The Seidel representation in quantum K-theory

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Flag varieties

$X = G/P_X$ flag variety. $T \subset B \subset P_X \subset G$

$W = N_G(T)/T$ Weyl group. $W_X = N_{P_X}(T)/T$ Weyl group of P_X.
$W^X \subset W$ minimal representatives of cosets in W/W_X.

Schubert varieties: For $w \in W$ set $X_w = \overline{Bw.P_X}$ and $X^w = \overline{B^- w.P_X}$
$w \in W^X \Rightarrow \dim(X_w) = \operatorname{codim}(X^w, X) = \ell(w)$

Quantum cohomology

Given $d \in H_2(X)$, $u, v, w \in W$, define the Gromov-Witten invariant

$$\langle [X^u], [X^v], [X_w] \rangle_d \;=\; \#\; f : \mathbb{P}^1 \to X \text{ of degree } d \text{ such that}$$
$$f(0) \in X^u,\ f(1) \in g.X^v, \text{ and } f(\infty) \in X_w$$
$$\text{(if finitely many, otherwise zero)}$$

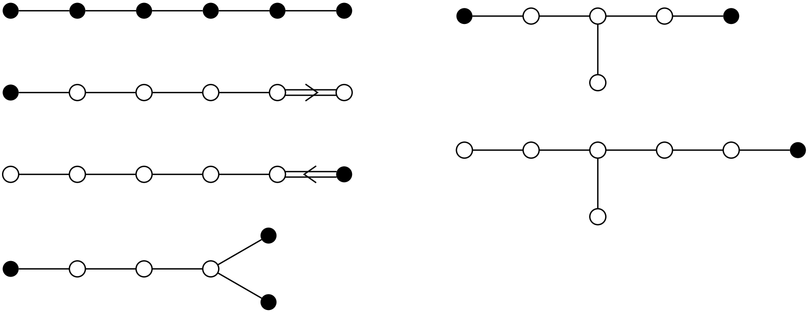
$QH(X) = H^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$

$$[X^u] \star [X^v] = \sum_{w,d} \langle [X^u], [X^v], [X_w] \rangle_d \, q^d \, [X^w]$$

Theorem (Ruan-Tian, Kontsevich-Manin): $QH(X)$ is an associative ring.

Cominuscule simple roots

A simple root β is **cominuscule** if coefficient of β in the highest root is 1.



Seidel representation

For β cominuscule, choose $v_\beta \in W$ minimal such that $v_\beta.\omega_\beta = w_0.\omega_\beta$

Theorem (Chaput, Manivel, Perrin)

$$[X^{v_\beta}] \star [X^w] = q^{d(\beta,w)} [X^{v_\beta w}]$$

$$\pi(\mathrm{Aut}(X)) \;\cong\; \{v_\beta : \beta \text{ cominuscule }\} \cup \{1\} \;\leq\; W$$

Group homomorphism: $\pi_1(\mathrm{Aut}(X)) \;\longrightarrow\; (QH(X)/\langle q=1\rangle)^\times$

$$v_\beta \;\mapsto\; [X^{v_\beta}]$$

Curve neighborhoods

Given $\Omega \subset X$ and $d \in H_2(X)$, define

$\Gamma_d(\Omega) \ = \ \{x \in X \mid \exists\, C \subset X$ of degree d connecting x to Ω $\}$

Theorem (BCMP) Ω Schubert variety $\Rightarrow \Gamma_d(\Omega)$ Schubert variety

$M_d = \overline{\mathcal{M}}_{0,3}(X,d) = \overline{\{f : \mathbb{P}^1 \to X \text{ of degree } d\}}$ Kontsevich moduli space.

Note: $\Gamma_d(\Omega) = \mathrm{ev}_3(\mathrm{ev}_1^{-1}(\Omega))$

Define: $M_d(X_u, X^v) \ = \ \mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v)$

$\Gamma_d(X_u, X^v) \ = \ \mathrm{ev}_3(M_d(X_u, X^v))$

Quantum K-theory (Givental, Lee)

A closed subvariety $\Omega \subset X$ defines a K-theory class $[\mathcal{O}_\Omega] \in K(X)$

Euler characteristic: $\chi : K(X) \to \mathbb{Z}$; $\chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{F})$

Define: $\mathrm{QK}(X) \;=\; K(X) \otimes_{\mathbb{Z}} \mathbb{Z}[[q]] \;=\; K(X)[[q]]$

$$[\mathcal{O}_{X_u}] \odot [\mathcal{O}_{X^v}] \;=\; \sum_d \mathrm{ev}_{3*}[\mathcal{O}_{M_d(X_u, X^v)}]\, q^d \quad \in \mathrm{QK}(X)$$

Note: $\chi\left([\mathcal{O}_{X_w}] \cdot \mathrm{ev}_{3*}[\mathcal{O}_{M_d(X_u,X^v)}]\right) = \chi(\,\mathcal{O}_{M_d(X_u, X^v, g.X_w)}\,)$
$= \;\#M_d(X_u, X^v, g.X_w)$ when finite.

Define: $\Psi : \mathrm{QK}(X) \to \mathrm{QK}(X)$; $\Psi = \sum_d q^d \, \mathrm{ev}_{3*}\, \mathrm{ev}_1^*$

$\Psi([\mathcal{O}_\Omega]) = \sum q^d [\mathcal{O}_{\Gamma_d(\Omega)}]$ if Ω has rational singularities.

Theorem (BCMP) Givental's quantum K-theory product is

$$[\mathcal{O}_{X_u}] \star [\mathcal{O}_{X^v}] \;=\; \Psi^{-1}([\mathcal{O}_{X_u}] \odot [\mathcal{O}_{X^v}])$$

Some results about $\mathrm{QK}(X)$

Structure constants $N_{u,v}^{w,d} \in K(X)$: $\quad [\mathcal{O}_{X^u}] \star [\mathcal{O}_{X^v}] = \sum_{w,d} N_{u,v}^{w,d} \, q^d \, [\mathcal{O}_{X^w}]$

Finiteness (BCMP [X comin], Kato [G/B], Anderson-Chen-Tseng [G/P]):

$$N_{u,v}^{w,d} = 0 \text{ for large } d.$$

Qauntum = affine (Kato):

$$\mathrm{QK}(G/B)_{\mathrm{loc}} \cong K_0(\mathrm{Gr})_{\mathrm{loc}}$$

Functoriality (Kato):

Ring homomorphism $\mathrm{QK}(G/B) \twoheadrightarrow \mathrm{QK}(G/P)$

Chevalley formula (BCMP [X comin], Lenart-Naito-Sagaki [G/B])

Expansion of $[\mathcal{O}_{X^{s_\beta}}] \star [\mathcal{O}_{X^w}]$

Challenges

$N_{u,v}^{w,d} \neq 0 \;\Rightarrow\; \ell(w) + \int_d c_1(T_X) \geq \ell(u) + \ell(v)$

Equality $\Rightarrow$ $N_{u,v}^{w,d} = \#$ curves in X = structure constant of $QH(X)$

Positivity Conjecture: $(-1)^{\ell(uvw)+\int_d c_1(T_X)} N_{u,v}^{w,d} \geq 0$

Questions When is $N_{u,v}^{w,d} \neq 0$?

Which powers q^d occur in $[\mathcal{O}_{X^u}] \star [\mathcal{O}_{X^v}]$?

Theorem (Postnikov, Fulton-Woodward)

$[X^u] \star [X^v]$ contains unique minimal power q^d

$d = \mathrm{dist}(w_0.X^u, X^v)$ = minimal degree of rat. curve from $w_0.X^u$ to X^v

Example: $X = \mathrm{Fl}(6)$, $w = 164532$. $[X^w]^2 \in QH(X)$ has no max q-degree, and q-degrees do not form an interval. QK(X) ???

Cominuscule quantum K-theory

Assume from now that $X = G/P_X$ is **cominuscule**:

P_X is maximal parabolic, and excluded simple root γ is cominuscule.

If in addition G is simply laced, then X is also **minuscule**.

Minuscule: $\mathrm{Gr}(m, n)$, $\mathrm{OG}(n, 2n)$, Q^{2n}, E_6/P_6, E_7/P_7

Cominuscule: $\mathrm{LG}(n, 2n)$, Q^{2n+1}

Theorem (BCMP) X minuscule or quadric $\Rightarrow$ Positivity Conjecture is true

Degrees in quantum products

Theorem (Postnikov, BCMP)

X cominuscule $\Rightarrow$ powers q^d in $[X^u] \star [X^v]$ form integer interval.

Theorem (BCMP)

X cominuscule $\Rightarrow$ powers q^d in $[\mathcal{O}_{X^u}] \star [\mathcal{O}_{X^v}]$ form integer interval.

X minuscule or quadric:

$[\mathcal{O}_{X^u}] \star [\mathcal{O}_{X^v}]$ contains exactly same powers q^d as $[X^u] \star [X^v]$

$X = \mathrm{LG}(n, 2n)$:

$[\mathcal{O}_{X^u}] \star [\mathcal{O}_{X^v}]$ contains same powers q^d as $[X^u] \star [X^v]$,
plus possibly one extra power.

$N_{u,v}^{w,d}$ has the conjectured sign when q^d occurs in $[X^u] \star [X^v]$.

Seidel representation in quantum K-theory

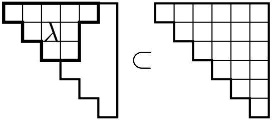
Recall: $v_\beta \in W$ is minimal such that $v_\beta.\omega_\beta = w_0.\omega_\beta$.

Example: $v_\gamma \in W^X$ largest element, $X^{v_\gamma} = \{v_\gamma.P_X\}$

Theorem (BCMP) $[\mathcal{O}_{X^{v_\beta}}] \star [\mathcal{O}_{X^w}] = q^{d(\beta,w)}\,[\mathcal{O}_{X^{v_\beta w}}]$

Application: Pieri formula for $X = \mathrm{OG}(n+1, 2n+2)$

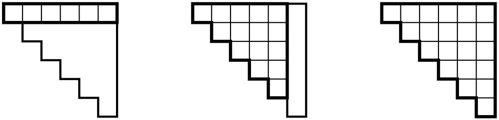
Schubert varieties in X can be indexed by strict partitions
$\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$



(n rows)

Shapes of Seidel elements v_β:

$(n) =$



Theorem (Kresch, Tamvakis): Pieri Formula for $[X^{(p)}] \star [X^\lambda] \in QH(X)$.

$$[\mathcal{O}_{X^{(n)}}] \star [\mathcal{O}_{X^\lambda}] \;=\; \begin{cases} [\mathcal{O}_{X^{(n,\lambda)}}] & \text{if } \lambda_1 < n, \\ q\,[\mathcal{O}_{X^{(\lambda_2,\ldots,\lambda_\ell)}}] & \text{if } \lambda_1 = n. \end{cases}$$

Pieri formula for $K(X)$

Let $\lambda \subset \nu$ be strict partitions. Skew shape: $\nu/\lambda = \nu \smallsetminus \lambda$.

A **KOG tableau** of shape ν/λ is a labeling of the boxes in ν/λ with integers such that

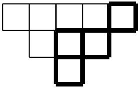
(1) All rows and columns are strictly increasing, and

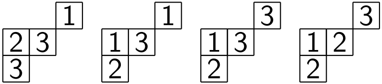
(2) Each label is either $\leq$ all labels south-west of it, or $\geq$ all labels south-west of it.

Theorem (B-Ravikumar) $[\mathcal{O}_{X^{(p)}}] \cdot [\mathcal{O}_{X^\lambda}] \;=\; \sum_{\nu} C^{\nu}_{p,\lambda}\, [\mathcal{O}_{X^\nu}]$ in $K(X)$

$C^{\nu}_{p,\lambda} = (-1)^{|\nu/\lambda|-p}$ # KOG-tableau of shape ν/λ with content $\{1,\ldots,p\}$

Example: $\nu = (5,3,1)$, $\lambda = (4,1)$, $p = 3$. Then $C^{\nu}_{3,\lambda} = -4$.

$\nu/\lambda =$ 



Pieri formula for $\mathrm{QK}(X)$

Compute $[\mathcal{O}_{X^{(p)}}] \star [\mathcal{O}_{X^\lambda}]$ in $\mathrm{QK}(X)$

Assume $\lambda_1 < n$:

$[X^{(p)}] \star [X^\lambda]$ has no q-terms $\Rightarrow$ $[\mathcal{O}_{X^{(p)}}] \star [\mathcal{O}_{X^\lambda}]$ has no q-terms.

$$[\mathcal{O}_{X^{(p)}}] \star [\mathcal{O}_{X^\lambda}] \;=\; [\mathcal{O}_{X^{(p)}}] \cdot [\mathcal{O}_{X^\lambda}] \;=\; \sum_\nu C^\nu_{p,\lambda} \, [\mathcal{O}_{X^\nu}]$$

Assume $\lambda_1 = n$:

Set $\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_\ell)$.

$$[\mathcal{O}_{X^{(p)}}] \star [\mathcal{O}_{X^\lambda}] \;=\; [\mathcal{O}_{X^{(p)}}] \star [\mathcal{O}_{X^{\overline{\lambda}}}] \star [\mathcal{O}_{X^{(n)}}] \;=\; \sum_\nu C^\nu_{p,\overline{\lambda}} \, [\mathcal{O}_{X^\nu}] \star [\mathcal{O}_{X^{(n)}}]$$

Proof Methods

$X = G/P_X$ cominuscule.

Diameter: $d_X(2) = \mathrm{dist}(1.P_X, w_0.P_X)$

Write $[\mathcal{O}_{X_u}] \star [\mathcal{O}_{X^v}] \;=\; \sum_{d \geq 0} ([\mathcal{O}_{X_u}] \star [\mathcal{O}_{X^v}])_d \, q^d$

Known: $([\mathcal{O}_{X_u}] \star [\mathcal{O}_{X^v}])_d = 0$ for $d > d_X(2)$.

Let $0 \leq d \leq d_X(2)$.

Choose $x, y \in X$ with $\mathrm{dist}(x, y) = d$.

$\Gamma_d(x, y) =$ union of curves of degree d through x, y.

Quantum = Classical Construction

Example: $X = \mathrm{Gr}(m,n) = \{V \subset \mathbb{C}^n \mid \dim(V) = m\}$

Let $V_1, V_2 \in X$ and set $d = \mathrm{dist}(V_1, V_2) = m - \dim(V_1 \cap V_2)$

Set $A = V_1 \cap V_2$, $B = V_1 + V_2$.

$\Gamma_d(V_1, V_2) = \{V \in X \mid A \subset V \subset B\} = \mathrm{Gr}(d, 2d)$

Note: $\Gamma_d(V_1, V_2)$ is determined by the point

$$\omega = (A, B) \text{ in } Y_d := \mathrm{Fl}(m-d, m+d; n).$$

Notation: $\Gamma_\omega = \mathrm{Gr}(d, B/A) \subset X$

Incidence variety:
$Z_d = \{(\omega, x) \in Y_d \times X \mid x \in \Gamma_\omega\} = \mathrm{Fl}(m-d, m, d+d; n)$

Projections: $q_d : Z_d \to Y_d$ and $p_d : Z_d \to X$.

Quantum = Classical Theorem

$$
\begin{aligned}
Z_d(X_u, X^v) &= q_d^{-1}(q_d p_d^{-1}(X_u) \cap q_d p_d^{-1}(X^v)) \\
&= \{(\omega, z) \in Z_d \mid \Gamma_\omega \cap X_u \neq \emptyset \text{ and } \Gamma_\omega \cap X^v \neq \emptyset\}
\end{aligned}
$$

Quantum = Classical Theorem for QH:

$$([X_u] \star [X^v])_d = p_{d*}[Z_d(X_u, X^v)]$$

$$
\begin{aligned}
Z_{d-1,1}(X_u, X^v) = \{(\omega, z) \in Z_d \mid \exists\, x \in \Gamma_\omega \cap X_u \text{ and } y \in \Gamma_\omega \cap X^v \\
\text{such that } \operatorname{dist}(x, y) \leq d-1\}.
\end{aligned}
$$

Quantum = Classical Theorem for QK:

$$([\mathcal{O}_{X_u}] \star [\mathcal{O}_{X^v}])_d = p_{d*}[\mathcal{O}_{Z_d(X_u, X^v)}] - p_{d*}[\mathcal{O}_{Z_{d-1,1}(X_u, X^v)}]$$

Main idea in proof

$\Gamma_d(X_u, X^v)$ = union of degree d curves connecting X_u to X^v.

$\Gamma_{d-1,1}(X_u, X^v) = \{z \in X \mid \exists$ degree $d-1$ curve C connecting X_u to X^v, and a line connecting z to C $\}$

Set $d_{\max}(u, v)$ = maximal power of q in $[X_u] \star [X^v]$.

We prove that:

(1) $p_{d*}[\mathcal{O}_{Z_d(X_u,X^v)}] = [\mathcal{O}_{\Gamma_d(X_u,X^v)}]$

(2) $p_{d*}[\mathcal{O}_{Z_{d-1,1}(X_u,X^v)}] = [\mathcal{O}_{\widetilde{\Gamma_{d-1,1}}(X_u,X^v)}]$ if $(X, d) \neq (\mathrm{LG}, d_{\max}(u, v)+1)$

(3) $d \leq d_{\max}(u, v) \Rightarrow \Gamma_{d-1,1}(X_u, X^v) \subset \Gamma_d(X_u, X^v)$ is a divisor.

(4) $d > d_{\max}(u, v) \Rightarrow \Gamma_{d-1,1}(X_u, X^v) = \Gamma_d(X_u, X^v)$

Brion's positivity theorem $\Rightarrow$ classes in (1) and (2) have alternating signs!

Seidel product with a point

$[\mathcal{O}_{\mathrm{pt}}] \star [\mathcal{O}_{X^v}] = [\mathcal{O}_{\Gamma_d(1.P_X, X^v)}] q^d$

where $d = \mathrm{dist}(1.P_X, X^v) = d_{\max}(\mathrm{pt}, v)$

Notice: $\Gamma_{d-1,1}(1.P_X, X^v) = \emptyset$

We show that $\Gamma_d(1.P_X, X^v)$ is a Schubert variety in X.