Solution to HW 7

2.4 5(f).

Definition: Let $\{A_i : i \in \mathbb{N}\}$ be an indexed family of sets.
Set $U_1 = A_1$.
For $n \in \mathbb{N}$, set $U_{n+1} = U_n \cup A_{n+1}$.
For $n \in \mathbb{N}$, we define $\bigcup_{i=1}^{n} A_i = U_n$.

2.4 5(g).

Definition: Let $x_1, x_2, \ldots$ be a sequence of real numbers.
Set $p_1 = x_1$.
For $n \in \mathbb{N}$, set $p_{n+1} = p_n \cdot x_{n+1}$.
For $n \in \mathbb{N}$, we define $\prod_{i=1}^{n} x_i = p_n$.

2.4 6(c).

Theorem: $\forall n \in \mathbb{N}$: $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$.

Proof. (i) Basis step: For $n = 1$ we have $\sum_{i=1}^{n} 2^i = 2 = 2^{n+1} - 2$.
(ii) Inductive step: Let $n \in \mathbb{N}$.
Assume that $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$.
[Note: We assume the identity for this specific n.]
Then we obtain
$\sum_{i=1}^{n+1} 2^i = \left(\sum_{i=1}^{n} 2^i\right) + 2^{n+1} = (2^{n+1} - 2) + 2^{n+1} = 2 \cdot 2^{n+1} - 2 = 2^{(n+1)+1} - 2$.
[Note: And now we have proved that the identity holds for $n + 1$.]
(iii) Conclude by PMI: $\forall n \in \mathbb{N}$: $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$. □

2.4 6(e).

Theorem: $\forall n \in \mathbb{N}$: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Proof. (i) Basis step: For $n = 1$ we have $\sum_{i=1}^{n} i^3 = 1 = \left(\frac{n(n+1)}{2}\right)^2$.
(ii) Inductive step: Let $n \in \mathbb{N}$.
Assume that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.
Then we obtain
$\sum_{i=1}^{n+1} i^3 = \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + \frac{(4n+4)(n+1)^2}{4}$
$= \frac{(n^2+4n+4)(n+1)^2}{4} = \frac{(n+2)^2(n+1)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2$.
(iii) Conclude by PMI: $\forall n \in \mathbb{N}$: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. □

2.4 7(h).
Theorem: $\forall n \in \mathbb{N}$: $3^n \geq 1 + 2^n$.

Proof. (i) Basis step: For $n = 1$ we have $3^n = 3 = 1 + 2^n$.
(ii) Inductive step: Let $n \in \mathbb{N}$.
Assume that $3^n \geq 1 + 2^n$.
Then we obtain
$3^{n+1} = 3 \cdot 3^n \geq 3(1 + 2^n) = 3 + 3 \cdot 2^n \geq 1 + 2 \cdot 2^n = 1 + 2^{n+1}$.
(iii) Conclude by PMI: $\forall n \in \mathbb{N}$: $3^n \geq 1 + 2^n$. □

2.4 7(m).
Theorem: $\forall n \in \mathbb{N}$: $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{Z}$.

Proof. (i) Basis step:
For $n = 1$ we have $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} = \frac{1}{3} + \frac{1}{5} + \frac{7}{15} = \frac{15}{15} = 1$, which is an integer.
(ii) Inductive step: Let $n \in \mathbb{N}$.
Assume that $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{Z}$.
Then we obtain
$\frac{(n+1)^3}{3} + \frac{(n+1)^5}{5} + \frac{7(n+1)}{15}$
$= \frac{n^3+3n^2+3n+1}{3} + \frac{n^5+5n^4+10n^3+10n^2+5n+1}{5} + \frac{7n+7}{15}$
$= \frac{n^3}{3} + \frac{3n^2}{3} + \frac{3n}{3} + \frac{1}{3} + \frac{n^5}{5} + \frac{5n^4}{5} + \frac{10n^3}{5} + \frac{10n^2}{5} + \frac{5n}{5} + \frac{1}{5} + \frac{7n}{15} + \frac{7}{15}$
$= \left(\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15}\right) + \left(\frac{3n^2}{3} + \frac{3n}{3} + \frac{5n^4}{5} + \frac{10n^3}{5} + \frac{10n^2}{5} + \frac{5n}{5}\right) + \left(\frac{1}{3} + \frac{1}{5} + \frac{7}{15}\right)$
$= \left(\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15}\right) + \left(n^2 + n + n^4 + 2n^3 + 2n^2 + n\right) + 1$.
The first parenthesis is an integer by the induction hypothesis,
and the second is an integer because n is an integer.
It follows that $\frac{(n+1)^3}{3} + \frac{(n+1)^5}{5} + \frac{7(n+1)}{15} \in \mathbb{Z}$.
(iii) Conclude by PMI: $\forall n \in \mathbb{N}$: $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{Z}$. □