Solution to HW 8

2.5 1(b).
Theorem: $\forall n \in \mathbb{N}$: $n > 33 \Rightarrow (\exists s, t \in \mathbb{Z} : s \geq 3 \wedge t \geq 2 \wedge n = 4s + 5t)$.

Proof. Define the predicate
$P(n)$: $n > 33 \Rightarrow (\exists s, t \in \mathbb{Z} : s \geq 3 \wedge t \geq 2 \wedge n = 4s + 5t)$
We must prove: $\forall n \in \mathbb{N}$: $P(n)$.
By the PCI, it is enough to show:
(*) $\forall n \in \mathbb{N}$: $(\, P(1) \wedge P(2) \wedge \cdots \wedge P(n-1)\,) \Rightarrow P(n)$.
Let $n \in \mathbb{N}$.
Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(n-1)$.
We must show that $P(n)$ is true.
Assume that $n > 33$.
We consider two cases.
Case 1: Assume that $34 \leq n \leq 37$.
Set $s = 40 - n$ and $t = n - 32$.
Then $s \geq 40 - 37 = 3$ and $t \geq 34 - 32 = 2$.
Furthermore, we have $4s+5t = 4(40-n)+5(n-32) = 5n-4n+4\cdot 40-5\cdot 32 = n$.
It follows that $P(n)$ is true.
Case 2: Assume that $n \geq 38$.
By assumption we know that $P(n-4)$ is true.
Since $P(n-4)$ holds and $n - 4 > 33$, we may choose $s, t \in \mathbb{Z}$ such that:
$s \geq 3$ and $t \geq 2$ and $n - 4 = 4s + 5t$.
Set $s' = s + 1$ and $t' = t$.
Then we have $s', t' \in \mathbb{Z}$, $s' \geq 3$, $t' \geq 2$, and $4s' + 5t' = (4s + 5t) + 4 = n$.
It follows that $P(n)$ is true.
We deduce that (*) is true, hence the theorem is true by the PCI. □

2.5 2. Let $a_1 = 2$, $a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 1$.
Theorem: $\forall n \in \mathbb{N}$: $a_n = 2^n$.

Proof. Define the predicate
$P(n) : a_n = 2^n$.
We must prove: $\forall n \in \mathbb{N} : P(n)$.
By the PCI, it is enouth to show:
(*) $\forall n \in \mathbb{N}$: $(\, P(1) \wedge P(2) \wedge \cdots \wedge P(n-1)\,) \Rightarrow P(n)$.
Let $n \in \mathbb{N}$.
Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(n-1)$.
We must show that $P(n)$ is true.
We consider 3 cases.
Case 1: If $n = 1$, then $a_n = 2 = 2^n$.
Case 2: If $n = 2$, then $a_n = 4 = 2^n$.
Case 3: Assume that $n \geq 3$.
Then $P(n-2)$ holds by assumption, so we have $a_{n-2} = 2^{n-2}$.
And $P(n-1)$ holds by assumption, so we have $a_{n-1} = 2^{n-1}$.
We therefore obtain:
$a_n = 5a_{n-1} - 6a_{n-2} = 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} = 5 \cdot 2^{n-1} - 3 \cdot 2^{n-1} = 2 \cdot 2^{n-1} = 2^n$.
This shows that $P(n)$ is true.
We deduce that (*) is true, hence the theorem is true by the PCI. □

2.5 4(b).
$f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 3,\ f_5 = 5,$
$f_6 = 8,\ f_7 = 13,\ f_8 = 21,\ f_9 = 34,\ f_{10} = 55.$

2.5 5(b).
Theorem: $\forall n \in \mathbb{N}$: $\gcd(f_n, f_{n+1}) = 1$.

Proof. (i) Basis step:
For $n = 1$ we have $\gcd(f_n, f_{n+1}) = \gcd(f_1, f_2) = \gcd(1, 1) = 1$.
(ii) Inductive step: Let $n \in \mathbb{N}$.
Assume that $\gcd(f_n, f_{n+1}) = 1$.
Then we obtain
$\gcd(f_{n+1}, f_{n+2}) = \gcd(f_{n+1}, f_n + f_{n+1}) = \gcd(f_{n+1}, f_n) = 1$.
(iii) Conclude by PMI: $\forall n \in \mathbb{N}$: $\gcd(f_n, f_{n+1}) = 1$. $\square$

2.5 8.
Theorem:
$\forall a, b \in \mathbb{Z}$: $(a, b) \neq (0, 0) \Rightarrow$ (there is a smallest positive linear comb. of a and b).

Proof. Let $a, b \in \mathbb{Z}$.
Assume that $(a, b) \neq (0, 0)$.
Consider the set of positive linear combinations of a and b:
$S = \{n \in \mathbb{N} \mid \exists s, t \in \mathbb{Z} : n = sa + tb\}$.
Since $(a, b) \neq (0, 0)$, we must have $a \neq 0$ or $b \neq 0$.
It follows that $|a| + |b| > 0$, hence $|a| + |b| \in \mathbb{N}$.
Notice that $|a| + |b|$ is a linear combination of a and b.
In fact, we may choose $s \in \{1, -1\}$ such that $|a| = sa$.
And we may choose $t \in \{1, -1\}$ such that $|b| = tb$.
Then we have $|a| + |b| = sa + tb$.
We deduce that $|a| + |b| \in S$.
This shows that S is not empty.
Since S is a non-empty subset of $\mathbb{N}$,
it follows from the WOP that S has a smallest element m.
This integer m is the smallest linear combination of a and b. $\square$

3.1 5(g,h).
Define the relations
$R = \{(1, 5), (2, 2), (3, 4), (5, 2)\}$,
$S = \{(2, 4), (3, 4), (3, 1), (5, 5)\}$, and
$T = \{1, 4), (3, 5), (4, 1)\}$.
Then $S \circ T = \{(3, 5)\}$ and $R \circ (S \circ T) = \{(3, 2)\}$.
And we have $R \circ S = \{(3, 5), (5, 2)\}$ and $(R \circ S) \circ T = \{(3, 2)\}$.

3.1 9.
Let $R \subset A \times B$ and $S \subset B \times C$ be relations.
Then $S \circ R \subset A \times C$ is a relation from A to C.
(a) Claim: $\mathrm{Dom}(S \circ R) \subset \mathrm{Dom}(R)$.
Let $x \in \mathrm{Dom}(S \circ R)$.
By definition of the domain of a relation,
we may choose $z \in C$ such that $(x, z) \in S \circ R$.
By definition of the composition of two relations,

we may choose $y \in B$ such that $(x, y) \in R$ and $(y, z) \in S$.
Since $(x, y) \in R$, it follows that $x \in \text{Dom}(R)$.

(b) Take $A = B = C = \{1, 2\}$.
Set $R = I_{\{1,2\}} = \{(1, 1), (2, 2)\}$ and $S = \{(1, 1)\}$.
Then $S \circ R = \{(1, 1)\}$.
We have $\text{Dom}(S \circ R) = \{1\} \subsetneq \{1, 2\} = \text{Dom}(R)$.

(c) We always have $\text{Rng}(S \circ R) \subset \text{Rng}(S)$.
The opposite inclusion is not true in the following example.
Take $A = B = C = \{1, 2\}$.
Set $R = \{(1, 1)\}$ and $S = I_{\{1,2\}} = \{(1, 1), (2, 2)\}$.
Then $S \circ R = \{(1, 1)\}$.
We have $\text{Rng}(S \circ R) = \{1\} \subsetneq \{1, 2\} = \text{Rng}(S)$.