## Solution to selected homework problems

Here are solutions to some selected problems from homework sets 2 and 3. Most of the proofs consists of skeletons. You might benefit from identifying these skeletons, for example by drawing boxes around them. And from completing the skeletons in cases where I did not include the last line of them. Enjoy!

— Anders.

**1.4 6(d).**
**Theorem:** $\forall a, b \in \mathbb{R} : \ |a + b| \le |a| + |b|$

*Proof.* Let $a, b \in \mathbb{R}$. We must show that $|a + b| \le |a| + |b|$.
We consider 4 cases:
Case 1: Assume that $a \ge 0$ and $b \ge 0$.
Then $a + b \ge 0$, and we have $|a| = a$, $|b| = b$, and $|a + b| = a + b$.
It follows that $|a + b| = |a| + |b|$.
Case 2: Assume that $a < 0$ and $b < 0$.
Then $a + b < 0$, and we have $|a| = -a$, and $|b| = -b$, and $|a + b| = -a - b$.
It follows that $|a + b| = |a| + |b|$.
Case 3: Assume that $a < 0 \le b$.
Then $|a| = -a$ and $|b| = b$.
We consider two subcases.
Case 3a: Assume that $a + b \ge 0$.
Then $|a + b| = a + b$.
It follows that $|a + b| = a + b = -|a| + |b| < |a| + |b|$ (since $|a| > 0$.)
Case 3b: Assume that $a + b < 0$.
Then $|a + b| = -a - b$.
It follows that $|a + b| = -a - b = |a| - |b| \le |a| + |b|$ (since $|b| \ge 0$.)
Case 4: Assume that $b < 0 \le a$:
By interchanging $a$ and $b$, we can use Case 3 to deduce that $|a + b| \le |a| + |b|$.
Since we have exhausted all possibilities for $a$ and $b$, we conclude that $|a + b| \le |a| + |b|$.
Since $a, b \in \mathbb{R}$ were arbitrary, we have proved: $\forall a, b \in \mathbb{R} : \ |a + b| \le |a| + |b|$ □

**1.4 9(c).**
**Theorem:** $\forall a, b, c \in \mathbb{R} : \ (ab > 0 \text{ and } bc < 0) \Rightarrow$
$(\exists x_1, x_2 \in \mathbb{R} : x_1 \ne x_2 \text{ and } ax_1^2 + bx_1 + c = ax_2^2 + bx_2 + c = 0)$

*Proof.* Let $a, b, c \in \mathbb{R}$.
Assume that $ab > 0$ and $bc < 0$.
It follows that $ab^2c = (ab)(bc) < 0$, hence $ac < 0$.
Set $D = b^2 - 4ac$. Since $ac < 0$ we deduce that $D > 0$.
Set $x_1 = \frac{-b+\sqrt{D}}{2a}$ and $x_2 = \frac{-b-\sqrt{D}}{2a}$.
Since $D > 0$ and $a \ne 0$, it follows that $x_1, x_2 \in \mathbb{R}$ and $x_1 \ne x_2$.
Finally, a calculation shows that $ax_1^2 + bx_1 + c = ax_2^2 + bx_2 + c = 0$. □

**1.5 7(b).**
**Theorem:** $\forall a, b, c \in \mathbb{N} : (a + 1 \text{ divides } b \text{ and } b \text{ divides } b + 3) \Leftrightarrow (a = 2 \text{ and } b = 3)$

*Proof.* Let $a, b, c \in \mathbb{N}$.
Assume that $a = 2$ and $b = 3$.
Then $a + 1 = 3$ and $b + 3 = 6$, so $a + 1$ divides $b$ and $b$ divides $b + 3$.
On the other hand, assume that $a + 1$ divides $b$ and $b$ divides $b + 3$.
Then $b$ divides 3, so $b = 1$ or $b = 3$.
Since $a + 1$ divides $b$ and $a + 1 \geq 2$, we also have $b \geq 2$.
It follows that $b = 3$.
Since $a + 1$ divides 3 and $a + 1 \geq 2$, we must have $a + 1 = 3$, hence $a = 2$. □

**1.6 3.**
**Conjecture 1:** $\forall n \in \mathbb{N} : (n \text{ is even and } n > 2) \Rightarrow (\exists p_1, p_2 \in \mathbb{N} : p_1 \text{ is prime and } p_2 \text{ is prime and } n = p_1 + p_2)$
**Conjecture 2:** $\forall m \in \mathbb{N} : (m \text{ is odd and } m > 5) \Rightarrow (\exists p_1, p_2, p_3 \in \mathbb{N} : p_1, p_2, p_3 \text{ are primes and } m = p_1 + p_2 + p_3)$
**Theorem:** Conjecture 1 implies Conjecture 2.

*Proof.* Assume that Conjecture 1 is true.
Let $m \in \mathbb{N}$.
Assume that $m$ is odd and $m > 5$.
Set $n = m - 3$.
Then $n$ is even and $n > 2$.
According to Conjecture 1 we may choose $p_1, p_2 \in \mathbb{N}$ such that $p_1$ and $p_2$ are primes and $n = p_1 + p_2$.
Take $p_3 = 3$.
Then $p_1, p_2, p_3$ are primes, and $m = n + p_3 = p_1 + p_2 + p_3$. □

**1.6 6(j).**
**Theorem:** $\exists L, G \in \mathbb{Z} : (L < G \text{ and } \forall x \in \mathbb{R} : (L < x < G \Rightarrow 40 > 10 - 2x > 12))$

*Proof.* Take $L = -2$ and $G = -1$.
Then $L < G$.
I will show that: $\forall x \in \mathbb{R} : (L < x < G \Rightarrow 40 > 10 - 2x > 12)$
Let $x \in \mathbb{R}$.
Assume that $L < x < G$.
This means that $-2 < x < -1$.
We deduce that $2 < -2x < 4$, and therefore $12 < 10 - 2x < 14 < 40$. □