Curve Neighborhoods of Schubert varieties

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(joint work with Leonardo C. Mihalcea)

1. The main result

The title of this talk refers to a recent paper [4] with Mihalcea, but my talk is also closely related to joint work with Chaput, Mihalcea, and Perrin [2].

Let X be a non-singular complex variety, let $\Omega \subset X$ be a closed subvariety, and let $d \in H_2(X) = H_2(X;\mathbb{Z})$ be a degree. The *curve neighborhood* $\Gamma_d(\Omega)$ is defined as the closure of the union of all rational curves in X of degree d that meet Ω. For example, if $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\Omega = \mathbb{P}^1 \times \{0\}$, then $H_2(X) = \mathbb{Z} \oplus \mathbb{Z}$, and we have $\Gamma_{(1,0)}(\Omega) = \Omega$ and $\Gamma_{(0,1)}(\Omega) = X$.

I will focus on the case where $X = G/P$ is a generalized flag variety, defined by a semisimple complex Lie group G and a parabolic subgroup P. I also fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. In this case it was proved in [2] that, if Ω is irreducible, then $\Gamma_d(\Omega)$ is irreducible. Notice also that $\Gamma_d(\Omega)$ is B-stable whenever Ω is B-stable. It follows that if Ω is a Schubert variety in X, then $\Gamma_d(\Omega)$ is also a Schubert variety.

It is natural to ask which Schubert variety this is. In other words, if we know the Weyl group element representing Ω, then what is the Weyl group element representing $\Gamma_d(\Omega)$? This question is related to several aspects of the quantum cohomology and quantum K-theory of homogeneous spaces, including two-point Gromov-Witten invariants, the (equivariant) quantum Chevalley formula [6, 7], the minimal powers of the deformation parameter q that appear in quantum products of Schubert classes [6], and a degree distance formula for cominuscule varieties [5] that played an important role in a generalization of the kernel-span technique from [1] and the quantum equals classical theorem from [3].

Let $W = N_G(T)/T$ be the Weyl group of G and let $W_P = N_P(T)/T \subset W$ be the Weyl group of P. We let $W^P \subset W$ denote the subset of minimal length representatives for the cosets in W/W_P. Each element $w \in W$ defines a Schubert variety $X(w) = \overline{Bw.P} \subset X$; if $w \in W^P$ then $\dim X(w) = \ell(w)$. The set of T-fixed points in X is $X^T = \{w.P \mid w \in W^P\}$. We let R be the root system of G, with positive roots R^+ and simple roots $\Delta \subset R^+$.

We describe the curve neighborhood of a Schubert variety in terms of the *Hecke product* of Weyl group elements, which can be defined as follows. For $w \in W$ and $\beta \in \Delta$ we set

$$w \cdot s_\beta = \begin{cases} w\, s_\beta & \text{if } \ell(ws_\beta) > \ell(w); \\ w & \text{if } \ell(ws_\beta) < \ell(w). \end{cases}$$

Given an additional element $w' \in W$ and a reduced expression $w' = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_\ell}$, we then define $w \cdot w' = w \cdot s_{\beta_1} \cdot s_{\beta_2} \cdot \ldots \cdot s_{\beta_\ell} \in W$, where the simple reflections are Hecke-multiplied to w in left to right order. This defines an associative monoid product on W. The Hecke product is compatible with the Bruhat order on W, for example we have $v \leq v' \Rightarrow u \cdot v \cdot w \leq u \cdot v' \cdot w$ for all $u, v, v', w \in W$.

Given a positive root $\alpha \in R^+$ with $s_\alpha \notin W_P$, let $C_\alpha \subset X$ be the unique T-stable curve that contains the points $1.P$ and $s_\alpha.P$. The main result of [4] is the following theorem, which makes it straightforward to compute the Weyl group element representing the curve neighborhood $\Gamma_d(X(w))$.

Theorem 1. *Assume that $0 < d \in H_2(X)$, and let $\alpha \in R^+$ be any positive root that is maximal with the property that $[C_\alpha] \leq d \in H_2(X)$. Then we have $\Gamma_d(X(w)) = \Gamma_{d-[C_\alpha]}(X(w \cdot s_\alpha))$.*

We remark that the homology group $H_2(X)$ can be identified with the coroot lattice of R modulo the coroots corresponding to P, in such a way that the class $[C_\alpha] \in H_2(X)$ is the image of the coroot $\alpha^\vee$. Theorem 1 therefore makes simultaneous use of the orderings of roots and coroots, which gives rise to interesting combinatorics.

2. Degree distance formula

Theorem 1 can be used to give simple proofs of several well known results concerning the quantum cohomology of generalized flag varieties. Here we will sketch a proof of the degree distance formula for cominuscule varieties due to Chaput, Manivel, and Perrin [5].

Assume that $X = G/P$ where P is a maximal parabolic subgroup of G, and let $\gamma \in \Delta$ be the unique simple root such that $s_\gamma \notin W_P$. Then $H_2(X) = \mathbb{Z}$ has rank one, and the generator $[X(s_\gamma)] \in H_2(X)$ can be identified with $1 \in \mathbb{Z}$. The variety X is called *cominuscule* if, when the highest root $\rho \in R^+$ is expressed as a linear combination of simple roots, the coefficient of γ is one. This implies that $\rho = w_P.\gamma$ where w_P denotes the longest element in W_P. In particular, since $\rho^\vee - \gamma^\vee$ is a linear combination of the coroots of P, we obtain $[C_\rho] = [C_\gamma] = 1 \in H_2(X)$. Given any effective degree $d \in H_2(X)$, it therefore follows from Theorem 1 that

$$\Gamma_d(X(w)) = \Gamma_{d-1}(X(w \cdot s_\gamma)) = \cdots = X(w \cdot s_\gamma \cdot s_\gamma \cdot \ldots \cdot s_\gamma)$$

where s_γ is repeated d times. Since $s_\rho = w_P s_\gamma w_P$, this identity is equivalent to the expression

$$\Gamma_d(X(w)) = X(w \cdot w_P s_\gamma \cdot w_P s_\gamma \cdot \ldots \cdot w_P s_\gamma)\,, \tag{1}$$

with $w_P s_\gamma$ repeated d times.

Given two points $x, y \in X$, let $d(x, y)$ denote the smallest possible degree of a rational curve in X from x to y. This number is determined by the following result from [5].

Corollary (Chaput, Manivel, Perrin)**.** *Let $u \in W^P$. Then $d(1.P, u.P)$ is the number of occurrences of s_γ in any reduced expression for u.*

Proof. For $d \in H_2(X)$, it follows from (1) that $u.P \in \Gamma_d(X(1))$ if and only if u has a reduced expression with at most d occurrences of s_γ. Now set $d = d(1.P, u.P)$ and observe that $u.P \in \Gamma_d(X(1)) \smallsetminus \Gamma_{d-1}(X(1))$. We deduce that u has a reduced expression with exactly d occurrences of of s_γ. The corollary now follows from Stembridge's result [8] that u is fully commutative, i.e. any reduced expression for u can be obtained from any other by interchanging commuting simple reflections. $\square$

References

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