Solution to HW 9

3.2 5(c).

Define a relation V on $\mathbb{R}$ by $V = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y \text{ or } xy = 1\}$.
Claim: V is an equivalence relation on $\mathbb{R}$.

Proof. We must show that V is reflexive on $\mathbb{R}$, symmetric, and transitive.
Reflexive on $\mathbb{R}$: Let $x \in \mathbb{R}$. Since $x = x$, we have $(x, x) \in V$.
Symmetric: Let $x, y \in \mathbb{R}$. Assume that $(x, y) \in V$. Then $x = y$ or $xy = 1$.
This implies that $y = x$ or $yx = 1$, hence $(y, x) \in V$.
Transitive: Let $x, y, z \in \mathbb{R}$. Assume that $(x, y) \in V$ and $(y, z) \in V$.
Then we have $x = y$ or $xy = 1$. And we have $y = z$ or $yz = 1$.
Case 1: Assume that $x = y$ and $y = z$. Then $x = z$, hence $(x, z) \in V$.
Case 2: Assume that $x = y$ and $yz = 1$. Then $xz = 1$, hence $(x, z) \in V$.
Case 3: Assume that $xy = 1$ and $y = z$. Then $xz = 1$, hence $(x, z) \in V$.
Case 4: Assume that $xy = 1$ and $yz = 1$. Then $x = y^{-1} = z$, hence $(x, z) \in V$.
This finishes the proof that V is an equivalence relation on $\mathbb{R}$. □

The equivalence class of 3 is
$3/V = \{x \in \mathbb{R} \mid (x, 3) \in V\} = \{x \in \mathbb{R} \mid x = 3 \text{ or } 3x = 1\} = \{3, \frac{1}{3}\}$.
The equivalence class of $\frac{-2}{3}$ is
$(\frac{-2}{3})/V = \{x \in \mathbb{R} \mid (x, \frac{-2}{3}) \in V\} = \{x \in \mathbb{R} \mid x = \frac{-2}{3} \text{ or } \frac{-2}{3}x = 1\} = \{\frac{-2}{3}, \frac{-3}{2}\}$.
The equivalence class of 0 is
$0/V = \{x \in \mathbb{R} \mid (x, 0) \in V\} = \{x \in \mathbb{R} \mid x = 0 \text{ or } 0x = 1\} = \{0\}$.

3.2 7.

Reflexive relations: (b), (c), (d).
Symmetric relations: (b), (c).
Transitive relations: (a), (b), (c).

3.2 12.

Let A be a set and let R and S be equivalence relations on A.
Claim: $R \cap S$ is an equivalence relation on A.

Proof. Since $R \subset A \times A$ and $S \subset A \times A$, it follows that $R \cap S \subset A \times A$. Therefore $R \cap S$ is a relation on A.

We must show that $R \cap S$ is reflexive on A, symmetric, and transitive.

Reflexive: Let $x \in A$. Since R is reflexive, we have $(x, x) \in R$. Since S is reflexive, we have $(x, x) \in S$. It follows that $(x, x) \in R \cap S$.

Symmetric: Let $x, y \in A$. Assume that $(x, y) \in R \cap S$. Since R is symmetric and $(x, y) \in R$, we have $(y, x) \in R$. Since S is symmetric and $(x, y) \in S$, we have $(y, x) \in S$. It follows that $(x, y) \in R \cap S$.

Transitive: Let $x, y, z \in A$. Assume that $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$. Since R is transitive and $(x, y) \in R$ and $(y, z) \in R$, we have $(x, z) \in R$. Since S is transitive and $(x, y) \in S$ and $(y, z) \in S$, we have $(x, z) \in S$. It follows that $(x, z) \in R \cap S$.

This completes the proof that $R \cap S$ is an equivalence relation on A. □

3.3 3(a). To be very descriptive, we need the following Lemma.

Lemma $\forall\ x \in \mathbb{R} : (x - 1, x] \cap \mathbb{Z} \neq \emptyset$.

Proof. Let $x \in \mathbb{R}$.
Choose $N \in \mathbb{Z}$ so large that $N > x$.
Set $S = \{m \in \mathbb{Z} \mid m \geq N - x\}$.
Then $S \subset \mathbb{N}$ and $S \neq \emptyset$.
By WOP, S contains a smallest element m_0.
Since $m_0 \in S$ we have $N - m_0 \leq x$.
Since $m_0 - 1 \notin S$ we have $N - m_0 > x - 1$.
It follows that $N - m_0 \in (x-1, x] \cap \mathbb{Z}$. □

Define $Q = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \in \mathbb{Z}\}$.
The exercise tells us that Q is an equivalence relation on $\mathbb{R}$; I will not prove this.
The corresponding partition of $\mathbb{R}$ is the family of subsets $\mathbb{R}/Q = \{x/Q \mid x \in \mathbb{R}\}$.
Set $I = [0, 1) \subset \mathbb{R}$.
For $z \in I$, set $A_z = \{z + m \mid m \in \mathbb{Z}\} = \{y \in \mathbb{R} \mid z - y \in \mathbb{Z}\}$.
Define the family $\mathcal{P} = \{A_z \mid z \in I\}$.

Theorem: $\mathbb{R}/Q = \mathcal{P}$.

Proof. Let $S \in \mathbb{R}/Q$.
Then we can choose $x \in \mathbb{R}$ such that $S = x/Q$.
By the Lemma, we may choose $n \in (x-1, x] \cap \mathbb{Z}$.
Set $z = x - n$. Then $z \in I$.
Since $x - z \in \mathbb{Z}$ we obtain
$S = x/Q = \{y \in \mathbb{R} \mid (x, y) \in Q\}$
$= \{y \in \mathbb{R} \mid x - y \in \mathbb{Z}\} = \{y \in \mathbb{R} \mid z - y \in \mathbb{Z}\} = A_z$.
It follows that $S \in \mathcal{P}$.
Now let $S \in \mathcal{P}$.
Then we can choose $z \in I$ such that $S = A_z$.
Since $S = A_z = z/Q$, we obtain $S \in \mathbb{R}/Q$. □

3.3 3(c).
Define $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \sin(x) = \sin(y)\}$.
I will not show that this is an equivalence relation on $\mathbb{R}$.
Set $I = [-1, 1]$.
For $z \in I$, set $B_z = \{y \in \mathbb{R} \mid \sin(y) = z\}$.
From calculus we know that the restriction of $\sin(x)$ to the interval $[-\pi/2, \pi/2]$ has an inverse function $\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$.
For each $z \in I$ we then have
$B_z = \{2\pi m + \sin^{-1}(z) \mid m \in \mathbb{Z}\} \cup \{\pi/2 + 2\pi m - \sin^{-1}(z) \mid m \in \mathbb{Z}\}$.
I will not prove this.
Set $\mathcal{P} = \{B_z \mid z \in I\}$.

Theorem: $\mathbb{R}/R = \mathcal{P}$.

Proof. Let $S \in \mathbb{R}/R$.
Choose $x \in \mathbb{R}$ such that $S = x/R$.
Set $z = \sin(x)$.
Then $z \in I$ and $S = x/R = \{y \in \mathbb{R} \mid \sin(x) = \sin(y)\} = B_z$.
Therefore $S \in \mathcal{P}$.
Let $S \in \mathcal{P}$.
Choose $z \in I$ such that $S = B_z$.

Then $S = B_z = z/R$.
Therefore $S \in \mathbb{R}/R$. $\square$

3.3 6(e).

Let $\mathcal{P} = \{A, B\}$ where $A = \{x \in \mathbb{Z} \mid x < 3\}$ and $B = \mathbb{Z} - A$.
Then $\mathcal{P}$ is a partition of $\mathbb{Z}$ (this will not be proved).
The corresponding equivalence relation is defined by:
$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid \exists S \in \mathcal{P} : x \in S \text{ and } y \in S\}$.
Set $Q = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid (x < 3 \text{ and } y < 3) \text{ or } (x \geq 3 \text{ and } y \geq 3)\}$.

Theorem: $R = Q$.

Proof. Let $(x, y) \in R$.
Choose $S \in \mathcal{P}$ such that $x \in S$ and $y \in S$.
By definition of $\mathcal{P}$ we must have $S = A$ or $S = B$.
Case 1: Assume that $S = A$.
Then $x < 3$ and $y < 3$, so $(x, y) \in Q$.
Case 2: Assume that $S = B$.
Then $x \geq 3$ and $y \geq 3$, so $(x, y) \in Q$.
This proves that $R \subset Q$.
The proof that $Q \subset R$ is similar, by considering the same two cases. $\square$

3.3 7(b).

For $a \in \mathbb{R}$, set $A_a = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = a - x^2\}$.
Set $\mathcal{P} = \{A_a \mid a \in \mathbb{R}\}$.

Theorem: $\mathcal{P}$ is a partition of $\mathbb{R} \times \mathbb{R}$.

Proof. According to the definition of a partition, we must prove claims 1-3 below.
Claim 1: $\forall S \in \mathcal{P} : S \neq \emptyset$.
Let $S \in \mathcal{P}$.
Choose $a \in \mathbb{R}$ such that $S = A_a$.
Since $(0, a) \in A_a$, it follows that $S \neq \emptyset$.
Claim 2: $\bigcup_{S \in \mathcal{P}} S = \mathbb{R} \times \mathbb{R}$
Let $(x, y) \in \bigcup_{S \in \mathcal{P}} S$.
Choose $S \in \mathcal{P}$ such that $(x, y) \in S$.
Choose $a \in \mathbb{R}$ such that $S = A_a$.
Since $(x, y) \in A_a$ and $A_a \subset \mathbb{R} \times \mathbb{R}$, we obtain $(x, y) \in \mathbb{R} \times \mathbb{R}$.
Let $(x, y) \in \mathbb{R} \times \mathbb{R}$.
Set $a = x^2 + y$.
Then $(x, y) \in A_a$.
Since $A_a \in \mathcal{P}$, this implies that $(x, y) \in \bigcup_{S \in \mathcal{P}} S$.
Claim 3: $\forall S, T \in \mathcal{P} :\ S = T$ or $S \cap T = \emptyset$.
Let $S, T \in \mathcal{P}$.
Choose $a, b \in \mathbb{R}$ such that $S = A_a$ and $T = A_b$.
Case 1: If $a = b$ then $S = T$ holds.
Case 2: Assume that $a \neq b$.
In this case I will show that $S \cap T = \emptyset$.
If this is false, then choose $(x, y) \in S \cap T$.
Since $(x, y) \in A_a$ we have $a = x^2 + y$.
Since $(x, y) \in A_b$ we have $b = x^2 + y$.
It follows that $a = b$, a contradiction.

We conclude that Claim 3 is true. □

3.3 7(c).

Let Q be the equivalence relation corresponding to the partition $\mathcal{P}$.

Then Q is a relation on the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, i.e. $Q \subset \mathbb{R}^2 \times \mathbb{R}^2$.

It is given by:

$$Q = \{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \exists a \in \mathbb{R} : (x_1, y_1) \in A_a \text{ and } (x_2, y_2) \in A_a\}$$
$$= \{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \exists a \in \mathbb{R} : y_1 + x_1^2 = a \text{ and } y_2 + x_2^2 = a\}$$
$$= \{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid y_1 + x_1^2 = y_2 + x_2^2\}.$$