Classes of quiver cycles and quiver coefficients

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Let Q be a quiver with vertex set $\{1,2,\dots,n\}$, and let $e=(e_1,\dots,e_n)$ be a dimension vector for Q. Set $E_i = \mathbb{C}^{e_i}$ for each i. The affine space of quiver representations $V = \bigoplus_{i\to j} \operatorname{Hom}(E_i,E_j)$ has a natural conjugation action of the group $G = \prod_{i=1}^n \operatorname{GL}(E_i)$. A **quiver cycle** is any G-stable closed irreducible subvariety $\Omega \subset V$. For example, any G-orbit closure is a quiver cycle. A quiver cycle Ω determines a G-equivariant cohomology class $[\Omega] \in H^*_G(V)$ and a G-equivariant Grothendieck class $[\mathcal{O}_\Omega] \in K_G(V)$. Notice that

$$H^*_G(V) = H^*_G(\text{point}) = \mathbb{Z}[c_{i,j}]_{1\leq i\leq n \text{ and } 1\leq j\leq e_i}$$

is a polynomial ring, where the variables $c_{i,1}, c_{i,2}, \dots, c_{i,e_i}$ are the Chern classes of $\operatorname{GL}(E_i)$. The cohomology class $[\Omega] \in H^*_G(V)$ is a polynomial in these variables. The K-theory ring $K_G(V)$ can be identified with the Grothendieck ring $\operatorname{Rep}(G)$ of virtual representations of G.

The classes $[\Omega]$ and $[\mathcal{O}_\Omega]$ can be interpreted as formulas for degeneracy loci as follows. Let X be a non-singular variety and let $\mathcal{E}_\bullet$ be a representation of Q on vector bundles over X, i.e. a collection of vector bundles $\mathcal{E}_i$ corresponding to the vertices $i \in \{1,2,\dots,n\}$ together with vector bundle maps $\mathcal{E}_i \to \mathcal{E}_j$ corresponding to the arrows $i \to j$ of Q. Assume that $\operatorname{rank}(\mathcal{E}_i) = e_i$ for each i. For each point $x \in X$, the fiber $\mathcal{E}_\bullet(x)$ is representation of Q of dimension vector e. We define a degeneracy locus $\Omega(\mathcal{E}_\bullet) \subset X$ by

$$\Omega(\mathcal{E}_\bullet) = \{x \in X \mid \mathcal{E}_\bullet(x) \in \Omega\}\,.$$

This degeneracy locus has a natural structure of subscheme of X. Examples of degeneracy loci of this type include determinantal varieties and Schubert varieties in flag manifolds GL_m/P.

Proposition. *Assume that Ω is Cohen-Macaulay and that $\operatorname{codim}(\Omega(\mathcal{E}_\bullet);X) = \operatorname{codim}(\Omega;V)$. Assume also that X admits an ample line bundle. Then the (Chow) cohomology class $[\Omega(\mathcal{E}_\bullet)] \in H^*(X)$ is obtained from $[\Omega] \in H^*_G(V)$ by setting $c_{i,j} = c_j(\mathcal{E}_i)$ for all i,j.*

The simplest interesting example is when $Q = \{1 \to 2\}$ has two vertices and one arrow. In this case any quiver cycle is a G-orbit closure defined by

$$\Omega = \{\phi \in \operatorname{Hom}(E_1,E_2) \mid \operatorname{rank}(\phi) \leq r\}$$

for some non-negative integer r. To describe the class $[\Omega]$ we need the following notation. Given an integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0)$ we define the Schur polynomial

$$S_\lambda(E_2 - E_1) = \det\left(h_{\lambda_i+j-i}\right)_{\ell\times\ell} \in H^*_G(V)$$

where the classes h_i are determined by the identity of power series

$$\sum_{i\geq 0} h_i T^i := \frac{1 - c_{1,1}T + c_{1,2}T^2 - \cdots \pm c_{1,e_1}T^{e_1}}{1 - c_{2,1}T + c_{2,2}T^2 - \cdots \pm c_{2,e_2}T^{e_2}}\,.$$

The classical Thom-Porteous formula states that $[\Omega] = S_\lambda(E_2 - E_1)$ in $H^*_G(V)$ for the partition $\lambda = (e_1 - r)^{e_2 - r} = (e_1 - r, \dots, e_1 - r)$ consisting of $e_2 - r$ copies of $e_1 - r$. The Grothendieck class of Ω is given by the analogous formula $[\mathcal{O}_\Omega] = \mathcal{G}_\lambda(E_2 - E_1) \in K_G(V)$ where $\mathcal{G}_\lambda$ denotes a stable Grothendieck polynomial. This formula is proved in [5].

Let Q be a quiver without oriented cycles, and let $\Omega \subset V = \bigoplus_{i\to j} \operatorname{Hom}(E_i, E_j)$ be a quiver cycle. For each vertex i, let $M_i = \bigoplus_{j \to i} E_j$ be the sum of all vertex vector spaces mapping to E_i. For example, the quiver $Q = \{1 \rightrightarrows 2 \leftarrow 3\}$ gives $M_2 = E_1 \oplus E_1 \oplus E_3$. The length $\ell(\lambda)$ of a partition λ is the number of non-zero parts of λ, and its weight is the sum $|\lambda| = \sum \lambda_i$ of its parts.

Definition. The **cohomological quiver coefficients** of Ω are the unique integers $c_\mu(\Omega) \in \mathbb{Z}$, indexed by sequences $\mu = (\mu^1, \dots, \mu^n)$ of partitions μ^i with $\ell(\mu^i) \leq e_i$, such that

$$[\Omega] = \sum_\mu c_\mu(\Omega) \prod_{i=1}^n S_{\mu^i}(E_i - M_i) \in H^*_G(V)\,.$$

More generally, the K**-theoretic quiver coefficients** of Ω are given by

$$[\mathcal{O}_\Omega] = \sum_\mu c_\mu(\Omega) \prod_{i=1}^n \mathcal{G}_{\mu^i}(E_i - M_i) \in K_G(V)\,. \tag{1}$$

Since $H^*_G(V)$ is a graded ring, it follows that the cohomological quiver coefficients of Ω are indexed by sequences μ for which $|\mu| := \sum |\mu^i| = \operatorname{codim}(\Omega; V)$. These coefficients are a subset of the K-theoretic quiver coefficients, which are defined for sequences μ with $|\mu| \geq \operatorname{codim}(\Omega; V)$. The cohomological quiver coefficients for equioriented quivers of type A were introduce in [8]. This was extended to K-theory and more general quivers in [5, 7]. Examples of quiver coefficients include the Littlewood-Richardson coefficients, Stanley coefficients, the monomial coefficients of Schubert polynomials, and the analogous K-theoretic constants [10, 11].

Conjecture. *Let $\Omega \subset V$ be any quiver cycle.*

(a) *The cohomological quiver coefficients of Ω are non-negative, i.e. $c_\mu(\Omega) \geq 0$ for $|\mu| = \operatorname{codim}(\Omega; V)$.*

(b) *The K-theoretic coefficient $c_\mu(\Omega)$ is non-zero for only finitely many sequences μ, i.e. the sum (1) is finite.*

(c) *If Ω has rational singularities, then the K-theoretic quiver coefficients of Ω have alternating signs, i.e. $(-1)^{|\mu| - \operatorname{codim}(\Omega;V)}\, c_\mu(\Omega) \geq 0$.*

This conjecture is motivated in part by Schubert calculus on flag varieties G/P. If $Y \subset G/P$ is any closed irreducible subvariety, then the cohomology class $[Y] \in H^*(G/P)$ can be uniquely written as a linear combination of Schubert classes, and the coefficients in this combination are non-negative integers. Furthermore, a result of Brion states that if Y has rational singularities, then its Grothendieck class $[\mathcal{O}_Y] \in K(G/P)$ is a linear combination of Schubert structure sheaves with alternating signs [3].

The Conjecture is known when $Q = \{1 \to 2 \to \cdots \to n\}$ is an equioriented quiver of type A. Special cases of (a) were proved Buch, Kresch, Tamvakis, and Yong [4, 10] after which the general case was proved by Knutson, Miller, and Shimozono [14]. Part (b) was proved by Buch [5], and part (c) was proved by Buch [6] and by Miller [17].

Now suppose that Q is a quiver of Dynkin type. In this case Fehér and Rimányi have given a set of linear equations that uniquely determine the cohomology class $[\Omega] \in H_G^*(V)$ [13]. These equations simply say that the restriction of $[\Omega]$ to any disjoint G-orbit in V is zero. Reineke has given an explicit resolution of the singularities of Ω [18]. Under the assumption that Ω has rational singularities, this resolution has been used to prove formulas for the K-theory class $[\mathcal{O}_\Omega]$, by Knutson and Shimozono [15] and by Buch [7]. The latter paper expresses the class $[\mathcal{O}_\Omega]$ in terms of quiver coefficients and proves part (b) of the conjecture, as well as part (c) when Q is of type A_3. All quiver cycles of Dynkin type A or D are known to have rational singularities by results of Bobiński and Zwara [1, 2] (see also [19, 16] for the case of equioriented quivers of type A).

We refer to [14, 9, 6, 12] for a different type of positivity of quiver cycle classes, which has been proved for the cohomology class of any quiver cycle of type A and for the K-theory class of equioriented quiver cycles of type A.

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