Solution to HW 6

2.2 9(g).
Theorem: $\forall$ sets $A, B, C : (A \cup B) \cap C \subset A \cup (B \cap C)$

Proof. Let A, B, C be sets.
Let $x \in (A \cup B) \cap C$.
Then $x \in A \cup B$ and $x \in C$.
In particular, we have $x \in A$ or $x \in B$.
If $x \in A$, then $x \in A \cup (B \cap C)$.
If $x \in B$, then $x \in B \cap C$, hence $x \in A \cup (B \cap C)$.
We conclude that $x \in A \cup (B \cap C)$, as required. □

2.2 10(d).
Theorem: $\forall$ sets $A, B, C, D : (C \subset A \text{ and } D \subset B) \Rightarrow (D - A \subset B - C)$.

Proof. Let A, B, C, D be sets.
Assume that $C \subset A$ and $D \subset B$.
Let $x \in D - A$.
Then $x \in D$ and $x \notin A$.
Since $x \in D$ and $D \subset B$, we have $x \in B$.
Since $x \notin A$ and $C \subset A$, we have $x \notin C$.
Therefore $x \in B - C$. □

2.2 11(d).
Theorem: Let $A = \{1, 2\}$ and $B = \{2\}$. Then $\mathcal{P}(A) - \mathcal{P}(B) \not\subset \mathcal{P}(A - B)$.

Proof. We have $\mathcal{P}(A) - \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} - \{\emptyset, \{2\}\} = \{\{1\}, \{1, 2\}\}$
and $\mathcal{P}(A - B) = \mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$.
It follows that $\mathcal{P}(A) - \mathcal{P}(B) \not\subset \mathcal{P}(A - B)$. □

2.2 11(f).
Theorem: Let $A = \{1, 2, 3\}$, $B = \{2, 3\}$, $C = \{3\}$. Then $A - (B - C) \neq (A - B) - C$.

Proof. We have $A - (B - C) = \{1, 2, 3\} - \{2\} = \{1, 3\}$ and $(A - B) - C = \{1\} - \{3\} = \{1\}$, hence $A - (B - C) \neq (A - B) - C$. □

2.3 1(h). Set $\Delta = (0, \infty)$. For $r \in \Delta$, set $A_r = [-\pi, r)$. Set $\mathcal{A} = \{A_r : r \in \Delta\}$.
Theorem: $\bigcup_{r \in \Delta} A_r = [-\pi, \infty)$ and $\bigcap_{r \in \Delta} A_r = [-\pi, 0]$.

Proof. The theorem is a consequence of the following four claims.
Claim 1: $\bigcup_{r \in \Delta} A_r \subset [-\pi, \infty)$.
Let $x \in \bigcup_{r \in \Delta} A_r$.
By definition of the union over $\mathcal{A}$, we may choose $r \in \Delta$ s.t. $x \in A_r = [-\pi, r)$.
Since $[-\pi, r) \subset [-\pi, \infty)$, it follows that $x \in [-\pi, \infty)$.
Claim 2: $[-\pi, \infty) \subset \bigcup_{r \in \Delta} A_r$.
Let $x \in [-\pi, \infty)$.
Set $r = x + 4$.
Then $r \in \Delta$ and $x \in A_r$.
It follows that $x \in \bigcup_{r \in \Delta} A_r$.
Claim 3: $\bigcap_{r \in \Delta} A_r \subset [-\pi, 0]$.
Let $x \in \bigcap_{r \in \Delta} A_r$.
Then $x \in A_1 = [-\pi, 1)$, so we must have $x \geq -\pi$.

We prove by contradiction that $x \leq 0$.
Suppose that $x > 0$.
Set $r = x/2$.
Since $x \in A_r$, we obtain $x < x/2$, a contradiction.
We conclude that $-\pi \leq x \leq 0$, so $x \in [-\pi, 0]$.
Claim 4: $[-\pi, 0] \subset \bigcap_{r\in\Delta} A_r$.
Let $x \in [-\pi, 0]$.
We will show that: $\forall r \in \Delta : x \in A_r$.
Let $r \in \Delta$.
Then $-\pi \leq x \leq 0 < r$, so we have $x \in [-\pi, r) = A_r$.
It follows that $x \in \bigcap_{r\in\Delta} A_r$. □

2.3 5(b). Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ be an indexed family of sets.
Theorem: $\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^c = \bigcap_{\alpha\in\Delta} A_\alpha^c$

Proof. Let x be any element of the universe. [Notice that a universe must be given, since otherwise the complement of a set has no meaning.]
The following list of statements are equivalent:
$x \in \left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^c \quad \Leftrightarrow$
$x \notin \bigcup_{\alpha\in\Delta} A_\alpha \quad \Leftrightarrow$
$\sim (\exists\alpha \in \Delta : x \in A_\alpha) \quad \Leftrightarrow$
$\forall\alpha \in \Delta : x \notin A_\alpha \quad \Leftrightarrow$
$\forall\alpha \in \Delta : x \in A_\alpha^c \quad \Leftrightarrow$
$x \in \bigcap_{\alpha\in\Delta} A_\alpha^c$.
Since x was arbitrary, we conclude that $\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^c = \bigcap_{\alpha\in\Delta} A_\alpha^c$. □

2.3 12.
Theorem: For each $n \in \mathbb{N}$ set $A_n = (0, 1/n)$. Then we have:
(1) $\forall n \in \mathbb{N} : A_n \subset (0, 1)$.
(2) $\forall n, m \in \mathbb{N} : A_n \cap A_m \neq \emptyset$.
(3) $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$.

Proof of (1). Let $n \in \mathbb{N}$. Since $1/n \leq 1$, it follows that $A_n = (0, 1/n) \subset (0, 1)$. □

Proof of (2). Let $n, m \in \mathbb{N}$.
Case 1: If $n \leq m$, then $A_m \subset A_n$, hence $A_n \cap A_m = A_m \neq \emptyset$.
Case 2: If $n > m$, then $A_n \subset A_m$, hence $A_n \cap A_m = A_n \neq \emptyset$. □

Proof of (3). Assume that $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.
Then we may choose $x \in \bigcap_{n\in\mathbb{N}} A_n$.
This implies: $\forall n \in \mathbb{N} : x \in A_n = (0, 1/n)$.
Since $x \in A_1$, we must have $0 < x < 1$.
Choose $n \in \mathbb{N}$ so that $n > 1/x$.
Since $x > 1/n$, it follows that $x \notin (0, 1/n) = A_n$.
This contradiction shows that our initial assumption was false.
We conclude that $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. □

2.3 14. Let $\mathcal{A}$ and $\mathcal{B}$ be two families of pairwise disjoint sets. [Make sure to review exactly what a pairwise disjoint family is!]
Set $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ and $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$.
Theorem (a): $\mathcal{C}$ is a pairwise disjoint family of sets.

Proof. Let $X, Y \in \mathcal{C}$. We must show that $X = Y$ or $X \cap Y = \emptyset$.
Since $\mathcal{C} \subset \mathcal{A}$, we have $X, Y \in \mathcal{A}$.
Since $\mathcal{A}$ is pairwise disjoint, we deduce that $X = Y$ or $X \cap Y = \emptyset$, as required. $\square$

Theorem (c): $\left(\bigcup_{A\in\mathcal{A}} A\right) \cap \left(\bigcup_{B\in\mathcal{B}} B\right) = \emptyset \Rightarrow \mathcal{D}$ is pairwise disjoint.

Proof. Assume that $\left(\bigcup_{A\in\mathcal{A}} A\right) \cap \left(\bigcup_{B\in\mathcal{B}} B\right) = \emptyset$.
Let $X, Y \in \mathcal{D}$. We must show that $X = Y$ or $X \cap Y = \emptyset$.
Since $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$, we have ($X \in \mathcal{A}$ or $X \in \mathcal{B}$) and ($Y \in \mathcal{A}$ or $Y \in \mathcal{B}$).
Case 1: Assume that $X \in \mathcal{A}$ and $Y \in \mathcal{A}$.
Since $\mathcal{A}$ is pairwise disjoint, we must have $X = Y$ or $X \cap Y = \emptyset$.
Case 2: Assume that $X \in \mathcal{B}$ and $Y \in \mathcal{B}$.
Since $\mathcal{B}$ is pairwise disjoint, we must have $X = Y$ or $X \cap Y = \emptyset$.
Case 3: Assume that $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.
Then $X \subset \bigcup_{A\in\mathcal{A}} A$ and $Y \subset \bigcup_{B\in\mathcal{B}} B$.
Since $\left(\bigcup_{A\in\mathcal{A}} A\right) \cap \left(\bigcup_{B\in\mathcal{B}} B\right) = \emptyset$, we deduce that $X \cap Y = \emptyset$.
Case 4: Assume that $X \in \mathcal{B}$ and $Y \in \mathcal{A}$.
By interchanging X and Y, it follows from Case 3 that $X \cap Y = \emptyset$.
Since we have exhausted all possibilities, we conclude $X = Y$ or $X \cap Y = \emptyset$. $\square$

Theorem: Let $\mathcal{A} = \{\{1\}\}$ and $\mathcal{B} = \{\{1, 2\}\}$, and set $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$. Then $\mathcal{A}$ and $\mathcal{B}$ are both pairwise disjoint families of sets, but $\mathcal{D}$ is not pairwise disjoint.

Proof. It follows directly from the definition that both $\mathcal{A}$ and $\mathcal{B}$ are pairwise disjoint families of sets.
We have $\mathcal{D} = \{\{1\}, \{1, 2\}\}$.
This family is not pairwise disjoint, since the members $X = \{1\}$ and $Y = \{1, 2\}$ of $\mathcal{D}$ do not satisfy that $X = Y$ or $X \cap Y = \emptyset$. $\square$