A Vafa-Intriligator formula for Fano varieties

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Notation

$\overline{\mathcal{M}}_{g,n}$ = moduli space of stable curves $/\mathbb{C}$ of genus g, n marked points.

$\dim \overline{\mathcal{M}}_{g,n} = 3g-3+n$ Stability: $2g-2+n>0$

X non-singular projective variety $/\mathbb{C}$.

$r = \dim(X)$.

$P \in H^{2r}(X) = H^{2r}(X,\mathbb{Q})$ point class.

Degree: $d \in H_2(X,\mathbb{Z})$

$$\overline{\mathcal{M}}_{g,n}(X,d) = \left\{ \text{stable } f : C \longrightarrow X \;\middle|\; \begin{array}{l} \text{genus } g,\ n \text{ markings} \\ f_*[C] = d \end{array} \right\}$$

$$\operatorname{vdim} \overline{\mathcal{M}}_{g,n}(X,d) = \int_d c_1(T_X) + (r-3)(1-g) + n$$

Gromov-Witten invariants with fixed markings

Given $\Omega_1, \Omega_2, \dots, \Omega_n \in H^*(X) = H^*(X, \mathbb{Q})$:

$$\langle \Omega_1 \otimes \Omega_2 \otimes \cdots \otimes \Omega_n \rangle_{g,d} = \langle \Omega_1 \otimes \Omega_2 \otimes \cdots \otimes \Omega_n \rangle^{X,\odot}_{g,d,n}$$

$$:= \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{vir}} ev^*(\Omega_1 \otimes \Omega_2 \otimes \cdots \otimes \Omega_n) \cdot \pi^*(P_{\overline{\mathcal{M}}_{g,n}}) \in \mathbb{Q}$$

= $\boxed{\text{virtual count of maps of degree } d \text{ from fixed } (C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_{g,n}: \; f: C \longrightarrow X, \; f(p_i) \in \Omega_i}$

$$\boxed{\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}(X,d) & \xrightarrow{\pi} & \overline{\mathcal{M}}_{g,n} \\ \downarrow ev & & \\ X^n & & \end{array}}$$

Properties

- $\langle \Omega_1 \otimes \cdots \otimes \Omega_n \rangle_{g,d} \neq 0 \Rightarrow$

$$\sum_{i=1}^{n} \deg(\Omega_i) = 2\,\mathrm{vdim}\,\overline{\mathcal{M}}_{g,n}(X,d) - 2\dim \overline{\mathcal{M}}_{g,n}$$
$$= 2\int_d c_1(T_X) + 2r(1-g)$$

- $\langle \Omega_1 \otimes \cdots \otimes \Omega_n \rangle_{g,d} = \langle \Omega_1 \otimes \cdots \otimes \Omega_n \otimes 1 \rangle_{g,d}$

  – so well defined for all $g, n \geq 0$, $d \in H_2(X, \mathbb{Z})$.

- $\langle \Omega_1 \otimes \cdots \otimes \Omega_n \rangle_{g,d} = \langle \Omega_1 \otimes \cdots \otimes \Omega_n \otimes \Delta^{\otimes g} \rangle^{X,\odot}_{0,d,n+2g}$

  where $\Delta \in H^*(X \times X) = H^*(X) \otimes H^*(X)$ is the diagonal class.

Quantum cohomology (small)

$\mathbb{Q}[q] = \mathrm{Span}_{\mathbb{Q}}\{ q^d \mid d \in H_2(X,\mathbb{Z})\}$ group ring.

$QH(X) = H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[q]$ as $\mathbb{Q}[q]$-module

$$\Omega_1 * \Omega_2 = \sum_d q^d (\Omega_1 * \Omega_2)_d$$

where $(\Omega_1 * \Omega_2)_d \in H^*(X)$ is defined by

$$\int_X (\Omega_1 * \Omega_2)_d \cdot \Omega_3 = \langle \Omega_1 \otimes \Omega_2 \otimes \Omega_3 \rangle_{0,d}$$

$\forall\, \Omega_3 \in H^*(X).$

Graded ring: $\deg(q^d) = 2 \int_d c_1(T_X)$

Quantum Euler class

Let $\{\gamma_j\}$ be any $\mathbb{Q}$-basis of $H^*(X)$.

Dual basis: $\{\gamma_k^\vee\}$ defined by $\int_X \gamma_j \cdot \gamma_k^\vee = \delta_{jk}$

Diagonal class: $\Delta = \sum_j \gamma_j^\vee \otimes \gamma_j \in H^*(X \times X)$

Quantum Euler class (Abrams):

$$E = \sum_j \gamma_j^\vee * \gamma_j \in QH(X)$$

$$= \chi(X) \cdot p + q\text{-corrections}$$

GW invariants from quantum cohomology

Bertram: For $\Omega_1, \dots, \Omega_n \in H^*(X)$,

$$\Omega_1 * \cdots * \Omega_n = \sum_{d,j} \langle \Omega_1 \otimes \cdots \otimes \Omega_n \otimes \gamma_j^\vee \rangle_{0,d,n+1}^{X,\odot} q^d \gamma_j \in QH(X).$$

Application (Kirillov-Maeno for $GL(n)/B$):

$$\langle \Omega_1 \otimes \cdots \otimes \Omega_n \rangle_{g,d,n}^{X,\odot} = \langle \Omega_1 \otimes \cdots \otimes \Omega_n \otimes \Delta^{\otimes g} \rangle_{0,d,n+2g}^{X,\odot}$$

$$= \langle \Omega_1 \otimes \cdots \otimes \Omega_n \otimes \Delta^{\otimes g} \otimes 1 \rangle_{0,d,n+2g+1}^{X,\odot}$$

$$= \mathrm{Coeff}(\Omega_1 * \cdots * \Omega_n * E^{*g}, \; q^d P)$$

Example

$$QH(\mathbb{P}^r) = \mathbb{Q}[H, q] / \langle H^{r+1} - q \rangle$$

$$E = 1 * H^r + H * H^{r-1} + \cdots + H^r * 1$$

$$= (r+1) H^r = (r+1) P.$$

$$\langle H^{p_1} \otimes \cdots \otimes H^{p_n} \rangle_{g,d} = \mathrm{Coeff}\left(H^{p_1} * \cdots * H^{p_n} * E^g , q^{dp} \right)$$

$$= \begin{cases} (r+1)^g & \text{if } p_1 + \cdots + p_n + rg = d(r+1) + r \\ 0 & \text{otherwise} \end{cases}$$

Integral formula

Assume X is Fano: $c_1(T_X)$ ample

$$\deg(q^d) = 2\int_d c_1(T_X) > 0 \quad \text{for } d \in H_2(X,\mathbb{Z}) \text{ effective.}$$

Given $A \in QH(X)$, write $A = \sum_d q^d (A)_d$, $(A)_d \in H^*(X)$

Def $$\int_X A = \sum_d q^d \int_X (A)_d \in \mathbb{Q}[q].$$

$$\langle \Omega_1 \otimes \cdots \otimes \Omega_n \rangle^{X,\odot}_{g,d,n} = \int_X \left(\Omega_1 * \cdots * \Omega_n * E^{*g}\right)_d \in \mathbb{Q}$$

$$\sum_d q^d \langle \Omega_1 \otimes \cdots \otimes \Omega_n \rangle^{X,\odot}_{g,d,n} = \int_X \Omega_1 * \cdots * \Omega_n * E^{*g} \in \mathbb{Q}[q]$$

Trace formula

For $\Omega \in H^*(X)$, set $\overline{\Omega} = (-1)^{\deg(\Omega)} \cdot \Omega \in H^*(X)$.

For $A = \sum_d q^d (A)_d \in QH(X)$, set $\bar{A} = \sum_d q^d \overline{(A)_d} \in QH(X)$.

$\mathbb{Q}[q]$-linear endomorphism:

$$QH(X) \longrightarrow QH(X) \quad ; \quad B \longmapsto A * \overline{B}$$

<u>Def</u> $\overline{\mathrm{tr}}(A) = \mathrm{tr}(B \mapsto A * \overline{B}) \in \mathbb{Q}[q]$.

<u>Note:</u> $\overline{\mathrm{tr}}(A) = \sum_j \int_X (A * \overline{\gamma_j}) * \gamma_j^\vee = \sum_j \int_X A * \gamma_j^\vee * \gamma_j = \int_X A * E$.

<u>Cor</u> (Chaput-Manivel-Perrin when $X = G/P$ comin. flag variety)

X Fano, $g \geq 1$, $\Omega_1, \ldots, \Omega_n \in H^*(X)$. Then

$$\sum_d q^d \langle \Omega_1 \otimes \cdots \otimes \Omega_n \rangle_{g,d} = \int_X \Omega_1 * \cdots * \Omega_n * E^{*g} = \overline{\mathrm{tr}}(\Omega_1 * \cdots * \Omega_n * E^{*g-1})$$

Weights of $QH(X)$

$$QH(X)^{even} = \{A \in QH(X) \mid \bar{A} = A\}$$

$$QH(X)^{odd} = \{A \in QH(X) \mid \bar{A} = -A\}$$

$$QH(X)_K = QH(X) \otimes_{\mathbb{Q}[q]} K, \quad \text{where} \quad K = \overline{\mathbb{Q}(q)}.$$

__Generalized eigenspaces__ Given K-linear $\lambda : QH(X)_K^{even} \to K$,

$$QH(X)_\lambda = \bigcap_{A \in QH(X)^{even}} \mathrm{Ker}\left([A*] - \lambda(A)\right)^{\dim H^*(X)} \subseteq QH(X)_K.$$

__Note:__ $QH(X)_K = \bigoplus_\lambda QH(X)_\lambda$ (since $QH(X)^{even}$ commutative.)

__Def__ λ is a __weight__ of $QH(X)_K$ if $QH(X)_\lambda \neq 0$.

__Note:__ λ weight $\Rightarrow$ $\lambda : QH(X)_K^{even} \longrightarrow K$ K-alg. hom.

Generalized Vafa-Intriligator formula

Note: $QH(X)_\lambda = QH(X)_\lambda^{even} \oplus QH(X)_\lambda^{odd}$

Def $\chi(\lambda) = \dim_K QH(X)_\lambda^{even} - \dim_K QH(X)_\lambda^{odd}$

Theorem X Fano, $g \geq 1$, $\Omega_1, \ldots, \Omega_n \in H^*(X)^{even}$:

$$\sum_d q^d \langle \Omega_1 \otimes \cdots \otimes \Omega_n \rangle_{g,d} = tr(\Omega_1 * \cdots * \Omega_n * E^{*\,g-1})$$

$$= \sum_\lambda \chi(\lambda)\, \lambda(E)^{g-1} \lambda(\Omega_1) \cdots \lambda(\Omega_n).$$

Note Many cases known when $X = G/P$ cominuscule flag variety.

Siebert-Tian, Bertram, Marian-Oprea: $X = Gr(m,N)$.

Chaput-Manivel-Perrin: $X = G/P$ cominuscule.

Rietsch: Found weights when $X = Gr(m,N)$.

Cheong: Found weights for $LG(N,2N)$, $OG(N,2N)$.

Weights of $QH(X)_{q=1}$

<u>Issue</u>: $K = \overline{\mathbb{Q}(q)}$ hard to work with!

$$QH(X)_{q=1} = QH(X) \otimes \overline{\mathbb{Q}} \Big/ \langle q^d - 1 : d \in H_2(X, \mathbb{Z}) \rangle$$

$$= \bigoplus_{\lambda} QH(X)_{\lambda}$$

sum over weights $\lambda : QH(X)^{\text{even}}_{q=1} \longrightarrow \overline{\mathbb{Q}}$.

<u>Cor</u> X Fano, $g \geq 1$, $\Omega_1, \dots, \Omega_n \in H^*(X)^{\text{even}}$:

$$\sum_d \langle \Omega_1 \otimes \cdots \otimes \Omega_n \rangle_{g,d} = \sum_{\lambda} \chi(\lambda)\, \lambda(E)^{g-1} \lambda(\Omega_1) \cdots \lambda(\Omega_n) \in \mathbb{Q}$$

<u>Note</u>: $H_2(X, \mathbb{Z}) = \mathbb{Z} \Rightarrow$ only one GW-invariant in sum.

Example $QH(\mathbb{P}^r)_{q=1} = \overline{\mathbb{Q}}[H]/\langle H^{r+1}-1\rangle$. $\quad E=(r+1)P$

$\zeta = \exp\left(\frac{2\pi i}{\ell}\right)$, $\quad \ell = r+1$ Fano index.

Eigenvectors: $\quad \sigma_j = \sum_{t=0}^{\ell-1} (\zeta^j)^{-t} H^t \quad , \quad 0 \le j < \ell$

$$H * \sigma_j = \zeta^j \cdot \sigma_j$$

Weights: $\quad \lambda_j : QH(\mathbb{P}^r)_{q=1} \longrightarrow \overline{\mathbb{Q}}, \quad \lambda_j(H^p) = (\zeta^j)^p$

$$\sum_d \langle H^{p_1} \otimes \cdots \otimes H^{p_n} \rangle_{g,d} = \sum_{j=0}^{\ell-1} \lambda_j(E)^{g-1} \lambda_j(H^{p_1}) \cdots \lambda_j(H^{p_n})$$

$$= \sum_{j=0}^{\ell-1} (r+1)^{g-1} (\zeta^j)^{r(g-1)+\sum p_i} = \begin{cases} (r+1)^g & \text{if } rg - r + \sum p_i \equiv 0 \pmod{\ell} \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture $\mathcal{O}$

X Fano.

$S \subseteq \overline{\mathbb{Q}}$ set of eigenvalues of

$$c_1(T_X) * : QH(X)_{q=1} \longrightarrow QH(X)_{q=1}$$

$$\mu_0 = \max \{ |\mu| : \mu \in S \}$$

Conjecture (Galkin, Golyshev, Iritani)

- $\mu_0 \in S$
- $\mu \in S$ and $|\mu| = \mu_0 \Rightarrow (\mu/\mu_0)^\ell = 1$,

$$\ell = \gcd \{ \textstyle\int_d c_1(T_X) \mid d \in H_2(X, \mathbb{Z}) \} \quad \text{Fano index.}$$

Known for flag varieties (Cheong-Li), complete intersections (Ke)

Conjecture $\mathcal{O}^+$ $\quad X$ Fano.

$QH(X)^+_{q=1} \subseteq QH(X)^{\text{even}}_{q=1}$ cone generated by irred. subvars.

Conjecture

- $QH(X)_{q=1}$ has a unique weight $\lambda_0 : QH(X)^{\text{even}}_{q=1} \longrightarrow \overline{\mathbb{Q}}$ defined by
$$\lambda_0(A) = \max\{|\lambda(A)| : \lambda \text{ weight of } QH(X)_{q=1}\}$$
for $A \in QH(X)^+_{q=1}$.

- $QH(X)_{\lambda_0} = \overline{\mathbb{Q}}\,\sigma_0$ where $\sigma_0 \in QH(X)^+_{q=1}$

- If λ is any weight of $QH(X)_{q=1}$ with $|\lambda(D)| = \lambda_0(D)$ for some ample divisor D, then $\exists\, \zeta \in \overline{\mathbb{Q}}$, $\zeta^{2\ell} = 1$, $\lambda(A) = \zeta^{\deg(A)}\lambda_0(A) \quad \forall\, A \in QH(X)^{\text{even}}_{q=1}$.

Thm True for most comin. flag vars & low deg. complete int.

Hypersurface

$X \subseteq \mathbb{P}^{r+1}$ nonsingular hypersurface of dimension $r \geq 3$, degree $m \leq r$.

$H_2(X, \mathbb{Z}) = \mathbb{Z} = \mathbb{Z} \cdot [\text{line}]$

X is Fano.

Fano index: $\ell = \int_{\text{line}} c_1(T_X) = r + 2 - m \geq 2.$

$H^*(X) = \mathbb{Q}[H] / \langle H^{r+1} \rangle \oplus H^r(X)_{\text{prim}}$ where

$H =$ restriction of hyperplane class on $\mathbb{P}^{r+1}$.

$H^r(X)_{\text{prim}} = \{ \Omega \in H^*(X) \mid H \cdot \Omega = 0 \} \subseteq H^r(X).$

Quantum Cohomology

$QH(X) = H^*(X) \otimes \mathbb{Q}[q]$, $\deg(q) = 2\ell = 2(r+2-m)$.

Magic relation (Givental):

$$H^{*(r+1)} = m^m q\, H^{*(r+1-\ell)} \in QH(X).$$

$QH(X)^{res} = \mathrm{Span}_{\mathbb{Q}}\{1, H, .., H^r\} \otimes \mathbb{Q}[q] \subseteq QH(X).$

Prop
- (Graber) $QH(X)^{res} \subseteq QH(X)$ subring.
- $\Omega \in H^r(X)_{prim} \Rightarrow H * \Omega = 0 \in QH(X)$.
- $\Omega_1, \Omega_2 \in H^r(X)_{prim} \Rightarrow \Omega_1 * \Omega_2 \in QH(X)^{res}$.

Weights of $QH(X)_{q=1}$

Assume $m \geq 3$. Then $\ell = r+2-m < r$.

<u>Magic</u>: $H^{*(r+1)} = m^m H^{*(r+1-\ell)} \in QH(X)_{q=1}$

$\zeta = \exp\left(\frac{2\pi i}{\ell}\right)$, $\varepsilon = \sqrt[\ell]{m^m}$

<u>Eigenvectors</u>: $\sigma_j = \sum_{t=0}^{\ell-1} (\zeta^j \varepsilon)^{-t} H^{*(r+1-\ell+t)}$, $0 \leq j < \ell$.

$\lambda_j : QH(X)^{even}_{q=1} \longrightarrow \overline{\mathbb{Q}}$,

$\lambda_j(H) = \zeta^j \varepsilon$, $\lambda_j(\Omega) = 0$ for $\Omega \in H^r(X)_{prim}$ (if r even).

Then $QH(X)_{\lambda_j} = \overline{\mathbb{Q}} \cdot \sigma_j \subseteq QH(X)_{q=1}$.

<u>Note</u>: $\mathrm{Span}_{\overline{\mathbb{Q}}}\{\sigma_0, \sigma_1, \ldots, \sigma_{\ell-1}\} = \mathrm{Span}_{\overline{\mathbb{Q}}}\{H^{*(r+1-\ell)}, \ldots, H^{*r}\}$

Nilpotent part

$$\lambda_{nil} : QH(X)^{even}_{q=1} \longrightarrow \overline{\mathbb{Q}}$$

$$\lambda_{nil}(H) = 0$$

$$\lambda_{nil}(\Omega) = 0, \quad \Omega \in H^r(X)_{prim} \text{ (if } r \text{ even)}.$$

$$QH(X)_{\lambda_{nil}} = H^r(X)_{prim} \oplus \mathrm{Span}_{\overline{\mathbb{Q}}}\{ H^{*\ell+i} - m^m H^{*i} \mid 0 \le i \le m \}$$

$$\chi(\lambda_{nil}) = \chi(X) - \ell$$

Gromov-Witten invariants

$\boxed{g \geq 1}$

$$\sum_d \left\langle H^{p_1} \otimes \cdots \otimes H^{p_n} \right\rangle_{g,d} = \sum_{j=0}^{\ell-1} \lambda_j(E)^{g-1} \lambda_j(H^{p_1}) \cdots \lambda_j(H^{p_n})$$
$$+ \chi(\lambda_{nil}) \cdot \lambda_{nil}(E)^{g-1} \cdot \lambda_{nil}(H^{p_1}) \cdots \lambda_{nil}(H^{p_n})$$

$$= \begin{cases} \ell\, \lambda_0(E)^{g-1} \lambda_0(H^{p_1}) \cdots \lambda_0(H^{p_n}) \\ + (\chi(X) - \ell)\, \lambda_{nil}(E)^{g-1} \lambda_{nil}(H^{p_1}) \cdots \lambda_{nil}(H^{p_n}) \\ 0 \quad \text{if } rg - r + \sum p_i \not\equiv 0 \pmod{\ell} \end{cases}$$

<u>Problem</u>: Find $\lambda_0(H^p)$, $\quad 0 \leq p \leq r$

Find $\lambda_0(E)$.

<u>Note</u>: $\lambda_j(H^{*p}) = \lambda_j(H)^p = (\zeta^j \varepsilon)^p \qquad$ known!

Quantum Euler class

Theorem

$$E \equiv \chi(X)\, m^{-1} H^{*r} + (\ell - \chi(X))\, m^{m-1} q\, H^{*\, r-\ell} \pmod{q^2}$$

Conjecture

$$E = \chi(X)\, m^{-1} H^{*r} + (\ell - \chi(X))\, m^{m-1} q\, H^{*\, r-\ell}$$

Theorem True if $r \leq 2\ell$ ($\Leftrightarrow \deg(P) \leq \deg(q^2)$.)

Consequence:

$$\lambda_j(E) = \chi(X)\, m^{-1} (\zeta^j \varepsilon)^r + (\ell - \chi(X))\, m^{m-1} (\zeta^j \varepsilon)^{r-\ell}$$

$$= \ell\, m^{-1} (\zeta^j \varepsilon)^r, \qquad \varepsilon = \sqrt[\ell]{m^m}$$

Restricted classes

$$\Psi_m = \prod_{j=0}^{m} \left(j H_1 + (m-j) H_2 \right) \in \mathbb{Z}[H_1, H_2].$$

$$c_i = m^{-1} \operatorname{Coeff}\left(\Psi_m , H_1^{i} H_2^{m+1-i} \right)$$

$$a_p = \sum_{j=1}^{1+p-\ell} c_i \qquad , \qquad 0 \le p \le r.$$

Theorem $H^{p} = H^{*p} - a_p\, q\, H^{*p-\ell} \pmod{q^2}.$

Corollary $p < 2\ell \Rightarrow$

$$\begin{aligned} \lambda_j(H^p) &= \lambda_j\left(H^{*p} - a_p H^{*p-\ell} \right) \\ &= (\zeta^j)^p \cdot \left(\varepsilon^p - a_p\, \varepsilon^{p-\ell} \right) \quad , \qquad \varepsilon = \sqrt[\ell]{m^m} \end{aligned}$$

Relation to Peterson variety

Assume $X = G/P$ flag variety.

<u>Thm</u> (Peterson)

$QH(X)$ is reduced.

Peterson variety: $\mathrm{Spec}\left(QH(X) \otimes \mathbb{C}\right)$

weights of $QH(X)_{q=1} \longleftrightarrow$ points in $\mathrm{Spec}\left(QH(X)_{q=1}\right)$

$\lambda(A) =$ value of A at λ.