Solutions to homework set 10-12 (selected problems)

3.4 1(b,d).

The relation R of (b) is antisymmetric. This follows directly form the definition of antisymmetric.

The relation R of (d) is not antisymmetric because $(1/2, 1) \in R$ and $(1, 1/2) \in \mathbb{R}$.

3.4 3(b).

Theorem: Let R be a relation on the set A that satisfies
(i) R is antisymmetric, (ii) R is symmetric, and (iii) $\mathrm{Dom}(R) = A$.
Then $R = I_A$.

Proof. Let $(x, y) \in R$.
Then $x \in A$ and $y \in A$.
Since R is symmetric we have $(y, x) \in R$.
Since R is antisymmetric, we must have $x = y$.
Therefore $(x, y) = (x, x) \in I_A$.
This proves that $R \subset I_A$.
Now let $(x, y) \in I_A$.
By definition of I_A we have $x \in A$ and $y = x$.
Since $x \in A = \mathrm{Dom}(R)$, we can choose $z \in A$ such that $(x, z) \in R$.
Since R is symmetric, we also have $(z, x) \in R$.
Since R is antisymmetric we must have $z = x$.
It follows that $(x, y) = (x, x) = (x, z) \in R$.
This proves $I_A \subset R$. □

3.4 6.

Set $P = \mathbb{R} \times \mathbb{R}$.
Define $R = \{((a, b), (x, y)) \in P \times P \mid a \leq x \text{ and } b \leq y\}$

Theorem: R is a partial order on P.

Proof. We must show that R is reflexive, antisymmetric, and transitive.
This is the following three claims.
Claim 1: $\forall p \in P : (p, p) \in R$.
Let $p \in P$. Choose $x, y \in \mathbb{R}$ such that $p = (x, y)$.
Since $x \leq x$ and $y \leq y$, we have $(p, p) = ((x, y), (x, y)) \in R$.
Claim 2: $\forall p, q \in P : (\ (p, q) \in R \text{ and } (q, p) \in R\ ) \Rightarrow p = q$
Let $p, q \in P$.
Assume that $(p, q) \in R$ and $(q, p) \in R$.
Choose $a, b \in \mathbb{R}$ such that $p = (a, b)$.
Choose $x, y \in \mathbb{R}$ such that $q = (x, y)$.
Since $(p, q) \in R$ we have $a \leq x$ and $b \leq y$.
Since $(q, p) \in R$ we have $x \leq a$ and $y \leq b$.
This implies that $a = x$ and $b = y$.
Therefore $p = q$.
Claim 3: $\forall p, q, r \in P$: $(\ (p, q) \in R \text{ and } (q, r) \in R\ ) \Rightarrow (p, r) \in R$
Let $p, q, r \in P$.
Assume that $(p, q) \in R$ and $(q, r) \in R$.
Choose $a, b \in \mathbb{R}$ such that $p = (a, b)$.
Choose $c, d \in \mathbb{R}$ such that $q = (c, d)$.
Choose $e, f \in \mathbb{R}$ such that $r = (e, f)$.

Since $(p,q) \in R$ we have $a \leq c$ and $b \leq d$.
Since $(q,r) \in R$ we have $c \leq e$ and $d \leq f$.
This implies that $a \leq e$ and $b \leq f$.
Therefore $(p,r) \in R$. □

3.4 12(b).
Let A be a non-empty set.
The inclusion relation on the power set $\mathcal{P}(A)$ is defined by
$R = \{(S,T) \in \mathcal{P}(A) \times \mathcal{P}(A) \mid S \subset T\}$
I will not prove that R is a partial order on $\mathcal{P}(A)$.

Theorem:
$\forall B \in \mathcal{P}(A)\ \forall x \in A : x \notin B \Rightarrow (\ B$ is an immediate predecessor of $B \cup \{x\}\ )$

Proof. Let $B \in \mathcal{P}(A)$ and let $x \in A$.
Assume that $x \notin B$.
Set $D = B \cup \{x\}$.
We must show that B is an immediate predecessor of D.
This is equivalent to the following three claims.
Claim 1: $B \neq D$.
This is true because $x \notin B$ and $x \in D$.
Claim 2: $(B,D) \in R$.
This is true because $B \subset D$.
Claim 3: $\forall C \in \mathcal{P}(A) : (\ (B,C) \in R$ and $(C,D) \in R\ ) \Rightarrow (\ C = B$ or $C = D\ )$
Let $C \in \mathcal{P}(A)$.
Assume that $(B,C) \in R$ and $(C,D) \in R$.
Then $B \subset C$ and $C \subset D$.
Case 1: Assume that $x \in C$.
Since $C \subset D$ and $D = B \cup \{x\} \subset C \cup \{x\} = C$, it follows that $C = D$.
Case 2: Assume that $x \notin C$. I will show that $C = B$.
Let $y \in C$.
Since $C \subset D = B \cup \{x\}$, we must have $y \in B \cup \{x\}$.
This implies that $y \in B$ or $y \in \{x\}$.
Since $y \in C$ and $x \notin C$, we have $y \neq x$, hence $y \notin \{x\}$.
Therefore $y \in B$.
This proves that $C \subset B$
Since we also have $B \subset C$ by assumption, we obtain $C = B$.
We conclude that $(\ C = B$ or $C = D\ )$ is true. □

3.4 13(a,d). Let R be a rectangle with horizontal and vertical sides of positive lengths.

Let H be the set of all rectangles with horizontal and vertical sides of positive lengths that are contained in R.

Consider the partial order $\subset$ on H given by inclusion of rectangles.

Theorem 1: $\forall S \in \mathcal{P}(H) : R$ is an upper bound of S.
This is true because for each rectangle $Q \in H$ we have $Q \subset R$.

Theorem 2: $\exists S \in \mathcal{P}(H) : S$ does not have a smallest upper bound.
Take $S = \emptyset$.
Then every rectangle $Q \in H$ is an upper bound for S.
Assume that Q_0 is a smallest upper bound for S.

Then Q_0 is a smallest element of H.
Therefore $Q_0 \subset \bigcap_{Q \in H} Q = \emptyset$.
It follows that $Q_0 = \emptyset \notin H$, a contradiction.

Theorem 2a: $\forall S \in \mathcal{P}(H) : S \neq \emptyset \Rightarrow S$ has a smallest upper bound.

This is a consequence of the fact that any non-empty bounded subset A of the real numbers $\mathbb{R}$ has a smallest upper bound $\sup A$ and a greatest lower bound $\inf A$.

Assume that R is placed in a coordinate system (with horizontal x-axis and vertical y-axis).

For any rectangle $Q \in H$ we denote the lower-left corner of Q by $(x_1(Q), y_1(Q))$ and we denote the upper-right corner of Q by $(x_2(Q), y_2(Q))$.

Given two rectangles $Q, Q' \in H$ we then have $Q \subset Q'$ if and only if
$(x_1(Q) \geq x_1(Q')$ and $y_1(Q) \geq y_1(Q')$ and $x_2(Q) \leq x_2(Q')$ and $y_2(Q) \leq y_2(Q'))$.
Let $S \in H$ and assume $S \neq \emptyset$.
Then the smallest upper bound for S is the unique rectangle Q' satisfying:
$x_1(Q') = \inf\{x_1(Q) \mid Q \in S\}$
$y_1(Q') = \inf\{y_1(Q) \mid Q \in S\}$
$x_2(Q') = \sup\{x_2(Q) \mid Q \in S\}$
$y_2(Q') = \sup\{y_2(Q) \mid Q \in S\}$

Since Theorem 2a is strictly speaking not necessary in order to answer problem 3.3(a), I will not prove this. However this is not hard, one simply have to work systematically with the definitions.

Theorem 3: $\exists S \in \mathcal{P}(H) : S$ does not have a smallest element.

Let $Q_1, Q_2 \in H$ be rectangles contained in R such that $Q_1 \not\subset Q_2$ and $Q_2 \not\subset Q_1$.
Take $S = \{Q_1, Q_2\}$.
Since no element of S is a lower bound for S, S has no smallest element.

4.1 1(b,c,d,e).

(b) The set is not a function because 1 is paired with more than one integer.

(c) The relation is a function with domain $\{1, 2\}$ and range $\{1, 2\}$. Another possible codomain is $\mathbb{Z}$.

(d) The relation is not a function because it contains $(0, 0)$ and $(0, \pi)$.

(e) The relation is not a function because it contains $(1, 1)$ and $(1, 2)$.

4.1 3(b). Let $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2 + 5\}$.
$\mathrm{Dom}(f) = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} : (x, y) \in f\} = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} : y = x^2 + 5\} = \mathbb{R}$.
$\mathrm{Rng}(f) = \{y \in \mathbb{R} \mid \exists x \in \mathbb{R} : y = x^2 + 5\} = \{y \in \mathbb{R} \mid y \geq 5\}$.
The set $\mathbb{R}$ is an alternative codomain.

4.1 13.

Theorem: $\emptyset$ is a function with domain $\emptyset$.

Proof. I will show that $\emptyset$ is a function from $\emptyset$ to $\emptyset$.
This means that:
$\forall x \in \emptyset\ \exists y \in \emptyset : (x, y) \in \emptyset$.
This is true because every statement of the form $\forall x \in \emptyset : P(x)$ is true. □

Theorem: Let A and B be sets and let $f : A \to B$ be a function. Then the following are equivalent:
(1) $A = \emptyset$
(2) $f = \emptyset$.

(3) $\mathrm{Rng}(f) = \emptyset$

Proof. (1) $\Rightarrow$ (2): Assume $A = \emptyset$.
Since $f \subset A \times B = \emptyset$, it follows that $f = \emptyset$.
(2) $\Rightarrow$ (3): Assume $f = \emptyset$.
Then $\mathrm{Rng}(f) = \{y \in B \mid \exists x \in A : (x, y) \in f\} = \emptyset$.
(3) $\Rightarrow$ (1): Assume $A \neq \emptyset$.
Choose $x \in A$.
Since f is a function we can choose $y \in B$ such that $(x, y) \in f$.
But then $y \in \mathrm{Rng}(f)$, so $\mathrm{Rng}(f) \neq \emptyset$. □

4.2 5(b).
Consider the function $f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = 2x^2 + 1\}$.
The inverse relation is $f^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = 2y^2 + 1\}$.
This is not a function because $(3, -1) \in f^{-1}$ and $(3, 1) \in f^{-1}$, but $-1 \neq 1$.

4.2 5(g).
Set $A = \mathbb{R} - \{1\}$ and $B = \mathbb{R} - \{0\}$.
Consider the relation $f = \{(x, y) \in A \times B \mid y = \frac{1}{1-x}\}$.
Then f is a function $f : A \to B$.
(I will not prove this and we do not need to know that f is a function.)
The inverse relation is given by:
$f^{-1} = \{(x, y) \in B \times A \mid x = \frac{1}{1-y}\} = \{(x, y) \in B \times A \mid x(1 - y) = 1\}$
$= \{(x, y) \in B \times A \mid 1 - y = x^{-1}\} = \{(x, y) \in B \times A \mid y = 1 - x^{-1}\}$.
Claim: $f^{-1} : B \to A$ is a function.
Must show: $\forall x \in B\ \exists! y \in A : (x, y) \in f$.
Let $x \in B$.
Since $x \in \mathbb{R}$ and $x \neq 0$, it follows that $x^{-1} \in \mathbb{R}$.
It follows that $1 - x^{-1} \in \mathbb{R}$.
Notice also that $1 - x^{-1} \neq 1$, hence $1 - x^{-1} \in A$.
Since $(x, 1 - x^{-1}) \in f$, we have shown: $\exists y \in A : (x, y) \in f$.
Let $y_1, y_2 \in A$. Assume $(x, y_1) \in f$ and $(x, y_2) \in f$.
Then we have $y_1 = 1 - x^{-1}$ and $y_2 = 1 - x^{-1}$, hence $y_1 = y_2$.
This finishes the proof that f^{-1} is a function.
Finally, for $x \in B$ we have $f^{-1}(x) = 1 - x^{-1}$.

4.2 15.
Let $f : A \to B$ and $g : C \to D$ be functions.
Define $f \times g = \{((a, c), (b, d)) \mid (a, b) \in f \text{ and } (c, d) \in g\}$.
(a) Claim: $f \times g : A \times C \to B \times D$ is a function.
We must show: $\forall x \in A \times C\ \exists! y \in B \times D$: $(x, y) \in f \times g$.
Let $x \in A \times C$.
Choose $a \in A$ and $c \in C$ such that $x = (a, c)$.
Set $b = f(a)$, $d = g(c)$, and $y = (b, d)$.
Since $(a, b) \in f$ and $(c, d) \in g$, we have $(x, y) \in f \times g$.
Let $y_1, y_2 \in B \times D$.
Assume $(x, y_1) \in f \times g$ and $(x, y_2) \in f \times g$.
Choose $b_1, b_2 \in B$ and $d_1, d_2 \in D$ such that $y_1 = (b_1, d_1)$ and $y_2 = (b_2, d_2)$.
Since $(x, y_1) \in f \times g$, we have $(a, b_1) \in f$ and $(c, d_1) \in g$.
Since $(x, y_2) \in f \times g$, we have $(a, b_2) \in f$ and $(c, d_2) \in g$.

Since $(a, b_1) \in f$ and $(a, b_2) \in f$ and f is a function, it follows that $b_1 = b_2$.
Since $(c, d_1) \in g$ and $(c, d_2) \in g$ and g is a function, it follows that $d_1 = d_2$.
Therefore $y_1 = (b_1, d_1) = (b_2, d_2) = y_2$.
(b) Let $(a, c) \in A \times C$.
Claim: $(f \times g)(a, c) = (f(a), g(c))$.
Set $b = f(a)$ and $d = g(c)$.
Since $(a, b) \in f$ and $(c, d) \in g$, we have $((a, c), (b, d)) \in f \times g$.
It follows that $(f \times g)(a, c) = (b, d) = (f(a), g(c))$.

4.3 1(d).
Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$.
Claim: f is onto $\mathbb{R}$.
Must show: $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}$: $f(x) = y$.
Let $y \in \mathbb{R}$.
Set $c = |y| + 1$.
Then $c^3 = |y|^3 + 3|y|^2 + 3|y| + 1 > |y|$.
It follows that $f(-c) < y < f(c)$.
Notice that f is continuous on the closed interval $[-c, c]$.
The intermediate value theorem therefore implies that:
$\exists x \in \mathbb{R}$: $f(x) = y$.
This is what we had to prove.

4.3 1(g).
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin(x)$.
Since we have $-1 \le \sin(x) \le 1$ for all $x \in \mathbb{R}$, it follows that $2 \notin \text{Rng}(f)$.
Therefore f is not onto $\mathbb{R}$.

4.3 1(h).
Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x, y) = x - y$.
Claim: f is onto $\mathbb{R}$.
Must show: $\forall z \in \mathbb{R}\ \exists a \in \mathbb{R} \times \mathbb{R} : f(a) = z$.
Let $z \in \mathbb{R}$.
Set $a = (z, 0) \in \mathbb{R} \times \mathbb{R}$.
Then $f(a) = f(z, 0) = z$.

4.3 10.
Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function.
This means: $\forall x_1, x_2 \in \mathbb{R} : x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
Claim: f is one-to-one.
We must show: $\forall x_1, x_2 \in \mathbb{R}$: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
I will prove the equivalent statement: $\forall x_1, x_2 \in \mathbb{R}$: $x_1 \ne x_2 \Rightarrow f(x_1) \ne f(x_2)$.
Let $x_1, x_2 \in \mathbb{R}$.
Assume $x_1 \ne x_2$.
Case 1: Assume $x_1 < x_2$.
Then $f(x_1) < f(x_2)$, hence $f(x_1) \ne f(x_2)$.
Case 2: Assume $x_2 < x_1$.
Then $f(x_2) < f(x_1)$, hence $f(x_1) \ne f(x_2)$.

4.4 3(d).

Define $G : (3, \infty) \to (5, \infty)$ by $G(x) = \dfrac{5x - 5}{x - 3}$.
Define $F : (5, \infty) \to (3, \infty)$ by $F(x) = \dfrac{3x - 5}{x - 5}$.
Claim: $F \circ G = I_{(3,\infty)}$ and $G \circ F = I_{(5,\infty)}$.
Let $x \in (3, \infty)$.
Set $y = G(x)$. Then we have:
$y = \frac{5x-5}{x-3}$.
$xy - 3y = 5x - 5$
$xy - 5x = 3y - 5$
$x = \frac{3y-5}{y-5}$.
It follows that $(F \circ G)(x) = F(G(x)) = F(y) = x$.
Let $x \in (5, \infty)$.
Set $y = F(x)$. Then we have:
$y = \frac{3x-5}{x-5}$
$xy - 5y = 3x - 5$
$xy - 3x = 5y - 5$
$x = \frac{5y-5}{y-3}$.
It follows that $(G \circ F)(x) = G(F(x)) = G(y) = x$.

4.4 6.

Let $F : A \to B$ and $G : B \to A$ be functions.
Claim:
$(G \circ F = I_A$ and $F \circ G = I_B) \Rightarrow (F$ is 1-1 and onto B, and G is 1-1 and onto $A)$
Proof: Assume that $G \circ F = I_A$ and $F \circ G = I_B$.
Then Theorem 4.4.4(a) implies that $G = F^{-1}$.
Since F^{-1} is a function, it follows from Theorem 4.4.2(a) that F is one-to-one.
Since $\text{Rng}(F) = \text{Dom}(F^{-1}) = \text{Dom}(G) = B$, it follows that F is onto B.
A similar argument shows that G is 1-1 and onto A.

Note: To get the most out of the solutions to section 4.6, you need to figure out what was on my scratch paper when I did the problems.

4.6 5(b).

Let (x_n) be the sequence defined by $x_n = \frac{n+1}{n}$.
Claim: $x_n \to 1$ as $n \to \infty$.
Must show: $\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N} :\ n > N \Rightarrow |x_n - 1| < \epsilon$.
Let $\epsilon > 0$.
Choose $N \in \mathbb{N}$ so large that $N > \frac{1}{\epsilon}$.
Will show: $\forall n \in \mathbb{N} :\ n > N \Rightarrow |x_n - 1| < \epsilon$.
Let $n \in \mathbb{N}$.
Assume $n > N$.
Then $|x_n - 1| = |\frac{n+1}{n} - 1| = \frac{1}{n} < \frac{1}{N} < \epsilon$.

4.6 5(c).

Define (x_n) by $x_n = n^2$.
Claim: The sequence (x_n) diverges.
Must show: $\sim (\ \exists L \in \mathbb{R}\ \forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N} :\ n > N \Rightarrow |x_n - L| < \epsilon\ )$
Equivalently: $\forall L \in \mathbb{R}\ \exists \epsilon > 0\ \forall N \in \mathbb{N}\ \exists n \in \mathbb{N} :\ n > N \wedge |x_n - L| \geq \epsilon$
Let $L \in \mathbb{R}$.

Set $\epsilon = 1$.
I will show: $\forall N \in \mathbb{N}\ \exists n \in \mathbb{N} : n > N \wedge |x_n - L| \geq \epsilon$
Let $N \in \mathbb{N}$.
Choose $n \in \mathbb{N}$ so large that $n > \max(N, L+1)$.
Then $n^2 \geq n > L + 1$.
It follows that $|x_n - L| = n^2 - L \geq n - L > 1 = \epsilon$.

4.6 5(f).
Define (x_n) by $x_n = \sqrt{n+1} - \sqrt{n}$.
Claim: $x_n \to 0$ as $n \to \infty$.
Must show: $\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N} : n > N \Rightarrow |x_n - 0| < \epsilon$
Let $\epsilon > 0$.
Choose $N \in \mathbb{N}$ so large that $N > \frac{1}{\epsilon^2}$.
Will show: $\forall n \in \mathbb{N} : n > N \Rightarrow |x_n - 0| < \epsilon$
Let $n \in \mathbb{N}$.
Assume $n > N$.
Then $1 < \epsilon^2 N < 4\epsilon^2 n$.
It follows that $1 < 2\epsilon\sqrt{n}$.
Therefore $n + 1 < n + 2\epsilon\sqrt{n} < n + 2\epsilon\sqrt{n} + \epsilon^2 = (\sqrt{n} + \epsilon)^2$.
We deduce that $\sqrt{n+1} < \sqrt{n} + \epsilon$.
Finally, we obtain $|x_n - 0| = \sqrt{n+1} - \sqrt{n} < \epsilon$.

4.6 5(h).
Define (x_n) by $x_n = \frac{6}{2^n}$.
Claim: $x_n \to 0$ as $n \to \infty$.
Must show: $\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N} : n > N \Rightarrow |x_n - 0| < \epsilon$
let $\epsilon > 0$.
Choose $N \in \mathbb{N}$ so large that $N > \frac{6}{\epsilon}$.
Will show: $\forall n \in \mathbb{N} : n > N \Rightarrow |x_n - 0| < \epsilon$
Let $n \in \mathbb{N}$.
Assume $n > N$.
Then $|x_n - 0| = \frac{6}{2^n} < \frac{6}{n} < \frac{6}{N} < \epsilon$.

4.6 6.
Let (x_n) and (y_n) be sequences of real numbers, and let $L, M, r \in \mathbb{R}$.
Assume that $x_n \to L$ for $n \to \infty$, and that $y_n \to M$ for $n \to \infty$.

(b) Define (z_n) by $z_n = x_n - y_n$.
Claim: $z_n \to L - M$ as $n \to \infty$.
Must show: $\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N} : n > N \Rightarrow |z_n - (L - M)| \leq \epsilon$
Let $\epsilon > 0$.
Since $x_n \to L$, I can choose $N_1 \in \mathbb{N}$ such that: $\forall n \in \mathbb{N} : n > N_1 \Rightarrow |x_n - L| \leq \frac{\epsilon}{2}$.
Since $y_n \to M$, I can choose $N_2 \in \mathbb{N}$ such that: $\forall n \in \mathbb{N} : n > N_2 \Rightarrow |y_n - M| \leq \frac{\epsilon}{2}$.
Set $N = \max(N_1, N_2)$.
Will show: $\forall n \in \mathbb{N} : n > N \Rightarrow |z_n - (L - M)| < \epsilon$.
Let $n \in \mathbb{N}$.
Assume $n > N$.
Then $n > N_1$ and $n > N_2$.
It follows that:
$|z_n - (L - M)| = |(x_n - L) + (M - y_n)| \leq |x_n - L| + |M - y_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

(e) Define (z_n) by $z_n = x_n y_n$.
Claim: $z_n \to LM$ as $n \to \infty$.
Must show: $\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N} : n > N \Rightarrow |z_n - LM| \leq \epsilon$
Let $\epsilon > 0$.
Since $x_n \to L$, I can choose $N_1 \in \mathbb{N}$ such that:
$\forall n \in \mathbb{N} : n > N_1 \Rightarrow |x_n - L| < \min(1, \frac{\epsilon}{2(|M|+1)})$.
Since $y_n \to M$, I can choose $N_2 \in \mathbb{N}$ such that:
$\forall n \in \mathbb{N} : n > N_2 \Rightarrow |y_n - M| < \frac{\epsilon}{2(|L|+1)}$
Set $N = \max(N_1, N_2)$.
Will show: $\forall n \in \mathbb{N} : n > N \Rightarrow |z_n - LM| < \epsilon$.
Let $n \in \mathbb{N}$.
Assume $n > N$.
Then we have $|x_n - L| < \min(1, \frac{\epsilon}{2(|M|+1)})$ and $|y_n - M| < \frac{\epsilon}{2(|L|+1)}$.
It follows that $|x_n| = |L + x_n - L| \leq |L| + |x_n - L| \leq |L| + 1$.
We obtain:
$|z_n - LM| = |x_n y_n - LM| = |x_n y_n - x_n M + x_n M - LM|$
$\leq |x_n y_n - x_n M| + |x_n M - LM| = |x_n| \cdot |y_n - M| + |x_n - L| \cdot |M|$
$< (|L| + 1)\frac{\epsilon}{2(|L|+1)} + \frac{\epsilon}{2(|M|+1)}|M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

4.6 8(c).
Let (x_n) be a sequences of real numbers, and let $L \in \mathbb{R}$.
Assume that $x_n \to L$ as $n \to \infty$.
Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function.
This means that we have: $\forall m, n \in \mathbb{N} : m < n \Rightarrow f(m) < f(n)$.
Define a new sequence (y_n) by setting $y_n = x_{f(n)}$ for each $n \in \mathbb{N}$.
Then (y_n) is a *subsequence* of (x_n).
Example: If $f(n) = 2n$, then $(y_n) = (x_2, x_4, x_6, \dots)$.

Claim 1: $\forall n \in \mathbb{N} : n \leq f(n)$.
We prove this by induction on n.
Basis step: Since $f(1) \in \mathbb{N}$, we have $1 \leq f(1)$.
Inductive step: Let $n \in \mathbb{N}$. Assume $n \leq f(n)$.
Since f is increasing, we have $f(n) < f(n+1)$.
It follows that $n + 1 \leq f(n) + 1 \leq f(n+1)$.
We conclude by the PMI that Claim 1 is true.

Claim 2: $y_n \to L$ as $n \to \infty$.
Must show: $\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N} : n > N \Rightarrow |y_n - L| \leq \epsilon$
Let $\epsilon > 0$.
Since $x_n \to L$, we may choose $N \in \mathbb{N}$ such that: $\forall n \in \mathbb{N} : n > N \Rightarrow |x_n - L| < \epsilon$.
Will show: $\forall n \in \mathbb{N} : n > N \Rightarrow |y_n - L| < \epsilon$.
Let $n \in \mathbb{N}$.
Assume $n > N$.
Then $f(n) \geq n > N$.
It follows that $|y_n - L| = |x_{f(n)} - L| < \epsilon$.