-n-i} \, \sigma_{(1^i)} = (-1)^{l-1} \, q \, \sigma_{p-n}$$

by induction, which proves the lemma. $\qquad\square$

If $I = (I_1, I_2, \ldots, I_p)$ is any sequence of integers we set $\sigma_I = \det(\sigma_{I_i+j-i})_{1 \leq i,j \leq p}$ in QH^*X. This is compatible with our notation σ_λ for partitions λ by Theorem 2. Any element σ_I can be rewritten as zero or plus or minus σ_μ for some partition μ by moves of the form $\sigma_{I,a,a+1,J} = 0$ and $\sigma_{I,a,b,J} = -\sigma_{I,b-1,a+1,J}$, which correspond to interchanging rows in the determinant defining σ_I. Notice that if λ is a partition then we have the formal identity $\sigma_\lambda = \det(c_{\lambda'_i+j-i})$ where λ' is the conjugate partition of λ, obtained by mirroring the Young diagram for λ in its diagonal. In particular, if $\ell(\lambda) > l$ then the top row of the latter determinant is zero, and $\sigma_\lambda = 0$.

Corollary 1. *Let λ be a partition with at most l parts. For each $1 \leq j \leq l$, choose $I_j \in \mathbb{Z}$ such that $I_j \equiv \lambda_j \pmod{n}$ and $j - l \leq I_j < j - l + n$. Then we have*

$$\sigma_\lambda = (-1)^{d(l-1)} \, q^d \, \sigma_I \in QH^*X$$

where $I = (I_1, \ldots, I_l)$ and $nd = |\lambda| - \sum I_j$.

Notice that the moves described above will rewrite the chosen element σ_I to zero or plus or minus a basis element σ_μ of QH^*X with μ contained in an $l \times k$ rectangle. The corollary is equivalent to the dual rim-hook algorithm of [2]. The formulation in terms of reduction modulo n is due to F. Sottile [19], who in turn was inspired by the work of Ravi, Rosenthal, and Wang [16]. This formulation gives an efficient algorithm to determine if two elements σ_λ and σ_μ are identical. In particular it shows that $\sigma_\lambda = 0$ if and only if $\lambda_i - i \equiv \lambda_j - j \pmod{n}$ for some $i \neq j$.

Corollary 1 makes it easy to do computations in QH^*X (cf. [11, 20, 2] and [12, Ex. 13.35]). In order to expand a product $\sigma_\lambda \cdot \sigma_\mu$, one uses the classical Littlewood-Richardson rule to write $\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda\mu}^\nu \sigma_\nu$ where the sum is over partitions ν of length at most l and $c_{\lambda\mu}^\nu$ is the Littlewood-Richardson coefficient. Each of the elements σ_ν can then be reduced to a basis element of QH^*X by the corollary.

8. Generators and relations

Finally, we show how to deduce Siebert and Tian's presentation of the quantum cohomology ring of X [18]. We claim that

$$(6) \qquad QH^*X = \mathbb{Z}[c_1, \ldots, c_l, q]/(\sigma_{k+1}, \ldots, \sigma_{n-1}, \sigma_n + (-1)^l q).$$

In fact, all of the given relations vanish in QH^*X by Lemma 4. On the other hand, the proof of Lemma 4 together with Corollary 1 shows that the linear map $H^*X \otimes \mathbb{Z}[q] \to \mathbb{Z}[c_1, \ldots, c_l, q]/(\sigma_{k+1}, \ldots, \sigma_{n-1}, \sigma_n + (-1)^l q)$ which sends $\Omega_\lambda \otimes q^d$ to $q^d \sigma_\lambda$ is surjective, which proves the isomorphism. Siebert and Tian gave this presentation in its dual form $QH^*X = \mathbb{Z}[\sigma_1, \ldots, \sigma_k, q]/(\tilde{Y}_{l+1}, \ldots, \tilde{Y}_{n-1}, \tilde{Y}_n + (-1)^k q)$ where the $\tilde{Y}_i$ are defined inductively by $\tilde{Y}_0 = 1$, $\tilde{Y}_i = 0$ for $i < 0$, and $\tilde{Y}_i = \sum_{j=1}^k (-1)^{j-1} \sigma_j \tilde{Y}_{i-j}$ for $i > 0$.

The presentation (6) can also be stated as $QH^*X = \mathbb{Z}[c_1, \ldots, c_l]/I^{(l,n)}$ where $I^{(l,n)} = (\sigma_{k+1}, \ldots, \sigma_{n-1})$. In this form the variable q is represented by $(-1)^{l-1}\sigma_n = \sigma_{(k+1, 1^{l-1})}$. In [11] other generators for the ideal $I^{(l,n)}$ are used, namely all elements σ_λ for which $\lambda_1 - \lambda_l = k + 1$. It is an easy exercise to show that these elements in fact generate the same ideal, and to show that the basis $\{q^d \sigma_\lambda \mid \lambda_1 \leq k\}$ of QH^*X is equal to the set $\{\sigma_\lambda \mid \lambda_1 - \lambda_l \leq k\}$ which is the basis used in [11]. We refer to [19] for a more detailed discussion of the relations of quantum cohomology of Grassmannians with other fields.

References

[1] A. Bertram, *Quantum Schubert calculus*, Adv. Math. **128** (1997), 289–305.

[2] A. Bertram, I. Ciocan-Fontanine, and W. Fulton, *Quantum multiplication of Schur polynomials*, J. Algebra **219** (1999), 728–746.

[3] A. S. Buch, A. Kresch, and H. Tamvakis, *Gromov-Witten invariants on Grassmannians*, preprint, 2002.

[4] A. S. Buch, *A direct proof of the quantum version of Monk's formula*, to appear in Proc. Amer. Math. Soc., 2002.

[5] _____, *Quantum cohomology of partial flag varieties*, in preparation, 2002.

[6] I. Ciocan-Fontanine, *On quantum cohomology rings of partial flag varieties*, Duke Math. J. **98** (1999), 485–524.

[7] S. Fomin, S. Gelfand, and A. Postnikov, *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10** (1997), 565–596.

[8] W. Fulton, *Young tableaux*, Cambridge University Press, 1997.

[9] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96.

[10] W. Fulton and C. Woodward, *On the quantum product of Schubert classes*, preprint, 2001.

[11] F. M. Goodman and H. Wenzl, *Littlewood-Richardson coefficients for Hecke algebras at roots of unity*, Adv. Math. **82** (1990), 244–265.

[12] V. G. Kac, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, 1990.

[13] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Mirror symmetry, II, Amer. Math. Soc., Providence, RI, 1997, pp. 607–653.

[14] A. Kresch and H. Tamvakis, *Quantum cohomology of the Lagrangian Grassmannian*, preprint, 2001.

[15] _____, *Quantum cohomology of orthogonal Grassmannians*, preprint, 2001.

[16] M. S. Ravi, J. Rosenthal, and X. Wang, *Degree of the generalized Plücker embedding of a Quot scheme and quantum cohomology*, Math. Ann. **311** (1998), 11–26.

[17] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, Math. Res. Lett. **1** (1994), 269–278.

[18] B. Siebert and G. Tian, *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, Asian J. Math. **1** (1997), 679–695.

[19] F. Sottile, *Rational curves on Grassmannians: systems theory, reality, and transversality*, Contemporary Mathematics **276** (2001), 9–42.

[20] M. A. Walton, *Fusion rules in Wess-Zumino-Witten models*, Nuclear Phys. B **340** (1990), 777–790.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, BUILDING 2, ROOM 275, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139

E-mail address: abuch@math.mit.edu