Puzzles for Projections from 3-step flag varieties

Guangzhou 2017

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Schubert varieties in 3-step flag manifold

$X = \mathrm{Fl}(a_1, a_2, a_3; n) = \{(A_1 \subset A_2 \subset A_3 \subset \mathbb{C}^n) \mid \dim(A_k) = a_k\}$

Def: A **Schubert string** for X is a permutation of $0^{a_1} 1^{a_2-a_1} 2^{a_3-a_2} 3^{n-a_3}$.

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$\mathbb{C}^n$ has basis $\{e_1, e_2, \ldots, e_n\}$. $u = (u_1, u_2, \ldots, u_n)$ Schubert string for X.

Def: $A^u = (A^u_1 \subset A^u_2 \subset A^u_3) \in X$ by $A^u_k = \mathrm{Span}_{\mathbb{C}}\{e_i : u_i < k\}$.

Example: $X = \mathrm{Fl}(1, 3, 4; 6)$

$$A^{130123} = (\mathbb{C}e_3 \;\subset\; \mathbb{C}e_1 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \;\subset\; \mathbb{C}e_1 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_5)$$

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$\mathbf{B}^+ \subset \mathrm{GL}(\mathbb{C}^n)$ upper triangular ; $\mathbf{B}^- \subset \mathrm{GL}(\mathbb{C}^n)$ lower triangular.

Schubert varieties: $X_u = \overline{\mathbf{B}^+.A^u}$; $X^u = \overline{\mathbf{B}^-.A^u} \subset X$

$\dim(X_u) = \mathrm{codim}(X^u, X) = \ell(u) = \#\{i < j \mid u_i > u_j\}$

Projection to Grassmannian

$\pi : \; X = \mathrm{Fl}(a_1, a_2, a_3; n) \longrightarrow Y = \mathrm{Gr}(a_2, n) \quad ; \quad \pi(A_1 \subset A_2 \subset A_3) = A_2$

Simple labels for X: 0 , 1 , 2 , 3

Simple labels for Y: 01 , 23 **Merged !!**

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Schubert string for Y: $w = (w_1, \ldots, w_n)$, $w_i \in \{01, 23\}$, $a_2 = \#\, 01$

Def: $V^w = \mathrm{Span}\{e_i \mid w_i = 01\} \in Y$

Note: $\pi(A^u) = A_2^u = V^w$ where $w_i = 01 \Leftrightarrow u_i \in \{0, 1\}$

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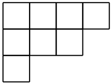
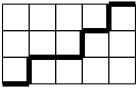
Translation to Young diagrams for $Y = \mathrm{Gr}(3, 8)$:

$w =$ 23-01-23-23-01-23-01-23 $\longleftrightarrow$  

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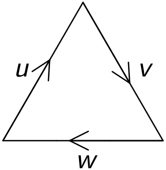
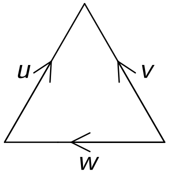
Goal: $\int_X [X^u] \cdot [X^v] \cdot \pi^*[Y^w] \; = \; \#$ 

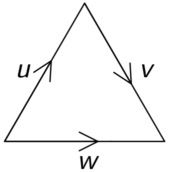
Product with pullback:

$$[X^u] \cdot \pi^*[Y^w] \;=\; \sum_v \left(\int_X [X^u] \cdot [X_v] \cdot \pi^*[Y^w] \right) [X^v] \qquad \text{in} \;\; H^*(X;\mathbb{Z})$$

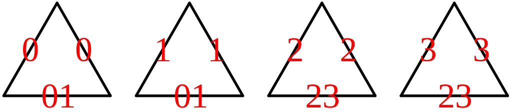


Pushforward of product:

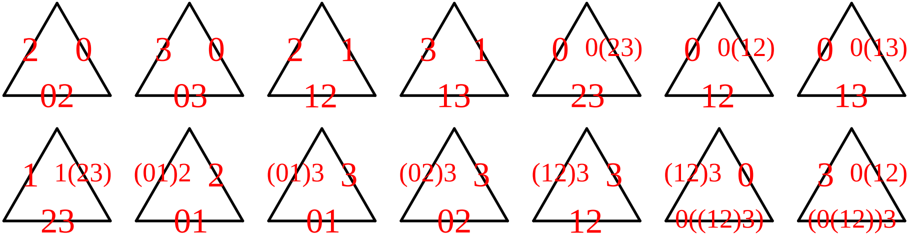
$$\pi_*([X^u] \cdot [X^v]) \;=\; \sum_w \left(\int_X [X^u] \cdot [X^v] \cdot \pi^*[Y_w] \right) [Y^w] \qquad \text{in} \;\; H^*(Y;\mathbb{Z})$$



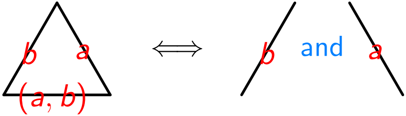
Simple puzzle pieces:



Composed puzzle pieces:

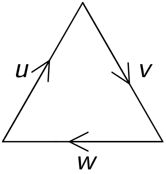


Definition of composed pieces:



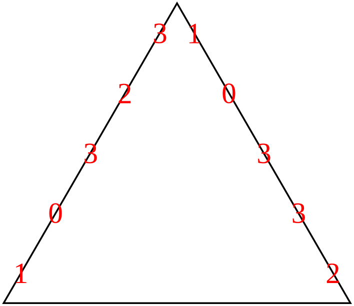
are **edges** **AND** $\max(a) < \min(b)$

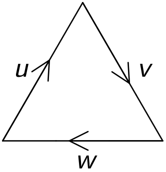
AND a and b are **not merged**.

Theorem: $$\int_X [X^u] \cdot [X^v] \cdot \pi^*[Y^w] \;=\; \#$$ 

Example: $\pi : \; X = \mathrm{Fl}(1,2,3;5) \longrightarrow \mathrm{Gr}(2,5) = Y$

$$\pi_*([X^{10323}] \cdot [X^{10332}]) \;=\; ?$$



Theorem: $\int_X [X^u]\cdot[X^v]\cdot\pi^*[Y^w] \;=\; \#$ 

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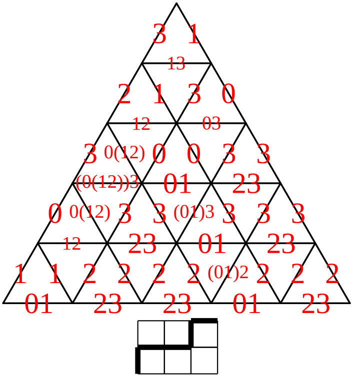

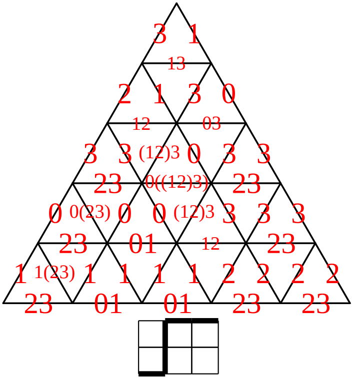

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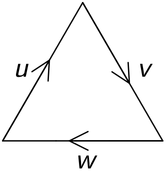

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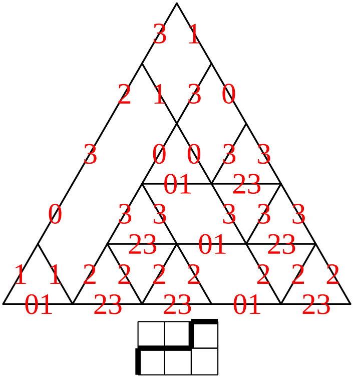

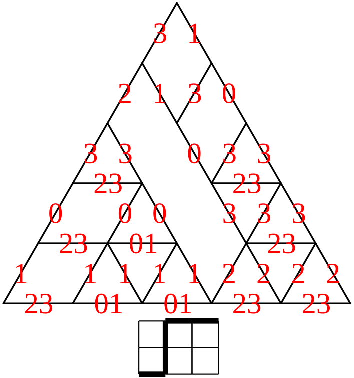

Quantum cohomology

Gromow-Witten invariants of $Y = \mathrm{Gr}(m,n)$:

$\langle Y^u, Y^v, Y_w \rangle_d \;=\; \#$ rational curves $C \subset Y$ of degree d
meeting Y^u , $g.Y^v$, and Y_w
where $g \in GL_n$ is a fixed general element.

$\langle Y^u, Y^v, Y_w \rangle_d \;=\; 0$ if infinitely many curves exist.

Small quantum cohomology ring

$$QH(Y) \;=\; H^*(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$$

$$[Y^u] \star [Y^v] \;=\; \sum_w \langle Y^u, Y^v, Y_w \rangle_d \, q^d \, [Y^w]$$

Quantum = classical

$$\begin{array}{ccc} X = \mathrm{Fl}(m-d, m, m+d; n) & \xrightarrow{\pi} & Y = \mathrm{Gr}(m,n) \\ \downarrow \phi & & \\ Z = \mathrm{Fl}(m-d, m+d; n) & & \end{array}$$

Theorem (B-Kresch-Tamvakis)

$$\langle Y^u, Y^v, Y_w \rangle_d \;=\; \#\, \phi\pi^{-1}(Y^u) \cap \phi\pi^{-1}(g.Y^v) \cap \phi\pi^{-1}(Y_w)$$

$$C \;\longleftrightarrow\; \big(\mathrm{Ker}(C), \mathrm{Span}(C)\big) := \left(\bigcap_{V \in C} V \,,\, \sum_{V \in C} V \right) \in Z$$

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Theorem (B-Mihalcea) Equivariant generalization:

$$\langle Y^u, Y^v, Y_w \rangle_d^T \;=\; \int_Z^T \phi_* \pi^* [Y^u] \cdot \phi_* \pi^* [Y^v] \cdot \phi_* \pi^* [Y_w]$$

$$=\; \int_X^T \phi^* \phi_* \pi^* [Y^u] \cdot \phi^* \phi_* \pi^* [Y^v] \cdot \pi^* [Y_w]$$

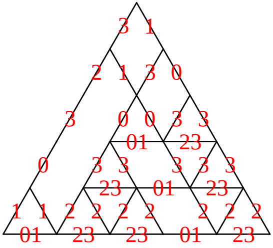
Example

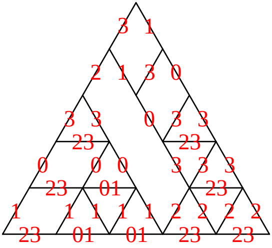
$$Z = \mathrm{Fl}(1,3;5) \xleftarrow{\ \phi\ } X = \mathrm{Fl}(1,2,3;5) \xrightarrow{\ \pi\ } Y = \mathrm{Gr}(2,5)$$

Compute coefficient of q^1 in quantum product $[Y^{(2,1)}] \star [Y^{(3,1)}] \in QH(Y)$

Quantum = classical implies:

$$\begin{aligned}([Y^{(2,1)}] \star [Y^{(3,1)}])_1 &= \pi_*\left(\phi^*\phi_*\pi^*[Y^{(2,1)}] \cdot \phi^*\phi_*\pi^*[Y^{(3,1)}]\right) \\ &= \pi_*([X^{10323}] \cdot [X^{10332}]) = [Y^{(2)}] + [Y^{(1,1)}]\end{aligned}$$





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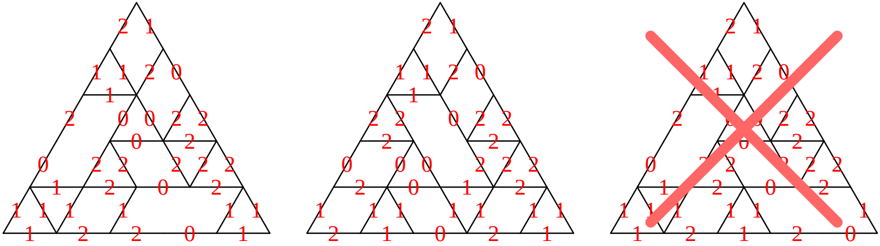
$$= \pi_*([X^{10323}] \cdot [X^{10332}]) = [Y^{(2)}] + [Y^{(1,1)}]$$

Now use 2-step puzzle formula:

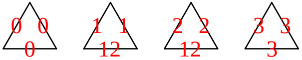
$$([Y^{(2,1)}] \star [Y^{(3,1)}])_1 = \pi_*\phi^* \left(\phi_*\pi^*[Y^{(2,1)}] \cdot \phi_*\pi^*[Y^{(3,1)}]\right)$$
$$= \pi_*\phi^*([Z^{10212}] \cdot [Z^{10221}]) = [Y^{(2)}] + [Y^{(1,1)}]$$



Projection to 2-step flag manifold

$\pi : \mathrm{Fl}(a_1, a_2, a_3; n) \to \mathrm{Fl}(a_1, a_3; n)$

Simple puzzle pieces:

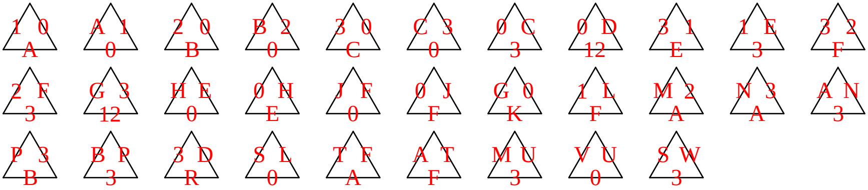


Simple labels:

$\mathrm{Fl}(a_1, a_2, a_3; n)$: 0 , 1 , 2 , 3

$\mathrm{Fl}(a_1, a_3; n)$: 0 , 12 , 3

Composed puzzle pieces:



Composed labels:

A=01 B=02 C=03 D=0(12) E=13 F=23 G=(12)3 H=0(13)
J=0(23) K=0((12)3) L=1(23) M=(01)2 N=(01)3 P=(02)3 R=(0(12))3
S=0(1(23)) T=(01)(23) U=((01)2)3 V=0(((01)2)3) W=(0(1(23)))3

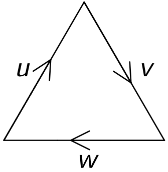
Rule: (a, b) can be a label **only if** $\max(a) < \min(b)$ **OR**
$\max(a) = \min(b)$ **AND** repetition separated by 3 parentheses.

Puzzle formula for projections

Let $\pi : X \to Y$ be a projection of partial flag manifolds.

Assume X has at most 3 steps, Y has at most 2 steps.

Theorem: $$\int_X [X^u] \cdot [X^v] \cdot \pi^*[Y^w] \;=\; \#$$



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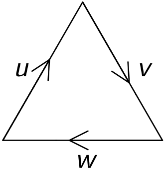


Known cases:

Puzzle rule for $H^*(\mathrm{Gr}(m,n))$ (Knutson, Tao, Woodward)

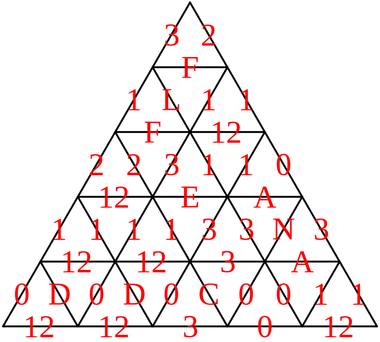
Puzzle rule for $H_T^*(\mathrm{Gr}(m,n))$ (Knutson, Tao)

Puzzle rule for $H^*(\mathrm{Fl}(a,b;n))$ (conjectured by Knutson,
proof in [B-Kresch-Purbhoo-Tamvakis] ,
different positive formula by Coskun.)

Conjecture (Knutson, Buch) / **Theorem** (Knutson - Zinn-Justin)

Formula holds for $X = Y = \mathrm{Fl}(a_1, a_2, a_3; n)$.

The mutation algorithm



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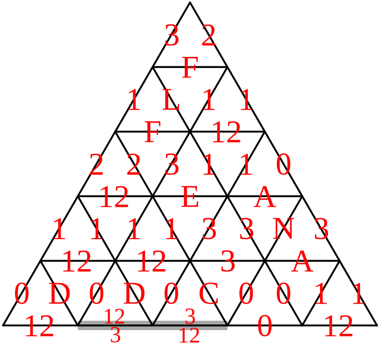

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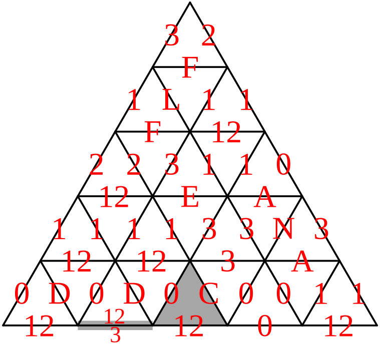

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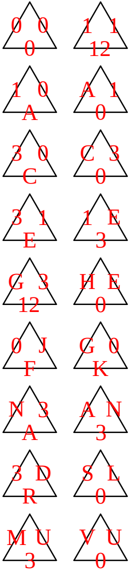

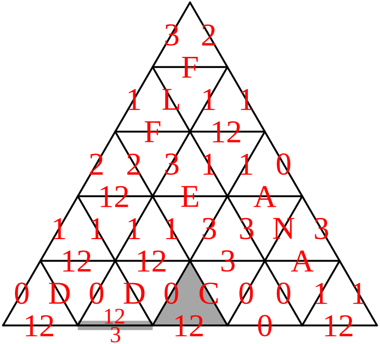

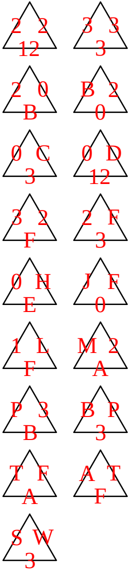

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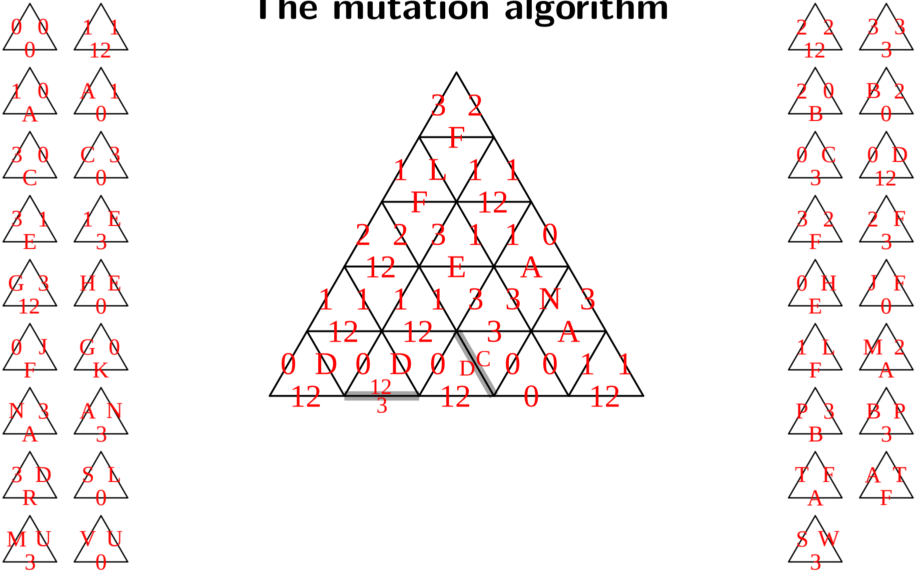

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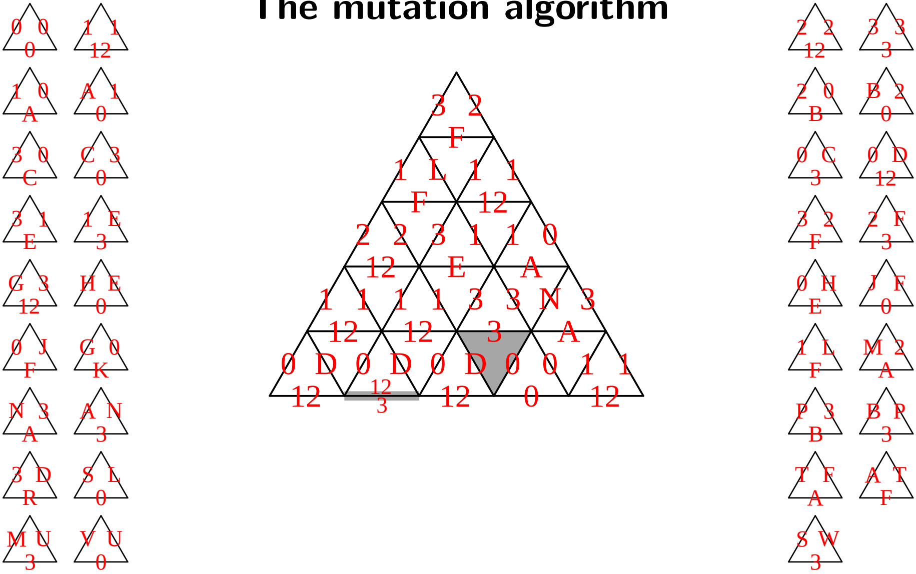

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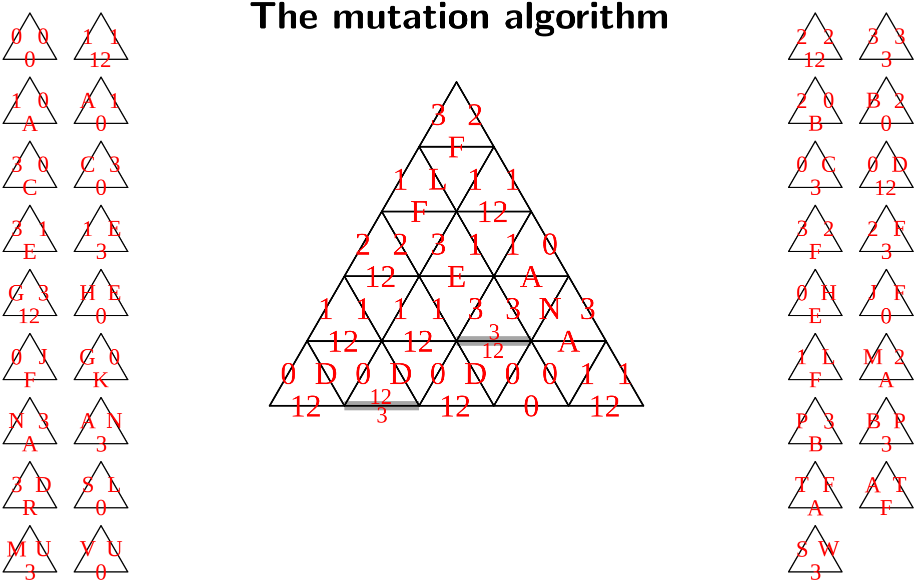

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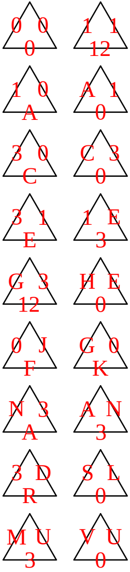

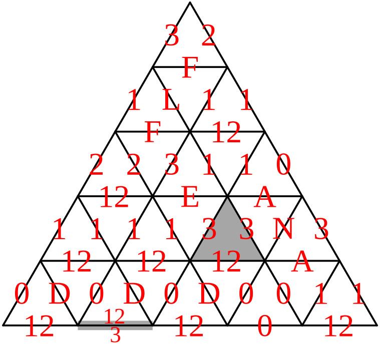

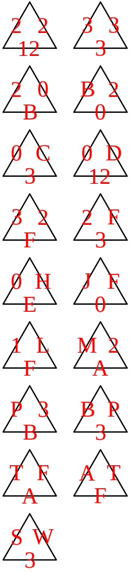

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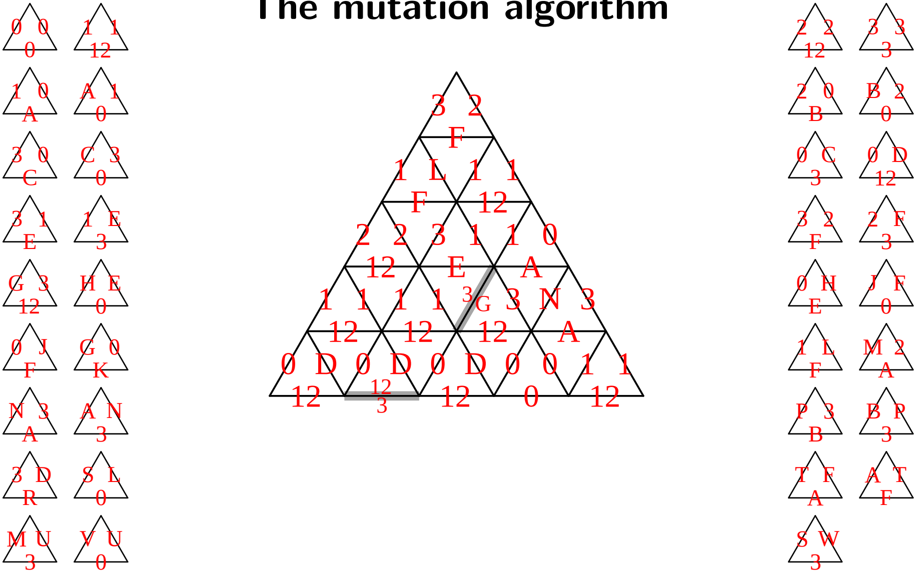

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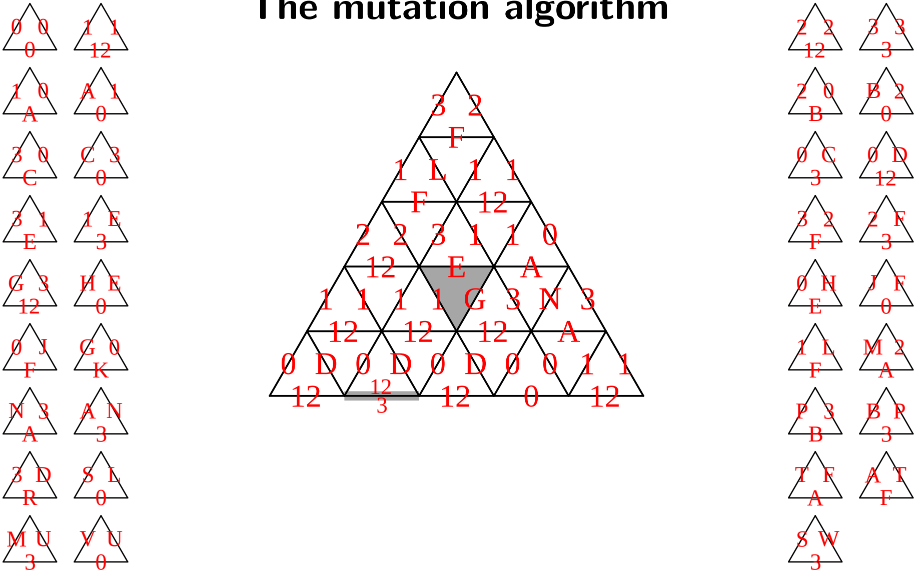

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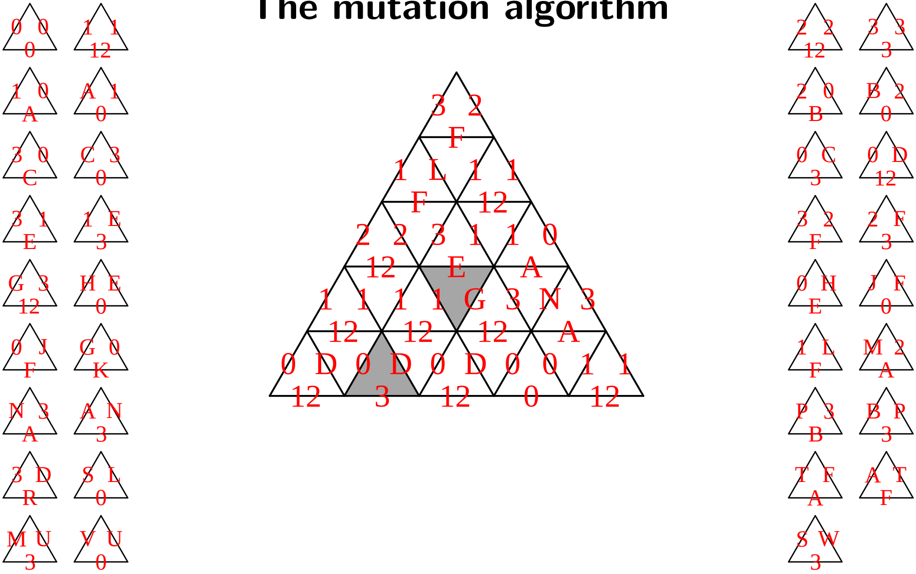

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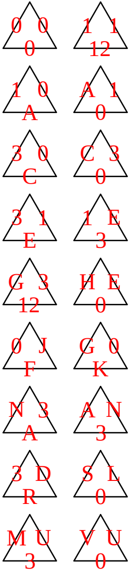

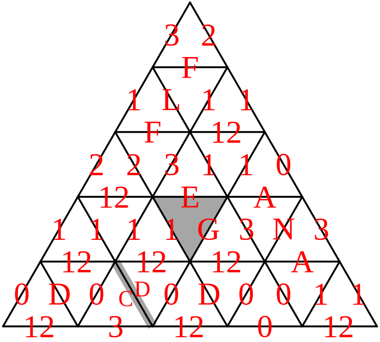

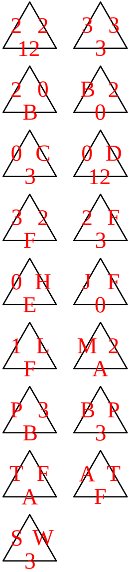

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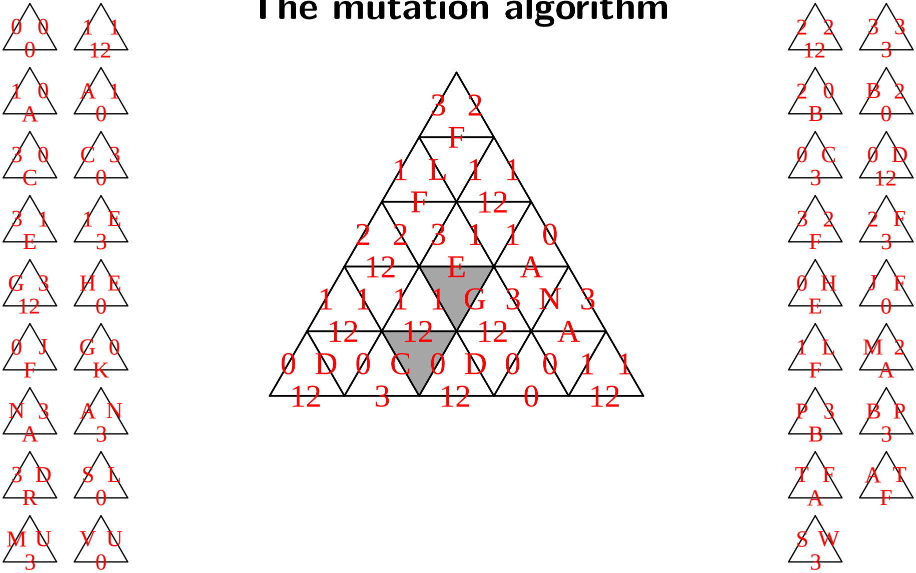

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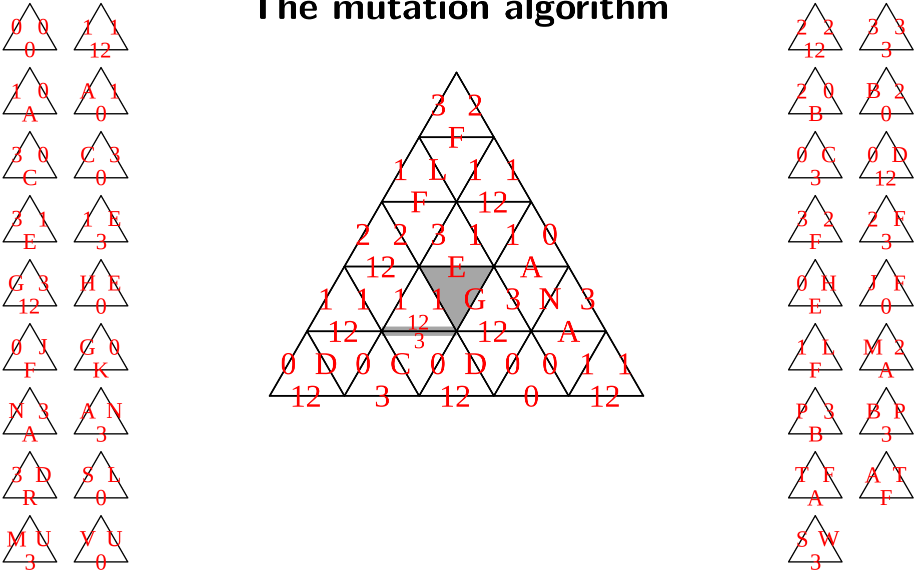

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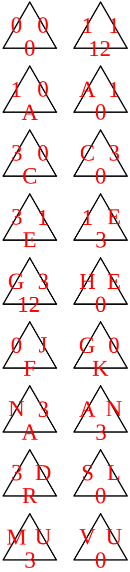

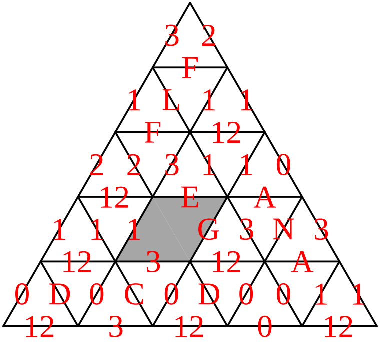

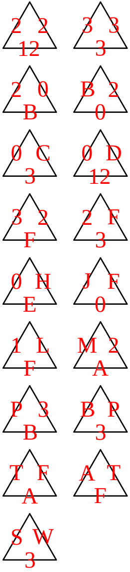

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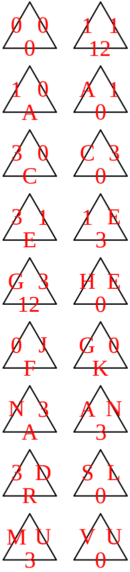

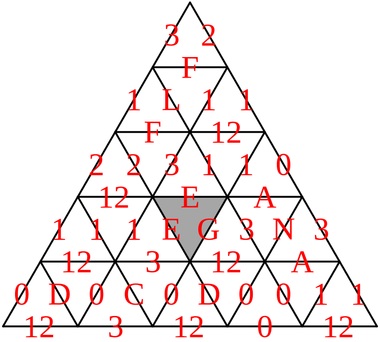

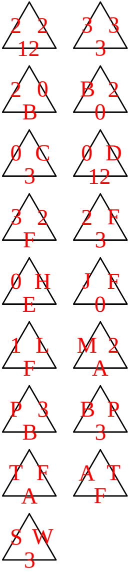

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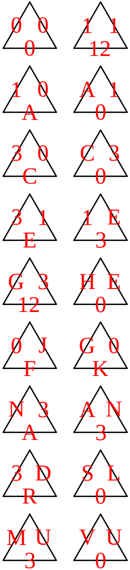

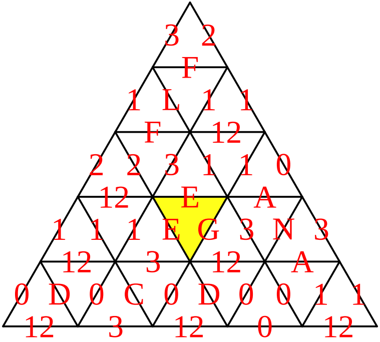

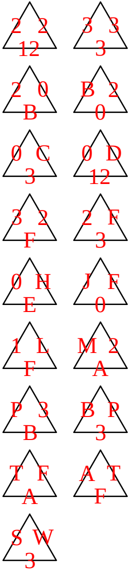

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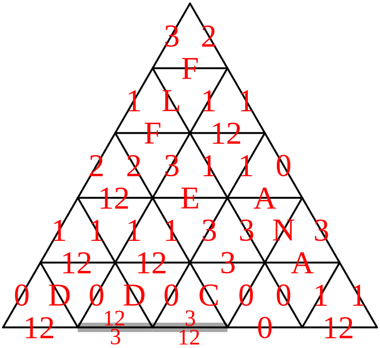

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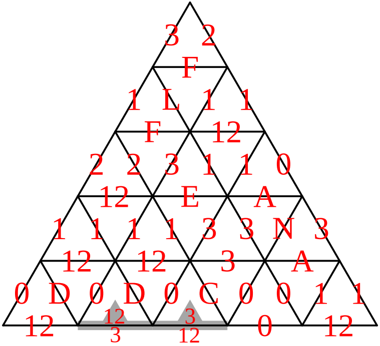

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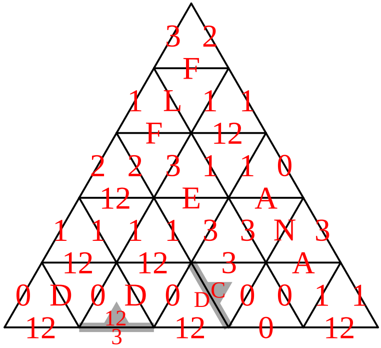

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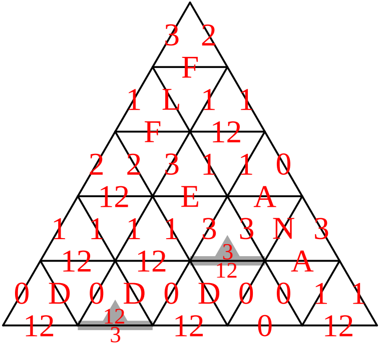

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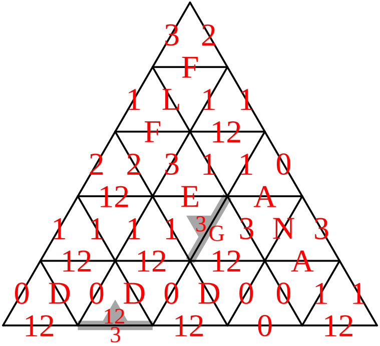

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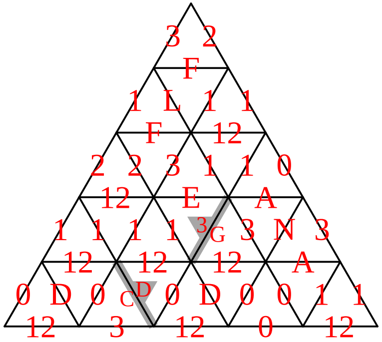

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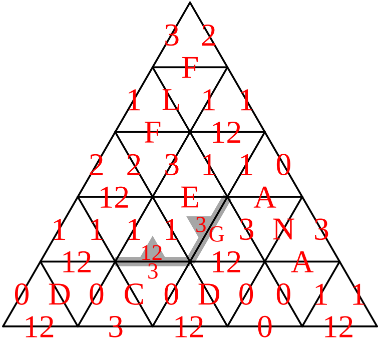

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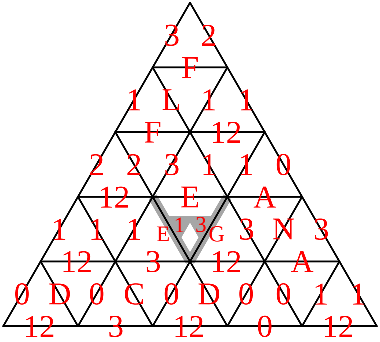

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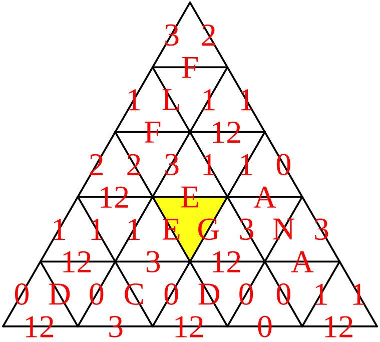

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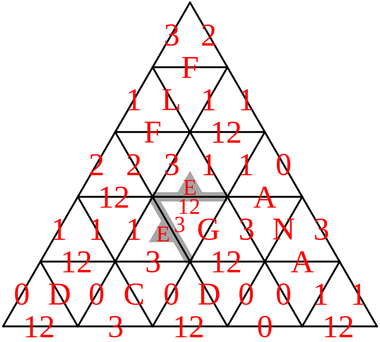
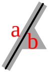
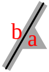


Resolutions of temporary puzzle piece

Def: Two gashes are **equivalent** if one can be propagated to the other.

Def: A gash is **opposite** to 

$\iff$ it is equivalent to 

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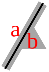
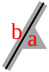
Def: A gash is **opposite** to 

$\Longleftrightarrow$ it is equivalent to 

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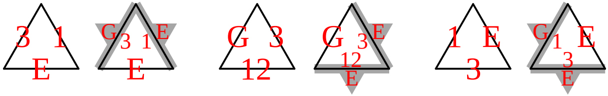
Def: A **resolution** of a temporary piece is a puzzle piece that creates two opposite gashes on replacement.

Example:

Resolutions of 

: 

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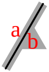
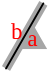
Def: A gash is **opposite** to 

$\iff$ it is equivalent to 

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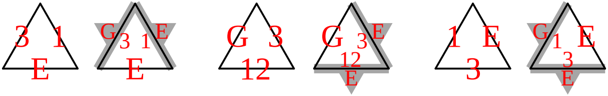
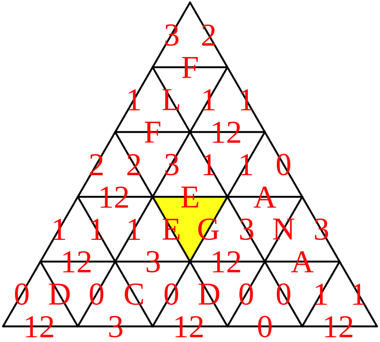
: 

Fact: Each temporary piece has exactly 3 resolutions.

Note: Every gash is either a **left gash** or a **right gash**.

The mutation algorithm



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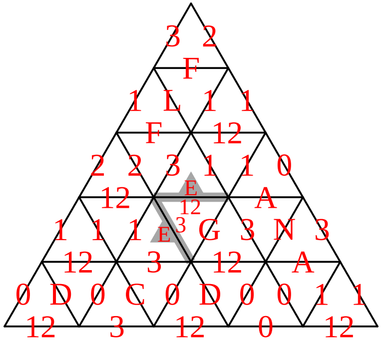

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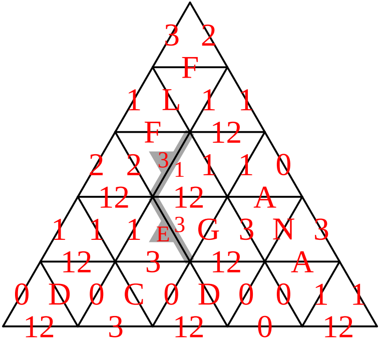

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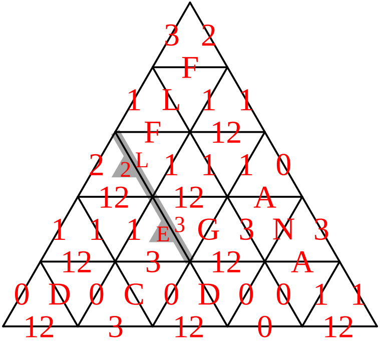

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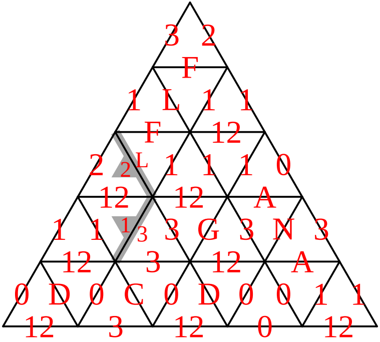

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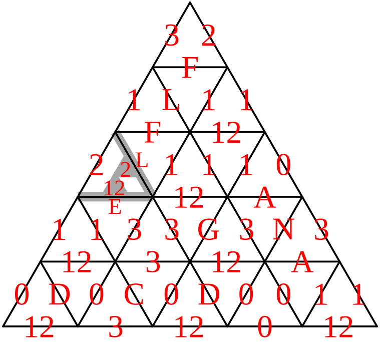

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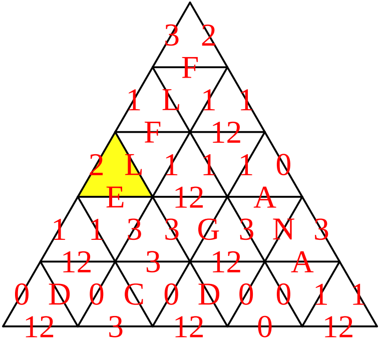

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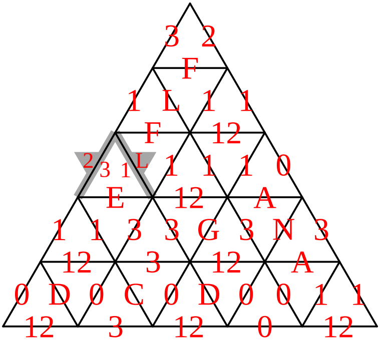

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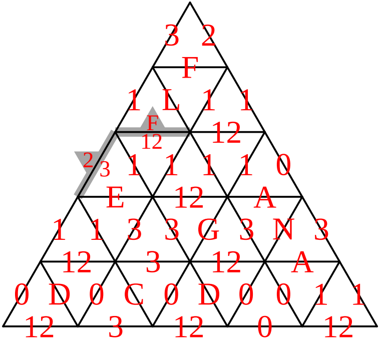

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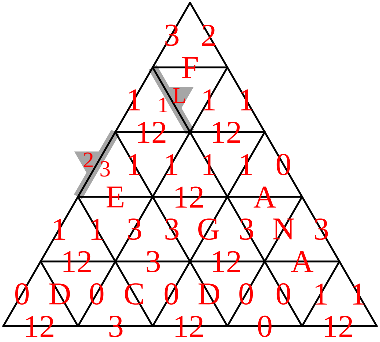

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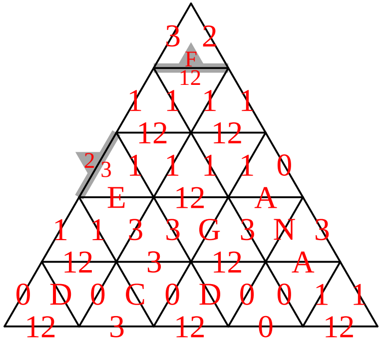

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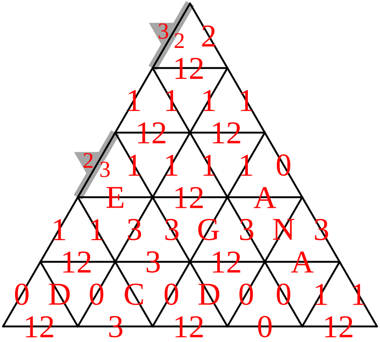

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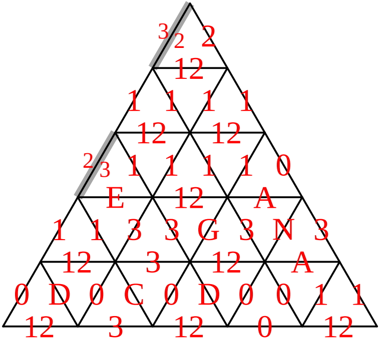

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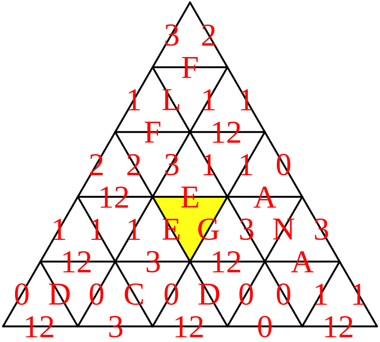

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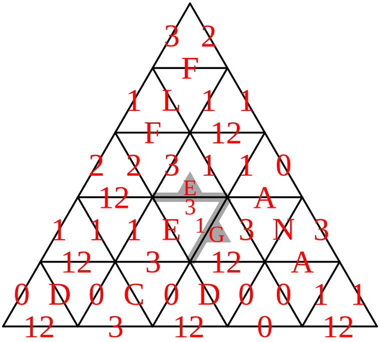

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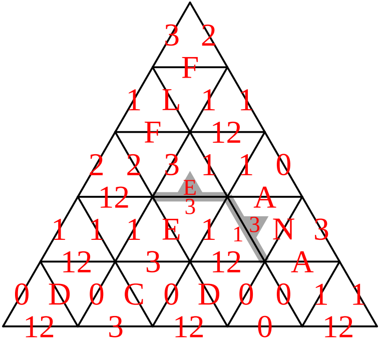

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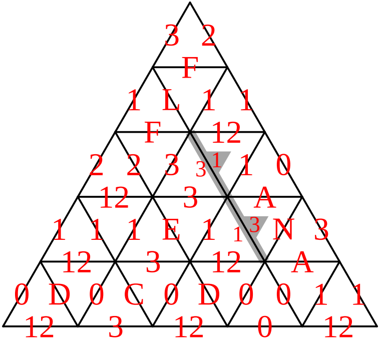

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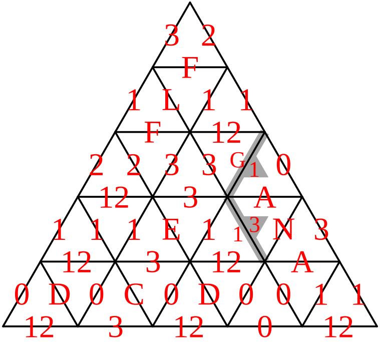

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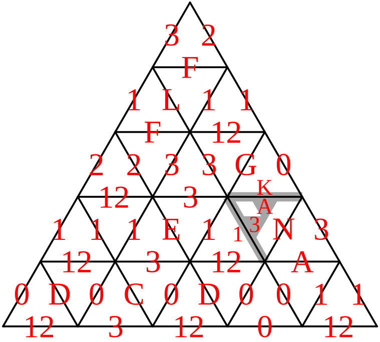

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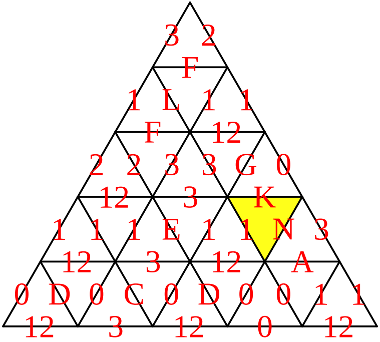
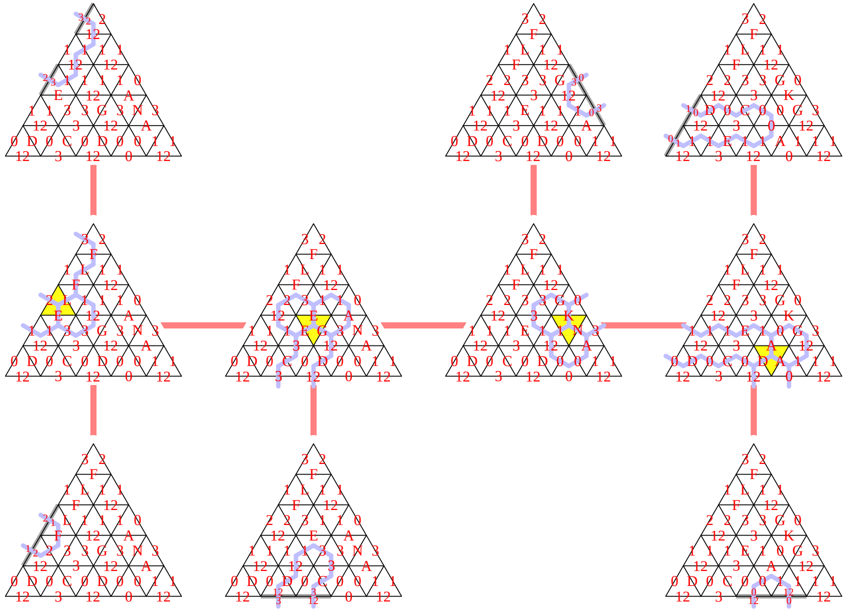
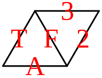


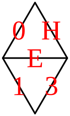
Component of the mutation graph

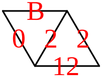


Borel construction for puzzle pieces

Def: A **scab** is a small rhombus consisting of two distinct puzzle pieces.







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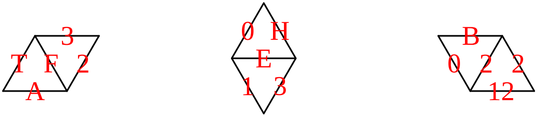
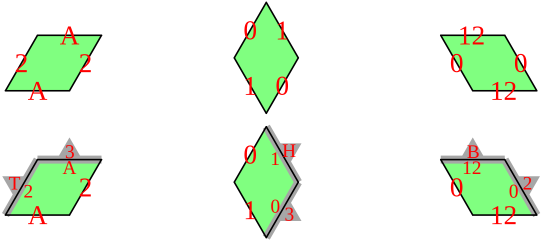


Def: A **resolution** of a scab is a symmetric rhombus that creates two opposite gashes on replacement, with left gash in left side, right gash in right side.



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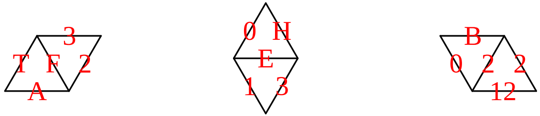

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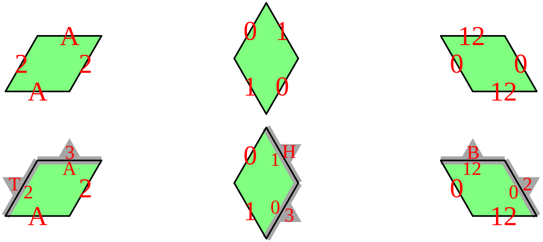
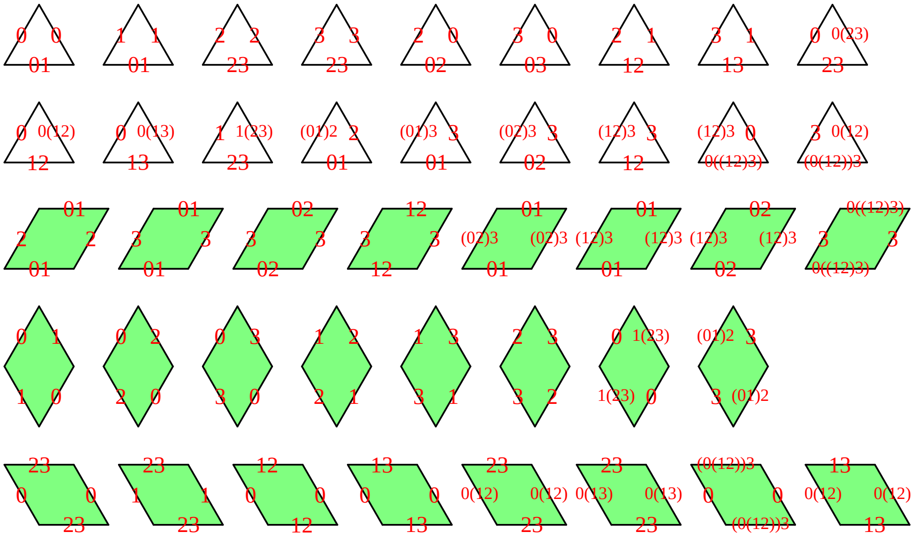


Fact: Any scab has at most one resolution.

Def: A resolution of a scab is also called an **equivariant puzzle piece**.

All puzzle pieces for $\pi : \mathrm{Fl}(a_1, a_2, a_3; n) \to \mathrm{Gr}(a_2, n)$



Equivariant cohomology

$T \subset \mathrm{GL}(\mathbb{C}^n)$ max torus of diagonal matrices.

$\Lambda = H_T^*(\mathrm{pt};\mathbb{Z}) = \mathbb{Z}[y_1, y_2, \ldots, y_n]$; $y_i = -c_1(\mathbb{C}e_i)$

$H_T^*(X;\mathbb{Z}) = \bigoplus_u \Lambda\,[X^u]$ is a Λ-algebra.

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where $\int_X^T : H_T^*(X; \mathbb{Z}) \to \Lambda$ is pushforward along $X \to \{\mathrm{pt}\}$.

Theorem (Graham): $C_{u,v}^w \in \mathbb{Z}_{\geq 0}[y_2 - y_1, \ldots, y_n - y_{n-1}]$

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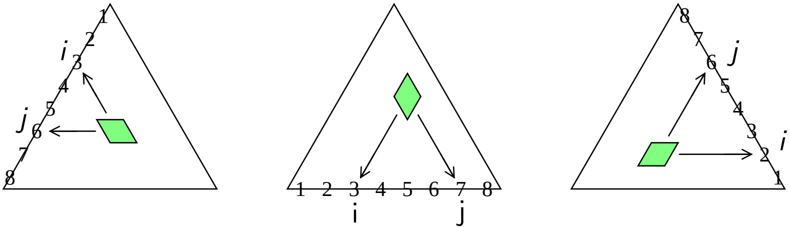
Theorem (Graham): $C_{u,v}^w \in \mathbb{Z}_{\geq 0}[y_2 - y_1, \dots, y_n - y_{n-1}]$

Def: weight(◇) $= y_j - y_i$ where $i < j$ are defined by



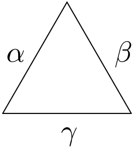
Equivariant puzzle formula

Let $\pi : X \to Y$ be a projection of partial flag manifolds.

Assume X has at most 3 steps, Y has at most 2 steps.

Let $\alpha, \beta \in H_T^*(X)$ and $\gamma \in H_T^*(Y)$ be Schubert classes,
such that one of α, β, γ is $\mathbf{B}^+$-stable, the other two are $\mathbf{B}^-$-stable.

Consider puzzles with all equivariant pieces pointing to $\mathbf{B}^+$-stable side:

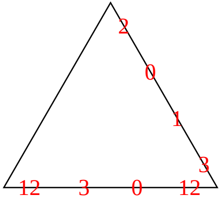
$P =$ 

Theorem: If all scabs pointing to $\mathbf{B}^+$-stable side have resolutions, then

$$\int_X^T \alpha \cdot \beta \cdot \pi^*(\gamma) \;=\; \sum_P \prod_{\lozenge \in P} \text{weight}(\lozenge)$$

Example $\quad \pi : \; X = \mathsf{Fl}(1,2,3;4) \;\longrightarrow\; Y = \mathsf{Fl}(1,3;4)$

$[X^{2013}] \cdot \pi^*[Y^{12\text{-}0\text{-}3\text{-}12}] \; = \;$ **?**



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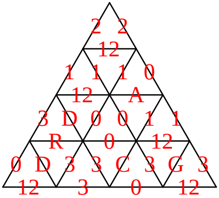
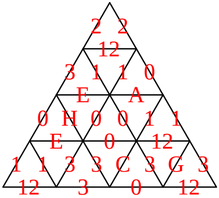
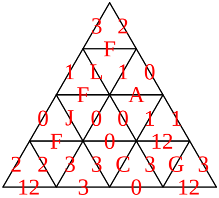
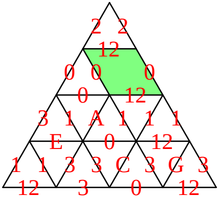
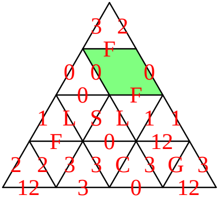
$$[X^{2013}] \cdot \pi^*[Y^{12\text{-}0\text{-}3\text{-}12}] \; = \; \textbf{?}$$











Example $\pi : \ X = \mathsf{Fl}(1,2,3;4) \longrightarrow Y = \mathsf{Fl}(1,3;4)$

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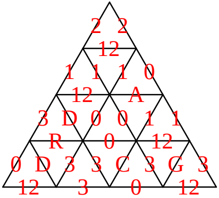
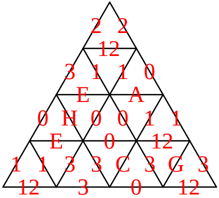
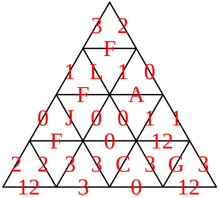


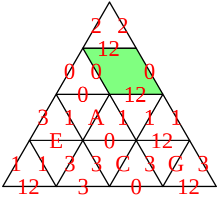
$[X^{2130}]$ $+$

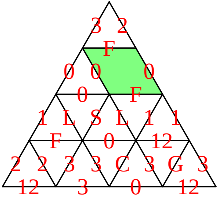


$[X^{2301}]$ $+$



$[X^{3102}]$





$+ \quad (y_2 - y_1)\,[X^{2031}] \quad + \quad (y_2 - y_1)\,[X^{3012}]$

Existence of equivariant puzzle pieces

Results: Formula for $[X^u] \cdot \pi^*[Y^v]$ in $H_T^*(X;\mathbb{Z})$ for every $\pi : X \to Y$

Formula for $\pi_*([X^u] \cdot [X^v])$ in $H_T^*(Y;\mathbb{Z})$ for every $\pi : X \to Y$

except $\pi : \mathrm{Fl}(a_1, a_2, a_3; n) \to \mathrm{Fl}(a_1, a_3; n)$

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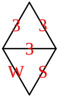
Reason: The scab 

has no resolution!

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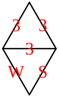
Reason: The scab 

has no resolution!

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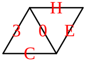
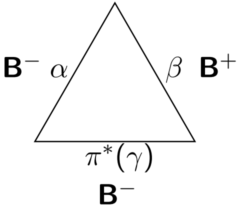
The scab 

has no resolution

$\Rightarrow$ No formula for $\pi^*(\gamma) \cdot \alpha$:



Existence of equivariant puzzle pieces

Results: Formula for $[X^u] \cdot \pi^*[Y^v]$ in $H_T^*(X;\mathbb{Z})$ for every $\pi : X \to Y$

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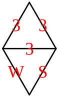
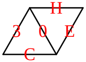
Reason: The scab 

has no resolution!

Example: $\pi : \mathrm{Fl}(a_1, a_2, a_3; n) \to \mathrm{Fl}(a_1, a_2; n)$

The scab 

has no resolution

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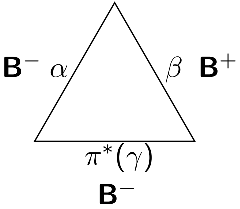
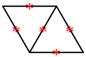
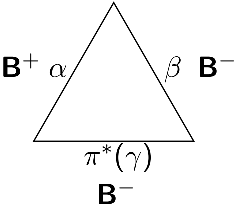


All scabs 

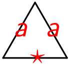
have resolutions

$\Rightarrow$ Obtain formula for $\beta \cdot \pi^*(\gamma)$:

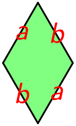


Projection to a point $\quad \pi : \mathrm{Fl}(n) \to \{\mathrm{pt}\}$

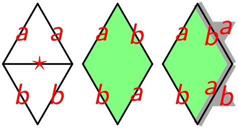
$$\int_X^T [X^u]\cdot[X^v]\cdot\pi^*[\mathrm{pt}] \;=\; \begin{cases} [X^{uv^{-1}w_0}]_{w_0} & \text{if } \ell(uv^{-1}w_0) = \ell(u) - \ell(v^{-1}w_0) \\ 0 & \text{otherwise.} \end{cases}$$

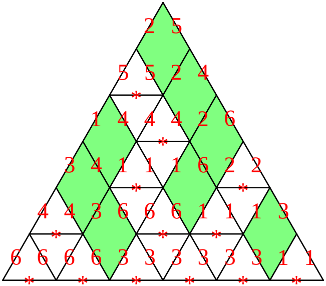
Puzzle pieces: 

for $1 \leq a \leq n$

Equivariant pieces: 

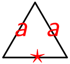
for $1 \leq a < b \leq n$



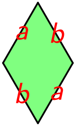


Projection to a point $\pi : \mathrm{Fl}(n) \to \{\mathrm{pt}\}$

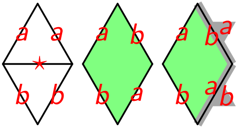
$$\int_X^T [X^u] \cdot [X^v] \cdot \pi^*[\mathrm{pt}] \;=\; \begin{cases} [X^{uv^{-1}w_0}]_{w_0} & \text{if } \ell(uv^{-1}w_0) = \ell(u) - \ell(v^{-1}w_0) \\ 0 & \text{otherwise.} \end{cases}$$

Puzzle pieces: 

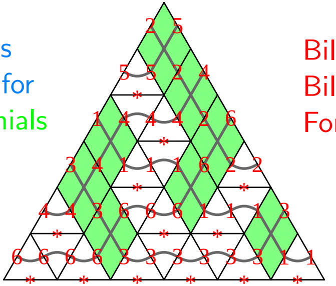
for $1 \le a \le n$

Equivariant pieces: 

for $1 \le a < b \le n$



Puzzle formula specializes to **pipe dream formula** for double Schubert polynomials



Billey-Jockusch-Stanley
Billey-Bergeron
Fomin-Kirillov

Highlights from proof $\quad X = G/P$; Fix $T \subset B \subset P \subset G$

Weyl groups

$W = N_G(T)/T$; $W_P = N_P(T)/T$

$W^P \subset W$ subset of minimal representatives for cosets in W/W_P

Schubert varieties

$X_u = \overline{Bu.P}$, $X^u = \overline{B^- u.P}$ for $u \in W$.

$\dim(X_u) = \operatorname{codim}(X^u, X) = \ell(u)$ whenever $u \in W^P$.

Schubert structure constants

$$C_{u,v}^w = \int_X^T [X^u] \cdot [X^v] \cdot [X_w] \in \Lambda = H_T^*(\mathrm{pt};\mathbb{Z})$$

$$[X^u] \cdot [X^v] = \sum_w C_{u,v}^w [X^w] \quad \text{in} \quad H_T^*(X;\mathbb{Z})$$

Chevalley formula

Let $D \in H^2_T(X;R)$, R commutative ring.

Write $u \to u'$ for covering relation in W^P: $u' = us_\alpha$ and $\ell(u') = \ell(u) + 1$

Define $(D, \frac{u'}{u}) \;:=\; \text{“}(D, \alpha^\vee)\text{”} \;=\; \int_{C_\alpha}^T D \quad \in H^*_T(\mathrm{pt};R)$

where $C_\alpha \subset X$ is the T-stable curve through $1.P$ and $s_\alpha.P$

Chevalley: $D \cdot [X^u] \;=\; D_u\,[X^u] \;+\; \sum_{u \to u'} (D, \frac{u'}{u})\,[X^{u'}]$ in $H^*_T(X;R)$

Molev-Sagan equations

Lemma: If $\eta \in R$ satisfies $\eta^2 + \eta + 1 = 0$, then

$$(-\eta^2 D_u - \eta D_v - D_w) C_{u,v}^{w} \;=$$

$$\eta^2 \sum_{u \to u'} (D, \tfrac{u'}{u}) C_{u',v}^{w} \;+\; \eta \sum_{v \to v'} (D, \tfrac{v'}{v}) C_{u,v'}^{w} \;+\; \sum_{w' \to w} (D, \tfrac{w}{w'}) C_{u,v}^{w'}$$

Proof: Expand and integrate

$$\eta^2 \,(D \cdot [X^u])\,[X^v]\,[X_w] \;+\; \eta\,[X^u]\,(D \cdot [X^v])\,[X_w] \;+\; [X^u]\,[X^v]\,(D \cdot [X_w]) \;=\; 0$$

Molev-Sagan equations

Lemma: If $\eta \in R$ satisfies $\eta^2 + \eta + 1 = 0$, then

$$(-\eta^2 D_u - \eta D_v - D_w) C_{u,v}^w \; = \; \eta^2 \sum_{u \to u'} (D, \tfrac{u'}{u}) C_{u',v}^w \; + \; \eta \sum_{v \to v'} (D, \tfrac{v'}{v}) C_{u,v'}^w \; + \; \sum_{w' \to w} (D, \tfrac{w}{w'}) C_{u,v}^{w'}$$

Proof: Expand and integrate

$$\eta^2 (D \cdot [X^u]) [X^v] [X_w] \; + \; \eta [X^u] (D \cdot [X^v]) [X_w] \; + \; [X^u] [X^v] (D \cdot [X_w]) \; = \; 0$$

Application:

Take $R = \mathbb{C}[\delta_\beta \mid \beta \in \Delta - \Delta_P]$, $D = \sum_{\beta \in \Delta - \Delta_P} \delta_\beta [X^{s_\beta}]$, $\eta = \exp(\frac{2\pi i}{3})$

Note: $(-\eta^2 D_u - \eta D_v - D_w) = 0 \iff u = v = w$

Molev-Sagan recursion for $\pi : X = G/P \longrightarrow Y = G/Q$

Want to compute $C_{u,v}^{w}$ for which $u \in W^{Q}$ or $v \in W^{Q}$ or $w \in W^{Q,\max}$

Use $D' = \sum_{\beta \in \Delta - \Delta_Q} \delta_\beta [X^{s_\beta}]$ divisor pulled back from Y !!

$$(-\eta^2 D'_u - \eta D'_v - D'_w) C_{u,v}^{w} =$$

$$\eta^2 \sum_{u \to u'} (D', \tfrac{u'}{u}) C_{u',v}^{w} + \eta \sum_{v \to v'} (D', \tfrac{v'}{v}) C_{u,v'}^{w} + \sum_{w' \to w} (D', \tfrac{w}{w'}) C_{u,v}^{w'}$$

Recursion involves only $C_{u',v'}^{w'}$, with $u' \in W^{Q}$ or $v' \in W^{Q}$ or $w' \in W^{Q,\max}$.

Molev-Sagan recursion for $\pi : X = G/P \longrightarrow Y = G/Q$

Want to compute $C^{w}_{u,v}$ for which $u \in W^Q$ or $v \in W^Q$ or $w \in W^{Q,\max}$

Use $D' = \sum_{\beta \in \Delta - \Delta_Q} \delta_\beta [X^{s_\beta}]$ divisor pulled back from Y !!

$$(-\eta^2 D'_u - \eta D'_v - D'_w) C^{w}_{u,v} =$$

$$\eta^2 \sum_{u \to u'} (D', \tfrac{u'}{u}) C^{w}_{u',v} + \eta \sum_{v \to v'} (D', \tfrac{v'}{v}) C^{w}_{u,v'} + \sum_{w' \to w} (D', \tfrac{w}{w'}) C^{w'}_{u,v}$$

Recursion involves only $C^{w'}_{u',v'}$ with $u' \in W^Q$ or $v' \in W^Q$ or $w' \in W^{Q,\max}$.

But: $(-\eta^2 D'_u - \eta D'_v - D'_w) = 0 \iff u^Q = v^Q = w^Q \in W^Q$

Here $u = u^Q u_Q$ is parabolic factorization: $u^Q \in W^Q$ and $u_Q \in W_Q$

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Use $D' = \sum_{\beta \in \Delta - \Delta_Q} \delta_\beta [X^{s_\beta}]$ divisor pulled back from Y !!

$$(-\eta^2 D'_u - \eta D'_v - D'_w) C^w_{u,v} = \eta^2 \sum_{u \to u'} (D', \tfrac{u'}{u}) C^w_{u',v} + \eta \sum_{v \to v'} (D', \tfrac{v'}{v}) C^w_{u,v'} + \sum_{w' \to w} (D', \tfrac{w}{w'}) C^{w'}_{u,v}$$

Recursion involves only $C^{w'}_{u',v'}$ with $u' \in W^Q$ or $v' \in W^Q$ or $w' \in W^{Q,\max}$.

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Here $u = u^Q u_Q$ is parabolic factorization: $u^Q \in W^Q$ and $u_Q \in W_Q$

Theorem: Let $u, v, w \in W^P$ satisfy $u^Q = v^Q = w^Q = \kappa \in W^Q$.

Then $C^w_{u,v}(X) = C^\kappa_{\kappa,\kappa}(Y)\, \kappa(C^{w_Q}_{u_Q,v_Q}(F))$ where $F = \pi^{-1}(1.Q) = Q/P$.

Example $\pi : \ X = \mathrm{Fl}(2,4,6;7) \ \longrightarrow \ Y = \mathrm{Gr}(4,7)$

$$F \ = \ \pi^{-1}(\mathrm{pt}) \ = \ \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,3)$$

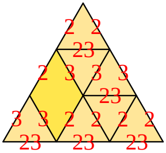
$u = 1301220$, $v = 1201320$, $w = 01\text{-}23\text{-}01\text{-}01\text{-}23\text{-}23\text{-}01$

$u_Q = 1010322$, $v_Q = 1010232$, $w_Q = 01\text{-}01\text{-}01\text{-}01\text{-}23\text{-}23\text{-}23$

$$C_{u,v}^{w}(X) \ = \ C_{w,w}^{w}(Y)\,\kappa(C_{u_Q,v_Q}^{w_Q}(F)) \ = \ C_{w,w}^{w}(Y)\,\kappa(C_{1010,1010}^{01\text{-}01\text{-}01\text{-}01})\,\kappa(C_{322,232}^{23\text{-}23\text{-}23})$$

where $\kappa = 1347256 \in S_7$

$C^{23\text{-}23\text{-}23}_{322,232} =$



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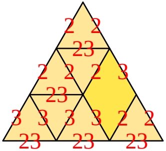
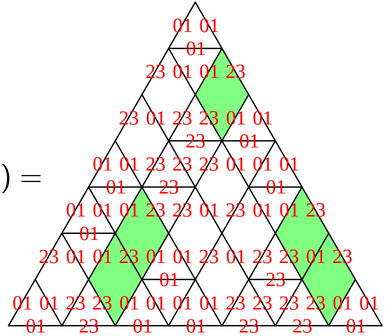
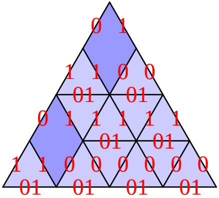


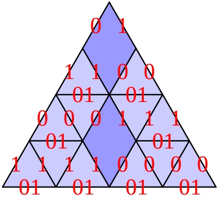
$C^{w}_{w,w}(Y) =$



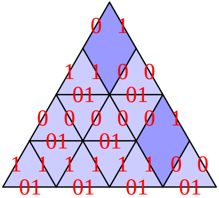
$C^{01\text{-}01\text{-}01\text{-}01}_{1010,1010} =$



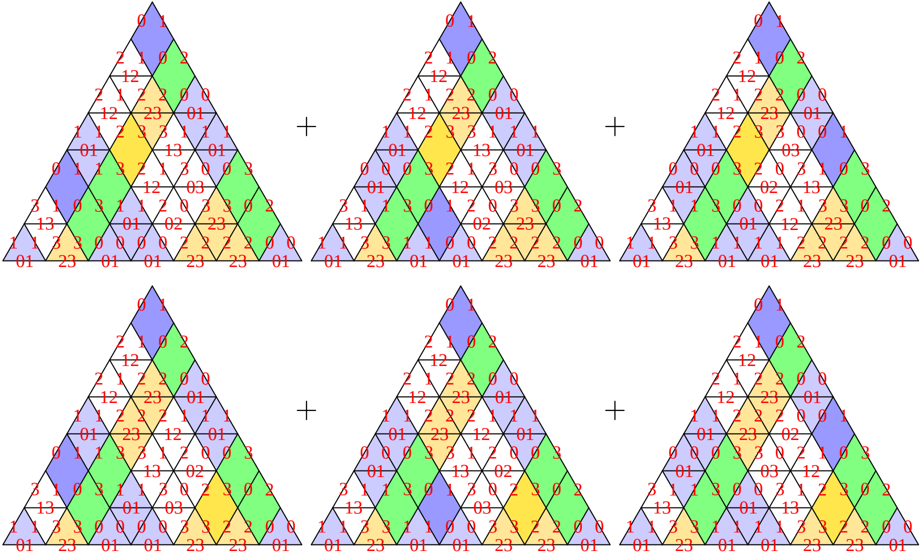
$+$



$+$



$$C^{w}_{u,v}(X) =$$


0 1
2 1 0 2
12
2 1 2 2 0 0
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