Giambelli formulas for orthogonal Grassmannians

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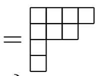
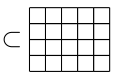
Type A: $X = \mathrm{Gr}(m, N) = \{V \subset \mathbb{C}^N \mid \dim(V) = m\}$

$B = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \subset \mathrm{GL}(N)$ acts on X

Schubert variety = B-orbit closure

$\leftrightarrow$ Partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0) =$  $\subset$  $m \times (N-m)$

$X_\lambda = \{V \in X \mid \dim(V \cap \mathbb{C}^{p_j}) \geq j \ \forall 1 \leq j \leq m\}$

$p_j = N - m + j - \lambda_j$

$\mathrm{codim}(X_\lambda) = |\lambda| = \sum \lambda_i$

$H^*(X;\mathbb{Z}) = \bigoplus_\lambda \mathbb{Z} \cdot [X_\lambda]$

Schubert calculus: $[X_\lambda] \cdot [X_\mu] = \sum_\nu c^\nu_{\lambda\mu} [X_\nu]$

say: applications to enumerative geometry

$c^\nu_{\lambda\mu}$ = Littlewood-Richardson coef.

say: Rep. theory of $\mathrm{GL}(m)$, sym. fcns., eigenvalues, etc. LR rule.

$0 \to S \to \mathbb{C}^N_X \to Q \to 0$

$c_p := c_p(Q) = [X_p] = [X_{\square\square\square\square}]$

$H^*(X;\mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_{N-m}]$ / (relations)

Pieri rule: $c_p \cdot [X_\lambda] = \sum_\mu [X_\mu]$

$\mu = \lambda +$ horiz strip of p boxes = PICTURE

Giambelli formula: $[X_\lambda] = DETERMINANT = \det\left[c_{\lambda_i + j - i}\right]_{m \times m} \in H^*(X)$

Set $c_0 = 1$ and $c_p = 0$ for $p < 0$

Compute LR coefs: $[X_\lambda] \cdot [X_\mu] = \det(c_{\lambda_i+j-i}) \cdot [X_\mu] = \sum_\nu c^\nu_{\lambda,\mu}[X_\nu]$ – compute with Pieri rule.

Orthogonal Grassmannians.

Orthogonal form $(-,-)$ on $\mathbb{C}^N$ def. by $(e_i, e_j) = \delta_{i+j,N+1}$

$Y = \mathrm{OG}(m, N) = \{V \subset \mathbb{C}^N \mid \dim(V) = m \text{ and } (V, V) = 0 \text{ (isotropic) }\}$

Write $N = 2m + K$, so $Y = \mathrm{OG}(m, 2m + K)$

Def: $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$ is K**-strict** if no part $> K/2$ is repeated.

Example: $\lambda = (6, 5, 3, 2, 2, 1, 1, 1)$ is 4-strict.

Assume $\lambda \subset m \times (N - m - 1)$

$Y_\lambda = \{V \in Y \mid \dim(V \cap \mathbb{C}^{p_j}) \geq j \ \forall 1 \leq j \leq m\}$

$p_j = N - m + j - \lambda_j - \#\{i \leq j \mid \lambda_i + \lambda_j \geq K + j - i \text{ and } \lambda_i > K/2\}$.

N odd: $Y = \mathrm{SO}(N)/P$ of Lie type B. $\{Y_\lambda\}$ = set of Schubert varieties.

N even: $Y = \mathrm{SO}(N)/P$ of type D.

$K/2 \notin \lambda$: Y_λ is a Schubert variety

$K/2 \notin \lambda$: $Y_\lambda = Y'_\lambda \cup Y''_\lambda$ union of two Schubert varieties

$Y^{(i)}_\lambda = \{V \in Y_\lambda \mid \dim(V \cap \widetilde{\mathbb{C}}^{N/2}) > \ell_K(\lambda)\}$

$\ell_K(\lambda) = \#\{j : \lambda_j > K/2\}$

$$\widetilde{\mathbb{C}}^{N/2} = \begin{cases} (*, \ldots, *, *, 0, 0, \ldots, 0) & \text{if } N/2 + \ell_K(\lambda) + i \text{ is even} \\ (*, \ldots, *, 0, *, 0, \ldots, 0) & \text{if } N/2 + \ell_K(\lambda) + i \text{ is odd} \end{cases}$$

DYNKIN DIAGRAM, involution ι, $\iota(Y_\lambda) = Y_\lambda$, $\iota(Y'_\lambda) = Y''_\lambda$

Say: Schubert varieties in even OG behave like roots of real polynomial!

$H^*(Y;\mathbb{Z})$ has $\mathbb{Z}$-basis $\{[Y_\lambda] : K/2 \notin \lambda\} \cup \{[Y'_\lambda], [Y''_\lambda] : K/2 \in \lambda\}$

Say: Combinatorics of Schubert basis for even OG is **special case soup**!

$$c_p = c_p(Q) = \begin{cases} [Y_p] & \text{if } p \leq K/2 \\ 2[Y_p] & \text{if } p > K/2 \end{cases}$$

N odd: $H^*(Y;\mathbb{Q}) = \mathbb{Q}[c_1, c_2, \ldots, c_L] \;/\; \text{(relations)}$, $L := N - m - 1$.

N even: $H^*(Y;\mathbb{Q}) = \mathbb{Q}[c_1, \ldots, c'_{K/2}, c''_{K/2}, \ldots, c_L] \;/\; \text{(relations)}$

$c'_{K/2} = [Y'_{K/2}]$, $c''_{K/2} = [Y''_{K/2}]$, $c_{K/2} = c'_{K/2} + c''_{K/2}$

Max orthogonal Grassmannian.

Assume $K = 1$, $Y = \mathrm{OG}(m, 2m+1)$

Schub varieties Y_λ given by strict $\lambda = (\lambda_1 > \cdots > \lambda_\ell > 0)$ with $\lambda_1 \leq m$.

Pieri rule (Hiller and Boe): $c_p \cdot [Y_\lambda] = \sum_\mu 2^{N(\lambda,\mu)} [Y_\mu]$

sum over **strict** $\mu = \lambda +$ horiz strip of p boxes

$N(\lambda, \mu) = \#$ components of μ/λ

Giambelli formula: Set $\sigma_\lambda = 2^{\ell_K(\lambda)} [Y_\lambda] \in H^*(Y)$

$\sigma_{a,b} = c_a c_b + 2 \sum_{j \geq 1} (-1)^j c_{a+j} c_{b-j}$

Schur: $\sigma_\lambda = \text{Pfaffian} \left[\sigma_{\lambda_i,\lambda_j}\right]_{m \times m} = \sqrt{\det \left[\sigma_{\lambda_i,\lambda_j}\right]}$

Other extreme.

$Y = \mathrm{OG}(m, 2m+K)$, $K \gg \lambda_1$: Behaves like type A.

Raising operators.

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$

Def: $R_{ij} \alpha = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots, \alpha_\ell)$

Set $c_\alpha = \prod_i c_{\alpha_i}$

If R monomial in R_{ij}'s, set $R c_\alpha = c_{R\alpha}$ ACTS ON INDICES!!!

Type A ($K \gg \lambda_1$) : $[Y_\lambda] = \det \left[c_{\lambda_i + j - i}\right]_{m \times m} = \left(\prod_{i<j} (1 - R_{ij})\right) c_\lambda$

Max OG ($K = 1$) : $[Y_\lambda] = 2^{-\ell_K(\lambda)} \sqrt{\det \left[\sigma_{\lambda_i,\lambda_j}\right]} = 2^{-\ell_K(\lambda)} \left(\prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}}\right) c_\lambda$

Def: For λ K-strict, set $R^\lambda_K = \prod_{\lambda_i + \lambda_j < K + j - i} (1 - R_{ij}) \prod_{\lambda_i + \lambda_j \geq K + j - i} \frac{1 - R_{ij}}{1 + R_{ij}}$

Thm (BKT): $[Y_\lambda] = 2^{-\ell_K(\lambda)} R^\lambda_K \, c_\lambda \in H^* \, \mathrm{OG}(m, 2m+K)$

Say: Thm also true in QH – no q correction terms.

Even orthogonal Grassmannians.

Assume N even.

$\iota : H^*(Y;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ involution.

$H^*(Y;\mathbb{Q}) = H^*(Y;\mathbb{Q})_1 \oplus H^*(Y;\mathbb{Q})_{-1}$

Prop: $H^*(Y;\mathbb{Q})_1 = \mathbb{Q}[c_1, \ldots, c_L] = \bigoplus_\lambda \mathbb{Q} \cdot [Y_\lambda]$

Set $\Delta_\lambda = \begin{cases} [Y'_\lambda] - [Y''_\lambda] & \text{if } K/2 \in \lambda \\ 0 & \text{else} \end{cases}$

Set $\overline{Y} = \mathrm{OG}(m-1, N-1)$

Note: $\mathbb{Q}[c_1, \ldots, c_L]$ acts on both $H^*(Y)$ and $H^*(\overline{Y})$.

Def: $\lambda + K/2 := (\lambda_1, \ldots, \lambda_j, K/2, \lambda_{j+1}, \ldots, \lambda_\ell)$ where $j = \ell_K(\lambda)$

Prop: The linear map $\phi : H^*(\overline{Y};\mathbb{Q}) \to H^*(Y;\mathbb{Q})_{-1}$ defined by $[\overline{Y}_\mu] \mapsto \Delta_{\mu+K/2}$ is an isomorphism of $\mathbb{Q}[c_1, \ldots, c_L]$-modules !!!!

$H^*(X;\mathbb{Q}) = \mathbb{Q}[c_1, \ldots, c_L] \cdot \Delta_{K/2} = \bigoplus_{\lambda \ni K/2} \mathbb{Q} \cdot \Delta_\lambda$

$\Delta_{K/2} = c'_{K/2} - c''_{K/2}$

General Giambelli formula.

Assume $\lambda = \mu + K/2$.
$[Y'_\lambda] = \frac{1}{2}[Y_\lambda] + \frac{1}{2}\Delta_\lambda = \frac{1}{2}[Y_\lambda] + \frac{1}{2}\phi([\overline{Y}_\mu])$
$= 2^{-\ell_K(\lambda)-1}\left(R^\lambda_K c_\lambda + \phi(R^\mu_{K+1} c_\mu)\right)$
$= 2^{-\ell_K(\lambda)-1}\left(R^\lambda_K c_\lambda + \Delta_{K/2}\, R^\mu_{K+1} c_\mu\right)$

Remarks.

1) Giambelli formula also true in quantum cohomology – no q corrections.

2) The Giambelli formula implies a new construction of Billey-Haiman polynomials for Grassmannian Weyl group elements, in terms of raising operators.