

Quantum posets and Tenelev degrees

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Flag varieties

$X = G/P$ flag variety.

$B \subseteq P \subseteq G$. $W_P \leq W$ Weyl groups.

$W^P \subseteq W$ min. representatives of cosets in W/W_P .

Schubert varieties

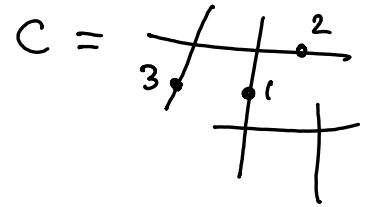
$u \in W$: $X_u = \overline{Bu.P}$, $X^u = \overline{B^-u.P} \subseteq X$

$u \in W^P \Rightarrow \dim(X_u) = \text{codim}(X^u, X) = \ell(u)$.

Curve neighborhoods

$$M_d = \overline{\mathcal{M}}_{0,3}(X, d) \quad \text{Kortsevich moduli space.}$$
$$= \{ \text{stable } f: C \rightarrow X \text{ of degree } d \}$$

$$d \in H_2(X, \mathbb{Z}).$$



$$M_d(X_u) = \text{ev}_1^{-1}(X_u) \subseteq M_d$$

$$\Gamma_d(X_u) = \text{ev}_3(M_d(X_u)) = \overline{\text{union of (rational) curves } C \subseteq X \text{ of deg. } d \text{ meeting } X_u.}$$

$$M_d(X_u, X^v) = \text{ev}_1^{-1}(X_u) \cap \text{ev}_2^{-1}(X^v)$$

$$\Gamma_d(X_u, X^v) = \text{ev}_3(M_d(X_u, X^v)) = \overline{\text{union of } C \subseteq X \text{ of deg. } d \text{ connecting } X_u, X^v.}$$

Quantum cohomology

$$QH(X) = H^*(X, \mathbb{Z}) \otimes \mathbb{Z}[q] \quad \text{algebra over } \mathbb{Z}[q]$$

$$[X_u] * [X_v] = \sum_{d \geq 0} (X_u * X_v)_d q^d$$

$$\text{where } (X_u * X_v)_d = (ev_3)_* [M_d(X_u, X_v)]$$

$$\text{Note: } \text{coeff}((X_u * X_v)_d; [X_w]) = \langle X_u, X_v, X_w \rangle_d$$

$$= \# \{ f: \mathbb{P}^1 \rightarrow X \text{ of deg. } d \text{ s.t.}$$

$$0, 1, \infty \mapsto X_u, X_v, g \cdot X_w \quad (g \in G \text{ general})$$

$$\text{Note: } [X^u] = [X_{u^v}] \quad , \quad u^v = w_0 u w_{0,p} \in W^p$$

Poincaré dual element.

Question: Which powers $q^d \in [X^u] * [X^v]$?

Fulton - Woodward, Postnikov:

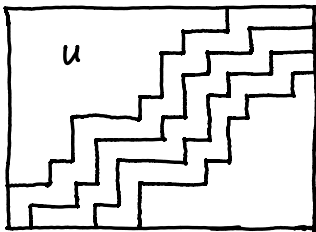
- \exists unique minimal power $q^{d_{\min}(u,v)}$.

$d_{\min}(u,v) = \text{min. degree of curve connecting } X_{u^v}, X^v.$

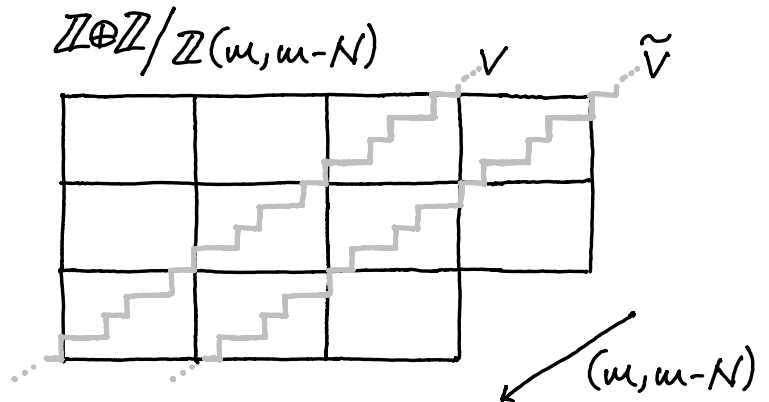
$X = Gr(u, N)$ Grassmannian of type A:

- q -degrees in $[X^u] * [X^v]$ form an interval.
- $q^d \in [X^u] * [X^v] \Leftrightarrow v \leq u(d) \leq \tilde{v}$ in Postnikov's cylinder:

$u(d) = u + d$ rim-hooks:



$d=3$



Example $X = Fl(\mathbb{C}^6)$. $w = 164532$.

The set $\{d \in H_2(X) \mid q^d \in [X^w] * [X^w]\}$

is not saturated and has no unique max element.

Cominuscule flag varieties

$Gr(m, N)$, $LG(N, 2N)$, $OG(N, 2N)$, Q^N , E_6/P_6 , E_7/P_7 .

Main result: (BCMP)

Assume X is cominuscule. Then

- q -degrees in $[X_u] * [X^v]$ form an interval.
- Determined by type-independent generalization of Postnikov's cylinder.

Quantum equals classical

(B, Kresch, Tamvakis, Chaput, Manivel, Perrin, Mihailescu)

$$X = \text{Gr}(m, N). \quad C \subseteq X \text{ curve.}$$

$$\text{Def } \ker(C) = \bigcap_{V \in C} V, \quad \text{Span}(C) = \sum_{V \in C} V \subseteq \mathbb{C}^N.$$

Exercise $C \subseteq X$ general of deg. $d \Rightarrow$

$$(\ker(C), \text{Span}(C)) \in Y_d = \text{Fl}(m-d, m+d; N)$$

$$Z_d = \text{Fl}(m-d, m, m+d; N) \xrightarrow{p} X$$

$\downarrow q$

$$Y_d = \text{Fl}(m-d, m+d; N)$$

Note: $q(p^{-1}(X_u)) = \left\{ \begin{array}{l} \text{ker-span pairs of general } C \subseteq X \\ \text{of degree } d \text{ meeting } X_u. \end{array} \right\}$

$$\text{Thm } \langle X^u, X^v, X^w \rangle_d = \int_{Y_d} q_* p^* [X^u] \cdot q_* p^* [X^v] \cdot q_* p^* [X^w].$$

Restatement

$$Z_d = \text{Fl}(m-d, m, m+d; N)$$

Def $Z_d(X_u, X_v) = \tilde{q}^{-1} \tilde{q} p^{-1}(X_u) \cap \tilde{q}^{-1} \tilde{q} p^{-1}(X_v)$.

Thm \exists birational map (for range of degrees d)

$$\varphi: M_d(X_u, X_v) \xrightarrow{\cong} Z_d(X_u, X_v)$$

$$\varphi(f) = (\text{Ker}(f), \text{ev}_3(f), \text{Span}(f))$$

Cor $\Gamma_d(X_u, X_v) = \text{ev}_3(M_d(X_u, X_v)) = \rho(Z_d(X_u, X_v))$ projected
Richardson
variety

Cor $(X_u * X_v)_d = \rho_*[Z_d(X_u, X_v)]$

$$= \begin{cases} [\Gamma_d(X_u, X_v)] & \text{if } \rho: Z_d(X_u, X_v) \rightarrow \Gamma_d(X_u, X_v) \text{ birational} \\ 0 & \text{otherwise.} \end{cases}$$

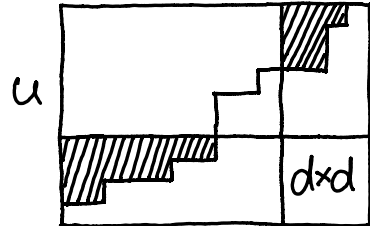
Fibers of $p: Z_d \rightarrow X_d:$

$$Z_d = Fl(m-d, m, m+d; N)$$

$$F_d = Gr(m-d, m) \times Gr(d, N-m)$$

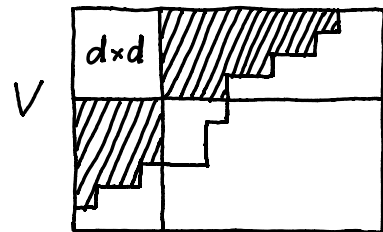
General fibers of $p: Z_d(X_u) \rightarrow \Gamma_d(X_u):$

$$(F_d)_{u_d} \text{ where } u_d = u \cap \square$$



General fibers of $p: Z_d(X^v) \rightarrow \Gamma_d(X^v):$

$$(F_d)^{v_d} \text{ where } v_d = v \cap \square$$



General fibers of $p: Z_d(X_u \cap X^v) \rightarrow \Gamma_d(X_u \cap X^v):$

THM: semi-transversal intersection of $(F_d)_{u_d}$ and $(F_d)^{v_d}$ in F_d .

$$\text{THM: } = (F_d)_{u_d} \cap (F_d)^{u_d \cap v_d}$$

Interval of q -degrees

Cor X cominuscle flag variety, $u, v \in W^P$

Assume $\Gamma_d(X_u, X_v) \neq \emptyset$ and $\boxed{dx_d} \subseteq u^v, v$. Then

$$\begin{aligned} & q^d \in [X_u] * [X^v] \\ \Leftrightarrow & p: Z_d(X_u, X^v) \rightarrow \Gamma_d(X_u, X^v) \text{ bi-rational} \\ \Leftrightarrow & (E_d)_{u_d} \cap (E_d)^{u_d \cap v^d} = \text{point} \\ \Leftrightarrow & u_d \leq v^d. \end{aligned}$$

Cor The q -degrees in $[X_u] * [X^v]$ form an interval.

Proof: Check that $u_{d+1} \leq v^{d+1} \Rightarrow u_d \leq v^d$.

Minimal and maximal powers of q

Consider $[X_u] * [X^v]$

Recall: $d \geq d_{\min}(u^v, v)$

\Leftrightarrow

\exists curve $C \subseteq X$ of degree d connecting X_u, X^v .

\Leftrightarrow

$\Gamma_d(X_u) \cap X^v \neq \emptyset$.

Thm $\Gamma_d(X_u) = X_{u(d)}$ Schubert variety, $u(d) \in W^p$

X cominuscule $\Rightarrow u(d) = u + d$ cominuscule rim-hooks.

Cor $d_{\min}(u^v, v) = \min \{d : u(d) \geq v\}$

Thm (Chaput - Manivel - Perrin)

$$[\text{pt}] * [X^v] = q^{d_{\min}(v, \text{pt})} [X^{w_0^p v}] \quad ; \quad w_0^p \in W^p \text{ max. elt.}$$

$$d_{\max}(u^v, v) = d_{\min}(v, \text{pt}) - d_{\min}((w_0^p v)^v, u)$$

Partial order on basis

\mathbb{Z} -basis of $QH(X)_q$: $\mathcal{B} = \{q^d[X^u] \mid u \in W^p, d \in \mathbb{Z}\}$

Def $q^e[X^v] \leq q^d[X^u] \Leftrightarrow X^u$ and X^v are connected by rational curve of deg. $d-e \geq 0$.

Theorem X cominuscule.

$$q^d \in [X^u] * [X^v] \Leftrightarrow [X^v] \leq q^d[X^u] \leq [pt] * [X^v].$$

Diagrams of boxes

$X = G/p$ minuscule.

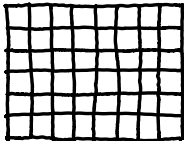
Thm (Proctor) (W^p, \leq) is a distributive lattice.

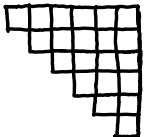
$u \vee v$ defined by $X^{u \vee v} = X^u \cup X^v$.

$u \wedge v = (u^v \cup v^u)^v$.

Def $\mathcal{P}_X \subseteq W^p$ subset of join irreducible elements.

$W^p \leftrightarrow$ (lower) order ideals in \mathcal{P}_X .

Examples: $X = Gr(m, N)$. $\mathcal{P}_X =$  $m \times (N-m)$

$X = LG(N, 2N)$ $\mathcal{P}_X =$  N rows

Quantum poset

$$\mathcal{B} = \{q^d[X^u]\} \quad q^e[X^v] \leq q^d[X^u] \Leftrightarrow \Gamma_{d-e}(X_u, X^v) \neq \emptyset$$

Thm (\mathcal{B}, \leq) is a distributive lattice. For $e \leq d$:

$$q^e[X^v] \cap q^d[X^u] = q^e[X^{v \cap u(d-e)}]$$

$$q^e[X^v] \cup q^d[X^u] = q^d[X^{v(e-d) \cup u}]$$

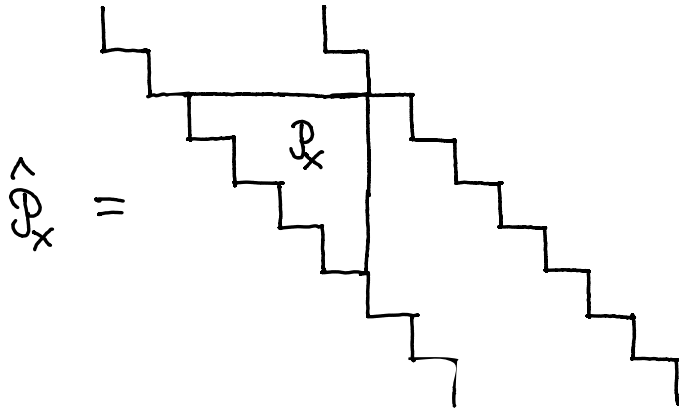
Def $\hat{\mathcal{P}}_X \subseteq \mathcal{B}$ subset of join-irreducible elements.

$\mathcal{B} \leftrightarrow$ (non-empty, proper, lower) order ideals in $\hat{\mathcal{P}}_X$.

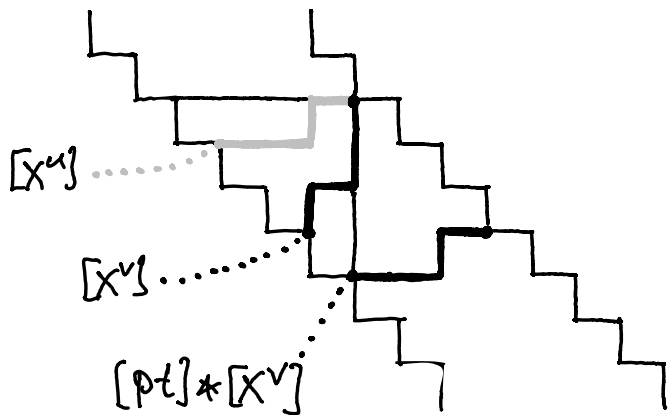
Also true: $\mathcal{P}_X \subseteq \hat{\mathcal{P}}_X$.

Example $X = \text{Gr}(m, N)$. $\hat{\mathcal{P}}_X = \mathbb{Z} \oplus \mathbb{Z} / \mathbb{Z}(m, m-N)$.

Example $X = LG(4, 8)$.



Example $X = LG(4, 8)$.



Multiply by q :

translate one box SE.

Multiply by $[pt]$:

reflect border in diagonal and attach to right side of \mathbb{P}_X .

$$u = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, \quad v = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \hline \end{array}$$

$$[X_u] * [X^v] = q^2 [X^{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \square & \square & \square \\ \hline \end{array}}] + q^2 [X^{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \square & \square & \square & \square \\ \hline \end{array}}] + q^3$$

Tevelev Degrees

Joint with Pandharipande.

X smooth projective variety / \mathbb{C} . $d \in H_2(X; \mathbb{Z})$.

$$\phi: \overline{M}_{g,n}(X, d) \longrightarrow \overline{M}_{g,n} \times X^n$$

$$\phi_* [\overline{M}_{g,n}(X, d)]^{\text{vir}} \in H^*(\overline{M}_{g,n} \times X^n)$$

"cohomological field theory of X ."

$$\text{vir dim } \overline{M}_{g,n}(X, d) = \int_d c_1(T_X) + (3 - \dim X)(g-1) + n$$

$$\text{Assume } \textcircled{*} \text{ vir dim } \overline{M}_{g,n}(X, d) = \dim(\overline{M}_{g,n} \times X^n)$$

Then $\text{vir deg}(\phi)$ is a Tevelev degree.

Note: $\textcircled{*} \Leftrightarrow \deg(q^d P) = \deg(P^{n+g})$ in $QH(X)$, $P = [\text{pt}]$.

Deformed Euler characteristics

$\{A_1, A_2, \dots, A_k\}$ basis of $H^*(X) = H^*(X, \mathbb{Q})$.

$\{B_1, B_2, \dots, B_k\}$ dual basis: $A_i \cup B_i = P \in H^*(X)$.

Def $\Delta = \sum_{i=1}^k B_i * A_i \in QH(X)$.

Note: $\Delta = \chi(X) P + q\text{-terms}$. $\chi(X) = \text{top. Euler char.}$

Thm (Pandharipande)

$$(*) \Rightarrow \text{virdeg}(\phi) = \text{coeff}(P^n \Delta^g; q^d P)$$

Work in progress: Compute RHS!

Example:

$X = \mathbb{P}^N$. Basis $\{1, H, H^2, \dots, H^N\}$, H hyperplane class.

$$\Delta = \sum_{i=0}^N H^i * H^{N-i} = (N+1)P \quad \text{since } \deg(q) = \deg(H^{N+1}).$$

$$\textcircled{*} \Leftrightarrow \deg(P^u \Delta^g) = \deg(q^d P) \Leftrightarrow (N+1)d = N(u+g-1)$$

$$\Rightarrow \text{coeff}(P^u \Delta^g; q^d P) = (N+1)^g$$

Tevelev degrees of cominuscule varieties

$X = G/P$ cominuscule.

$$\Delta = \sum_{u \in W^P} [X_u] * [X^u] \in QH(X).$$

Note: $P = [pt]$ is invertible in $QH(X)_q$.

P has finite order in $QH(X)/\langle q=1 \rangle$.

Cor $\text{coeff}(P^u \Delta^g; q^d P) = \text{coeff}(\Delta^g; q^d P^{1-u})$
depends on u modulo $\text{ord}(P)$.

Example ($g=0$) $\otimes \text{deg}(P^u) = \text{deg}(q^d P) \Rightarrow$

$$\text{coeff}(P^u; q^d P) = \text{coeff}(P^{u-1}; q^d) = \begin{cases} 1 & \text{if } u \equiv 1 \pmod{\text{ord}(P)} \\ 0 & \text{else.} \end{cases}$$

Genus one

Note: $\text{coeff}([X_u] * [X^v]; P) = \delta_{u,v}$

$$\begin{aligned} \Rightarrow \text{coeff}([X_u] * [X^u] * P^u; q^d P) \\ = \text{coeff}([X^u] * P^u; q^d [X^u]) \in \{0,1\}. \end{aligned}$$

Thm ($g=1$) $\circledast \text{deg}(P^u \Delta) = \text{deg}(q^d P) \Rightarrow$

$$\text{coeff}(P^u \Delta; q^d P) = \# \{ u \in W^p \mid P^u * [X^u] = q^d [X^u] \}$$

Example $X = LG(N, 2N)$, $g=1$.

$$p^2 = q^N \text{ in } QH(X).$$

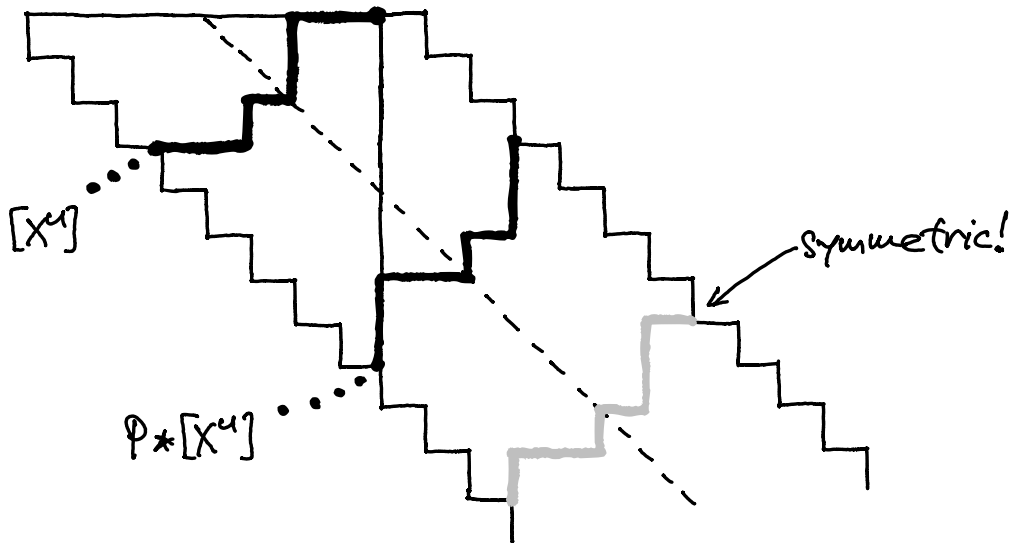
$$\text{coeff}(p^u \Delta; q^d p) = \text{coeff}(\Delta; q^d p^{1-u})$$

WLOG $u=0$ or $u=1$.

$$u=0: \text{coeff}(\Delta; p) = \chi(X) = \#W^p = 2^N.$$

$u=1: (N \text{ even})$

$$\text{coeff}(p \Delta; q^{N/2} p) = \# \{ u \in W^p \mid p_*[X^u] = q^{N/2} [X^u] \}$$



$$p * [X^u] = q^{N/2} [X^u]$$

\Leftrightarrow

u symmetric about diagonal.

$$\text{coeff}(\Delta p; q^{N/2} p) = \#\{u \text{ with symmetric border}\} = 2^{N/2}.$$

Quadric hypersurface

$X \subseteq \mathbb{P}^{N+1}$ quadric hypersurface of dim. $N \geq 3$.

$\deg(q) = \deg(p)$ in $QH(X)$.

$$\Delta = (N+a)p + (N-a)q$$

$$\text{where } a = \begin{cases} 1 & \text{if } N \text{ odd} \\ 2 & \text{if } N \text{ even.} \end{cases}$$

Thm \otimes $d+1 = n+g \Rightarrow$

$$\text{coeff}(\mathbb{P}^n \Delta^g; q^d p) = \frac{(2N)^g - (-1)^{n+g} (2a)^g}{2}$$

Hypersurface

$X \subseteq \mathbb{P}^{N+1}$ hypersurface of degree $m \geq 3$, dimension $N \geq 3$.

$H_2(X; \mathbb{Z}) = \mathbb{Z} \Rightarrow$ one deformation parameter q .

$$\frac{1}{2} \deg(q) = \int_{\text{line}} c_1(T_X) = N+2-m.$$

Thm Assume $\deg(\mathbb{P}^n \Delta^g) = \deg(q^d \mathbb{P})$ and $\deg(q^2) > \deg(\mathbb{P})$.

$$\text{coeff}(\mathbb{P}^n \Delta^g; q^d \mathbb{P}) = ((m-1)!)^n \left((N+2-m) m^{m-1} \right)^g m^{(d-n-g)m-1}$$

Complete intersection

$X = V(f_1, \dots, f_L) \subseteq \mathbb{P}^{N+L}$ complete intersection of dim. $N \geq 3$
and degrees (m_1, \dots, m_L) .

$$|m| = \sum_{i=1}^L m_i ; \quad m^{am+b} = \prod_{i=1}^L m_i^{am_i+b} ; \quad (m-1)! = \prod_{i=1}^L (m_i-1)!$$

$$\frac{1}{2} \deg(q) = N+L+1 - |m|$$

Thm Assume $\deg(\mathbb{P}^n \Delta^g) = \deg(q^d \mathbb{P})$,

$$\deg(q^2) > \deg(\mathbb{P}),$$

$$|m| \geq L+2 \quad (X \text{ not quadric}).$$

$$\text{coeff}(\mathbb{P}^n \Delta^g; q^d \mathbb{P}) = ((m-1)!)^n ((N+L+1-|m|) m^{m-1})^g m^{(d-n-g)m+1}$$