Mutations of Puzzles and Equivariant cohomology of two-step flag varieties

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Two-step flag varieties

Fix $0 \leq a \leq b \leq n$.

$$X = \mathrm{Fl}(a, b; n) = \{(A, B) \mid A \subset B \subset \mathbb{C}^n ; \dim(A) = a ; \dim(B) = b\}$$

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Def: A **012-string** for X is a permutation of $0^a\, 1^{b-a}\, 2^{n-b}$.

$\mathbb{C}^n$ has basis $\{e_1, e_2, \ldots, e_n\}$. $\quad u = (u_1, u_2, \ldots, u_n)$ 012-string.

Def. $(A_u, B_u) \in X$ by $A_u = \mathrm{Span}\{e_i : u_i = 0\}$ and $B_u = \mathrm{Span}\{e_i : u_i \leq 1\}$.

Example: $X = \mathrm{Fl}(1, 3; 5)$. $u = 10212$. $(A_u, B_u) = (\mathbb{C}e_2\,,\, \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_4)$.

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Schubert variety: $\quad X_u = \overline{\mathbf{B}.(A_u, B_u)}\quad ;\quad \mathbf{B} \subset \mathrm{GL}(\mathbb{C}^n)$ lower triangular.

$\mathrm{codim}(X_u, X) \;=\; \ell(u) \;=\; \#\{i < j \mid u_i > u_j\}$

Equivariant cohomology

$T \subset \mathrm{GL}(\mathbb{C}^n)$ maximal torus of diagonal matrices.

$H_T^*(\text{point}) = \mathbb{Z}[y_1, \ldots, y_n]$, where $y_i = -c_1(\mathbb{C}e_i)$.

$$H_T^*(X) = \bigoplus_u \mathbb{Z}[y_1, \ldots, y_n] \cdot [X_u]$$ is an algebra over $H_T^*(\text{point})$.

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The **equivariant Schubert structure constants** of X are the polynomials $C_{u,v}^w \in \mathbb{Z}[y_1, \dots, y_n]$ defined by

$$[X_u] \cdot [X_v] = \sum_w C_{u,v}^w \, [X_w]$$

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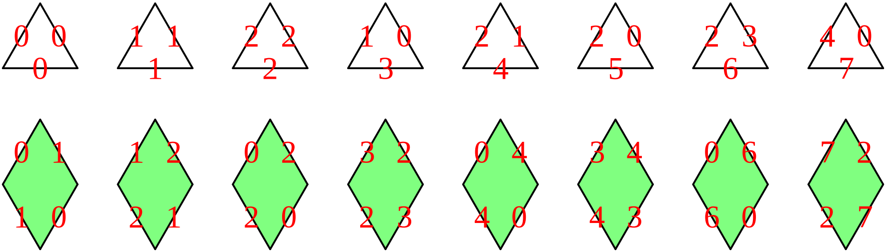
$H_T^*(X)$ graded ring $\Rightarrow$ $C_{u,v}^w$ homogeneous of degree $\ell(u) + \ell(v) - \ell(w)$.

$\ell(w) = \ell(u) + \ell(v) \;\Rightarrow\; C_{u,v}^w = \#(\, g_1.X_u \cap g_2.X_v \cap g_3.X_{w^\vee} \,)$; $g_i \in \mathrm{GL}(\mathbb{C}^n)$.

Theorem (Graham) $\quad C_{u,v}^w \;\in\; \mathbb{Z}_{\geq 0}[y_2 - y_1, \ldots, y_n - y_{n-1}]$

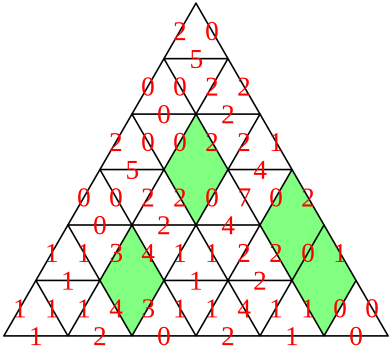
Puzzle pieces



Simple labels: 0 , 1 , 2

Composed labels: $3 = 10$, $4 = 21$, $5 = 20$, $6 = 2(10)$, $7 = (21)0$

Equivariant puzzles



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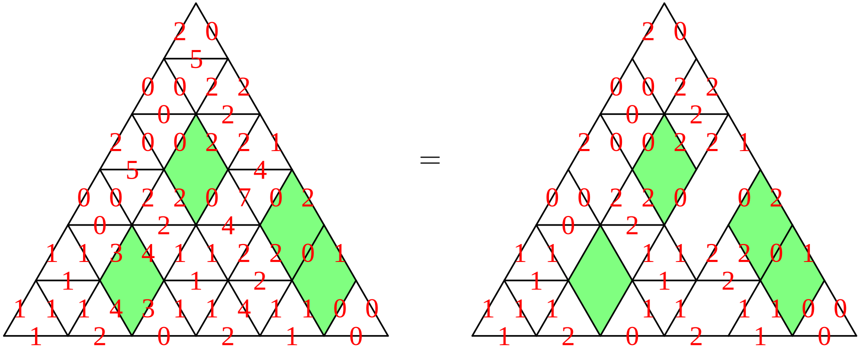
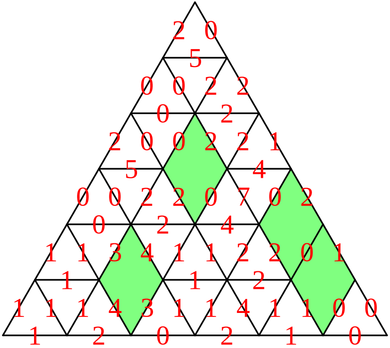


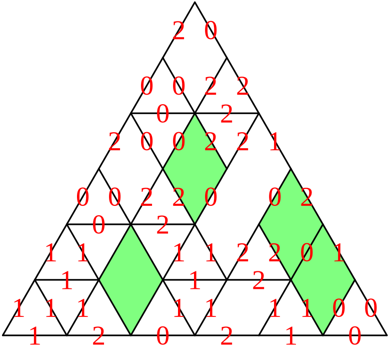
Note: The composed labels are uniquely determined by the simple labels.

Equivariant puzzles

$P =$



$=$



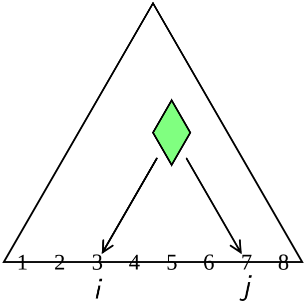
Note: The composed labels are uniquely determined by the simple labels.

Boundary: $\partial P = \triangle_w^{u,v}$ where $u = 110202$, $v = 021210$, $w = 120210$.

Equivariant puzzle formula

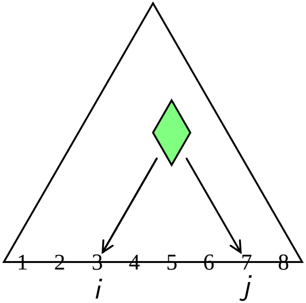
Theorem

$$C_{u,v}^{w} \;=\; \sum_{\partial P = \triangle_{w}^{u,v}} \; \prod_{\lozenge \in P} \text{weight}(\lozenge)$$



$$\text{weight}(\lozenge) = y_j - y_i$$

Equivariant puzzle formula



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Theorem

$$C^{w}_{u,v} = \sum_{\partial P = \triangle^{u,v}_{w}} \prod_{\lozenge \in P} \text{weight}(\lozenge)$$

Known cases:

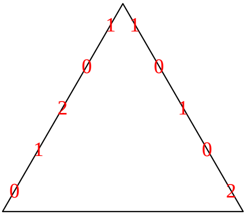
Puzzle rule for $H^*(\mathrm{Gr}(m,n))$ (Knutson, Tao, Woodward)

Puzzle rule for $H^*_T(\mathrm{Gr}(m,n))$ (Knutson, Tao)

Puzzle rule for $H^*(\mathrm{Fl}(a,b;n))$ (conjectured by Knutson,
proof in [B-Kresch-Purbhoo-Tamvakis] ,
different positive formula by Coskun.)

Example: Let $X = \mathrm{Fl}(2,4;5)$. In $H_T^*(X)$ we have:

$[X_{01201}] \cdot [X_{10102}] \; = \; \textbf{?}$



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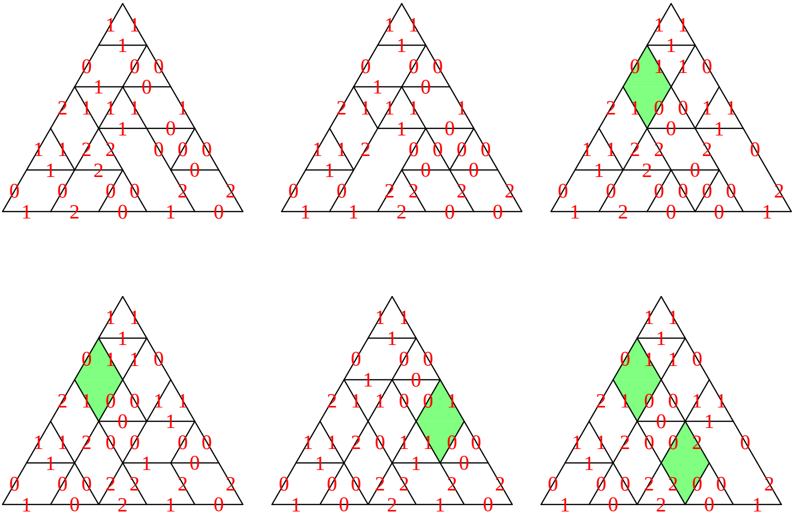

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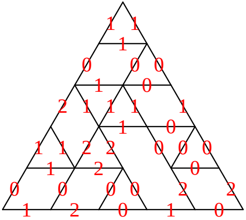
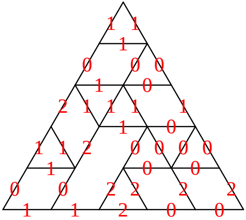


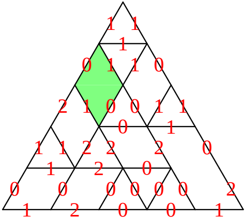
$[X_{12010}]$

$+$

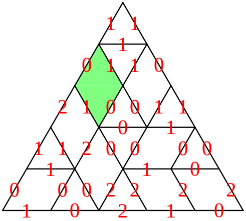


$[X_{11200}]$

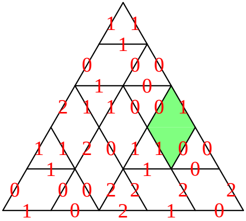
$+$

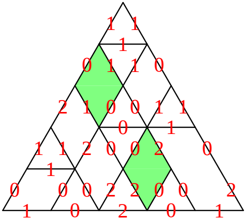


$(y_4 - y_1)[X_{12001}]$



$+$ $(y_5 + y_4 - y_3 - y_1)[X_{10210}]$





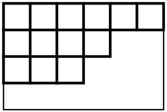
$+$ $(y_4 - y_3)(y_4 - y_1)[X_{10201}]$

Quantum cohomology of Grassmannians

$$X = \mathrm{Gr}(m,n) = \{V \subset \mathbb{C}^n \mid \dim(V) = m\} = \mathrm{Fl}(m,m;n)$$

Schubert varieties $\longleftrightarrow$ **02-strings** $\longleftrightarrow$ **Young diagrams**

$X_{0222020220}$ $\longleftrightarrow$ 0222020220 $\longleftrightarrow$



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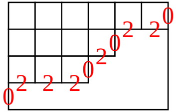

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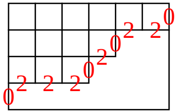

(Small) equivariant quantum ring:

$$QH_T(X) \;=\; H_T^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q] \;=\; \bigoplus_{\lambda} \mathbb{Z}[y_1,\dots,y_n,q] \cdot [X_\lambda]$$

Ring structure is defined by **equivariant Gromov-Witten invariants** $N^{\nu,d}_{\lambda,\mu} \in \mathbb{Z}[y_1,\dots,y_n]$:

$$[X_\lambda] \star [X_\mu] = \sum_{\nu,\, d \geq 0} N^{\nu,d}_{\lambda,\mu}\, q^d\, [X_\nu]$$

Gromov-Witten invariants of $X = \mathrm{Gr}(m, n)$

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$N^{\nu,0}_{\lambda,\mu} = C^{\nu}_{\lambda,\mu}$ ($QH_T(X)$ is a deformation of $H^*_T(X)$.)

$N^{\nu,d}_{\lambda,\mu} \in \mathbb{Z}[y_1, \ldots, y_n]$ is homogeneous of degree $|\lambda| + |\mu| - |\nu| - nd$.

$|\lambda| + |\mu| = |\nu| + nd \quad \Rightarrow$

$N^{\nu,d}_{\lambda,\mu} = \#$ rational curves $C \subset X$ of degree d meeting $g_1.X_\lambda$, $g_2.X_\mu$, $g_3.X_{\nu^\vee}$.

Thm (Mihalcea) $N^{\nu,d}_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}[y_2 - y_1, \ldots, y_n - y_{n-1}]$

Quantum equals classical theorem

Def: (B) Given curve $C \subset X = \mathrm{Gr}(m, n)$ set

$$\mathrm{Ker}(C) = \bigcap_{V \in C} V \subset \mathbb{C}^n \quad \text{and} \quad \mathrm{Span}(C) = \sum_{V \in C} V \subset \mathbb{C}^n$$

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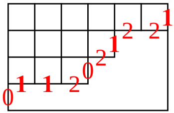
Fix degree d. Set $Y = \mathrm{Fl}(m-d, m+d; n)$.

Given a 02-string λ for X, let $\lambda(d)$ be the 012-string for Y obtained from λ by replacing the first d occurrences of 2 and the last d occurrences of 0 with 1.

$\lambda =$0**22**202**0**22**0** and $d = 2$ gives $\lambda(d) =$ 0**11**202**1**22**1**.



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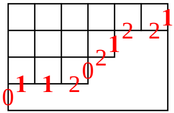

$$Y_{\lambda(d)} = \{(A, B) \in Y \mid \exists V \in X_\lambda : A \subset V \subset B\}$$

$= $ Set of Kernel-Span pairs of general curves of degree d meeting X_λ.

Quantum equals classical theorem

Theorem (B–Kresch–Tamvakis) For $|\lambda| + |\mu| = |\nu| + nd$ we have bijection

$$\left\{\begin{matrix}\text{rational curves in } X \\ \text{of degree } d \text{ meeting} \\ g_1.X_\lambda,\ g_2.X_\mu,\ g_3.X_\nu\end{matrix}\right\} \longleftrightarrow g_1.Y_{\lambda(d)} \cap g_2.Y_{\mu(d)} \cap g_3.Y_{\nu(d)}$$

$$C \longmapsto (\mathrm{Ker}(C), \mathrm{Span}(C))$$

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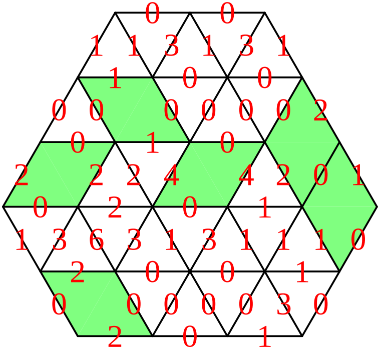
$$C \quad \longmapsto \quad (\mathrm{Ker}(C)\,,\ \mathrm{Span}(C))$$

Theorem (B-Mihalcea) $\quad N^{\nu^\vee,d}_{\lambda,\mu} \;=\; C^{\nu(d)^\vee}_{\lambda(d),\mu(d)} \quad \in \; \mathbb{Z}[y_1,\dots,y_n]$

Corollary: $\quad N^{\nu^\vee,d}_{\lambda,\mu} \;=\; \displaystyle\sum_{\partial P = \triangle^{\lambda(d),\mu(d)}_{\nu(d)^\vee}} \;\prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$

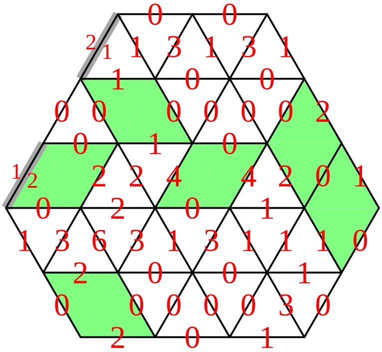
The mutation algorithm



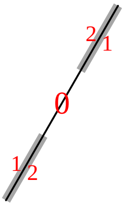
Puzzle:

- Shape is a hexagon.
- All pieces may be rotated.
- Boundary labels are simple.

The mutation algorithm



Flawed puzzle containing the **gash pair**:



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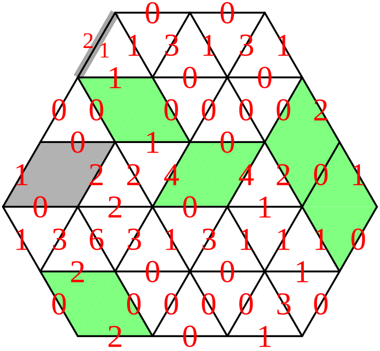
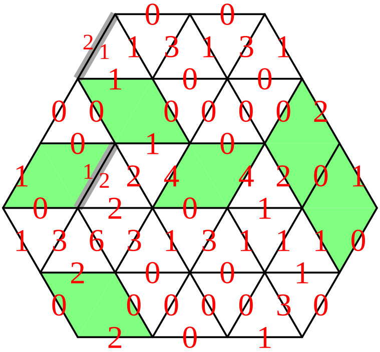
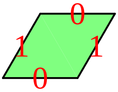


Remove problematic piece.

The mutation algorithm



Replace with:



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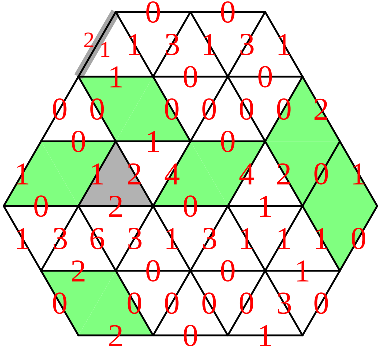
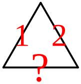
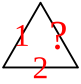
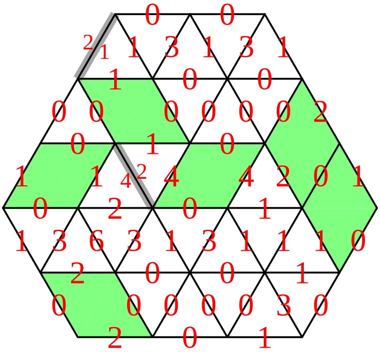


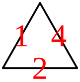
Replace with 

OR 

?

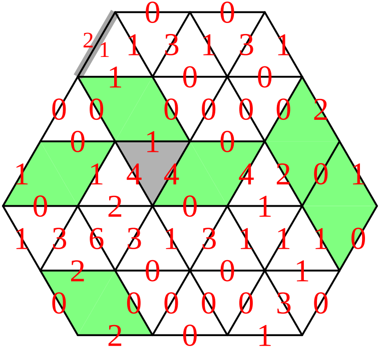
The mutation algorithm



The piece 

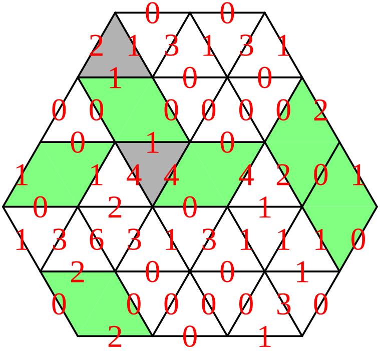
fits. Always at most one choice !!!

The mutation algorithm



But no puzzle piece fits this time.

The mutation algorithm



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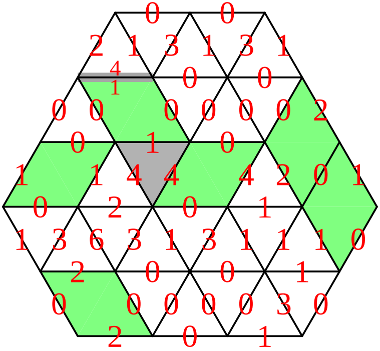

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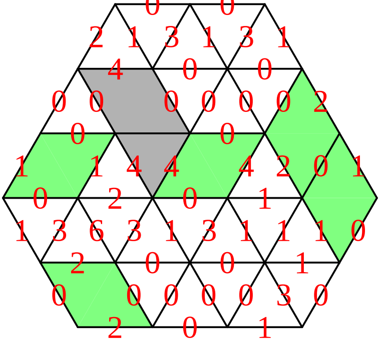

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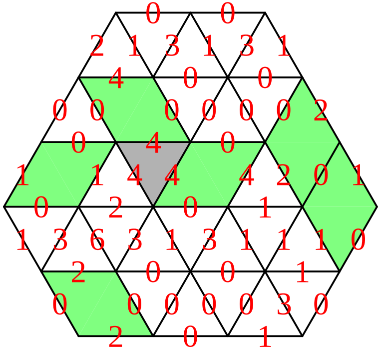

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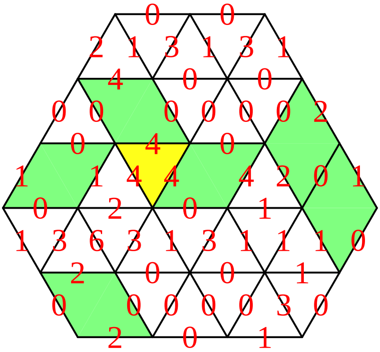
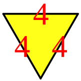
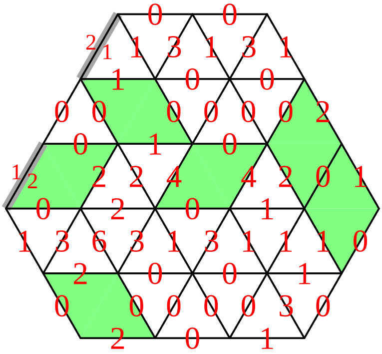
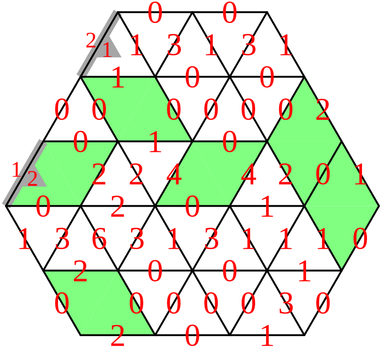


Flawed puzzle containing the **illegal puzzle piece**: 

The mutation algorithm

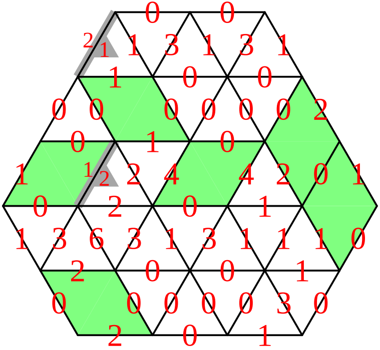


The mutation algorithm

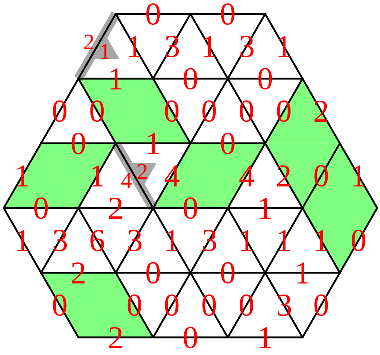


Use **directed gashes**.

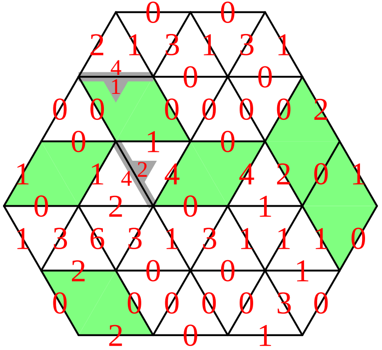
The mutation algorithm



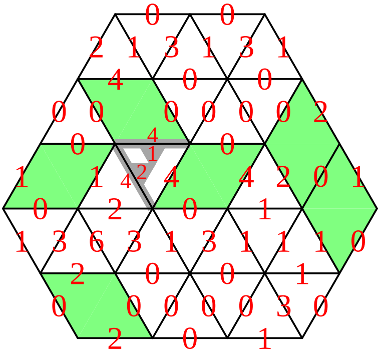
The mutation algorithm



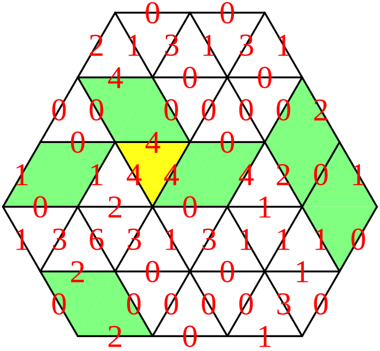
The mutation algorithm



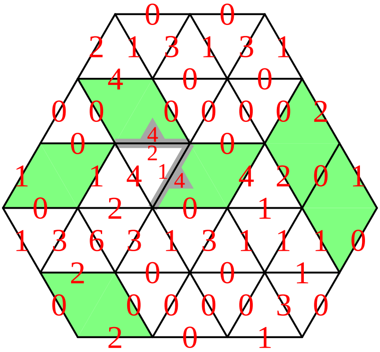
The mutation algorithm



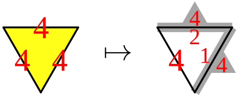
The mutation algorithm



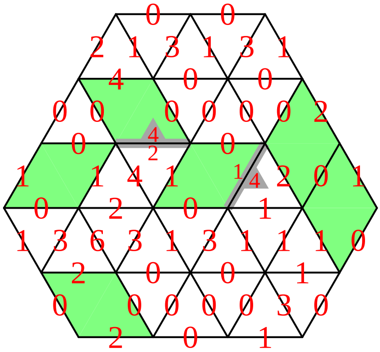
The mutation algorithm



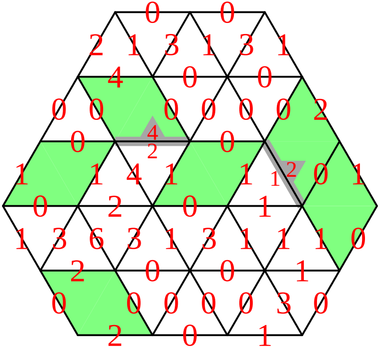
Resolution:



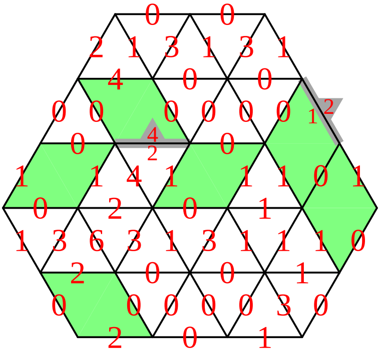
The mutation algorithm



The mutation algorithm



The mutation algorithm



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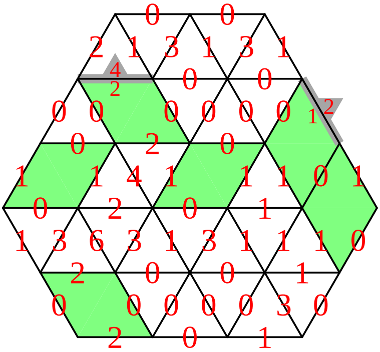
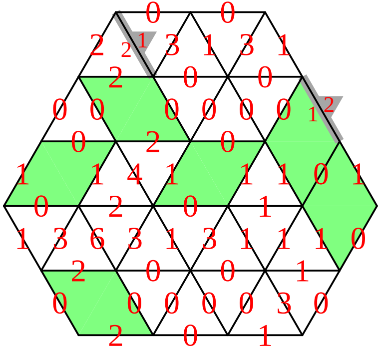
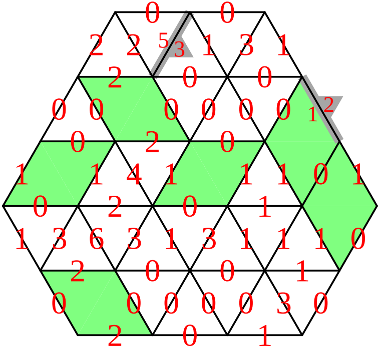


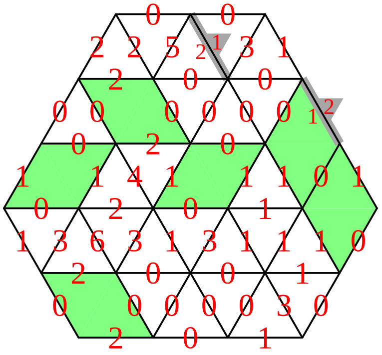
The mutation algorithm



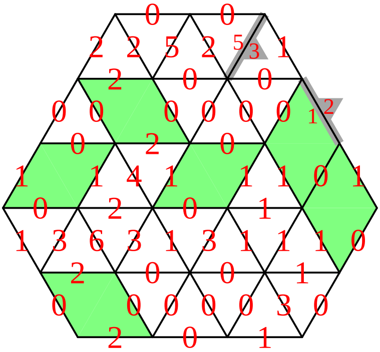
The mutation algorithm



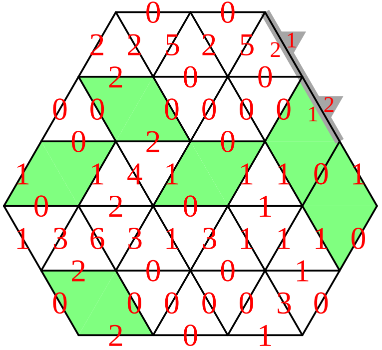
The mutation algorithm



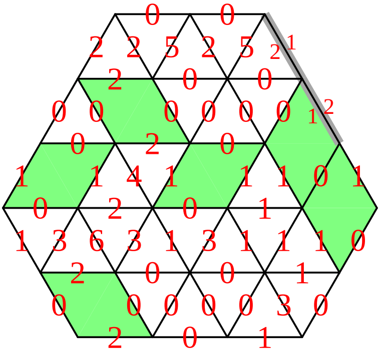
The mutation algorithm



The mutation algorithm

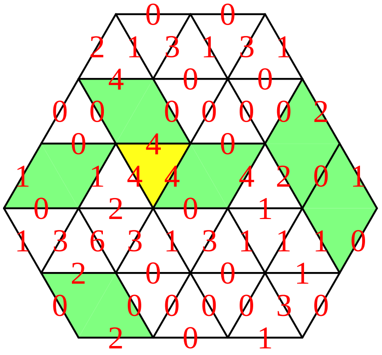


The mutation algorithm

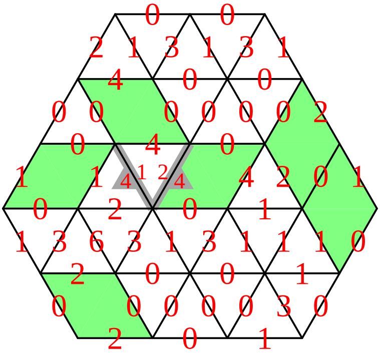


Flawed puzzle containing a gash pair.

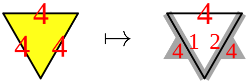
The mutation algorithm



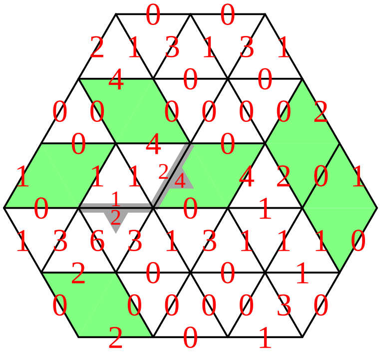
The mutation algorithm



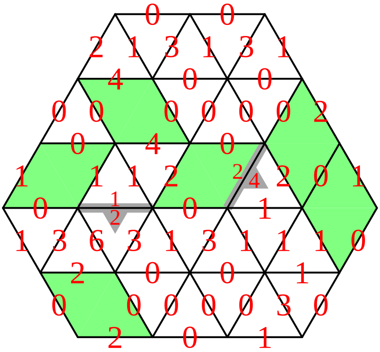
Resolution:



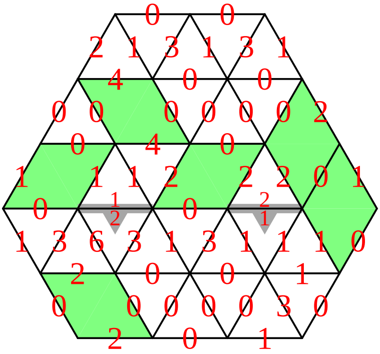
The mutation algorithm



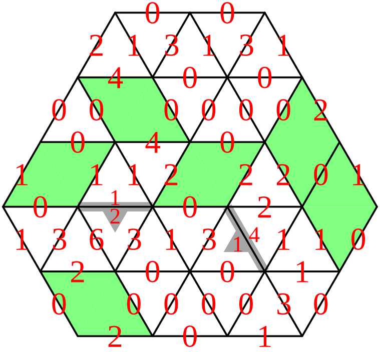
The mutation algorithm



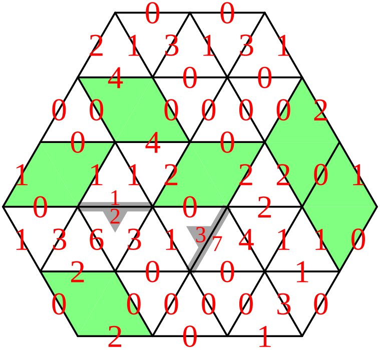
The mutation algorithm



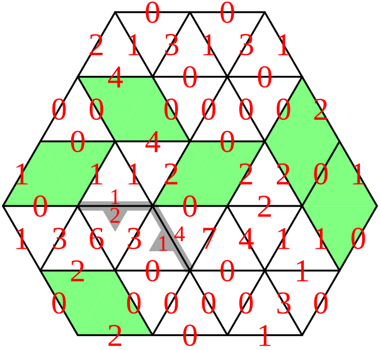
The mutation algorithm



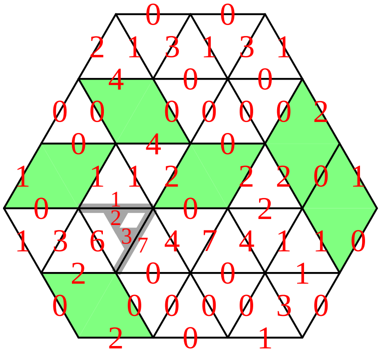
The mutation algorithm



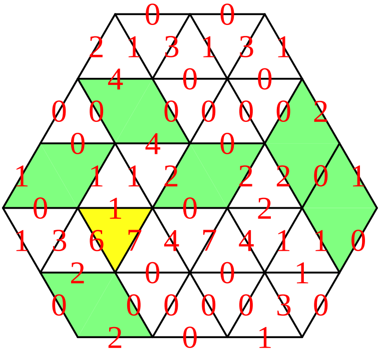
The mutation algorithm

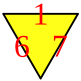


The mutation algorithm

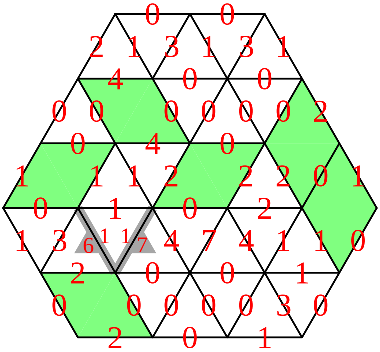


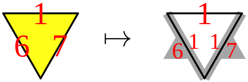
The mutation algorithm



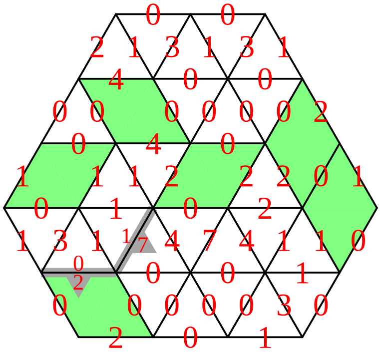
Flawed puzzle containing the **illegal puzzle piece**: 

The mutation algorithm

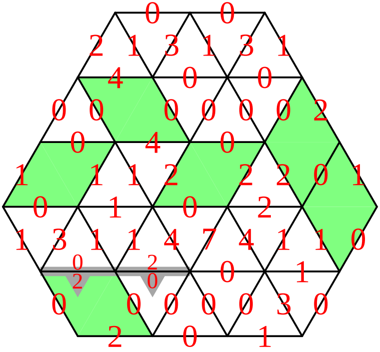


Resolution: 

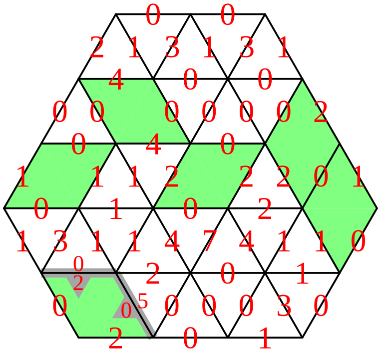
The mutation algorithm



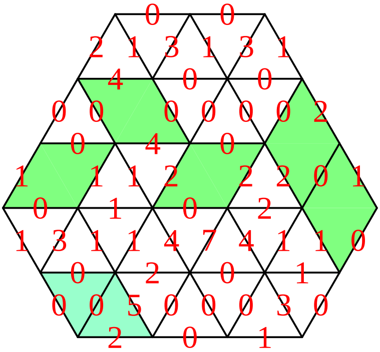
The mutation algorithm



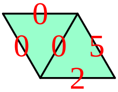
The mutation algorithm



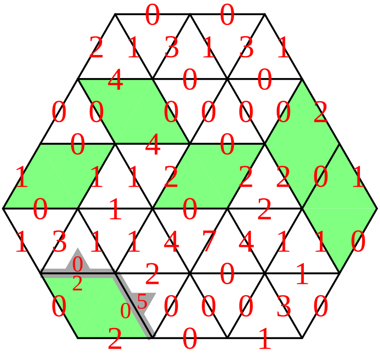
The mutation algorithm



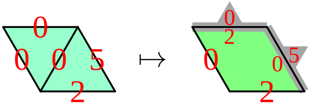
Flawed puzzle containing the **marked scab**:



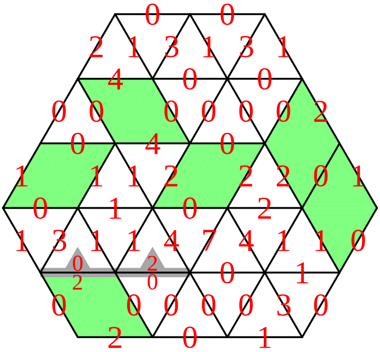
The mutation algorithm



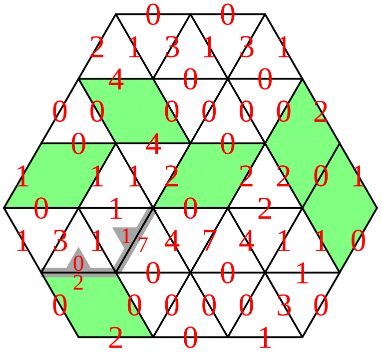
Resolution:



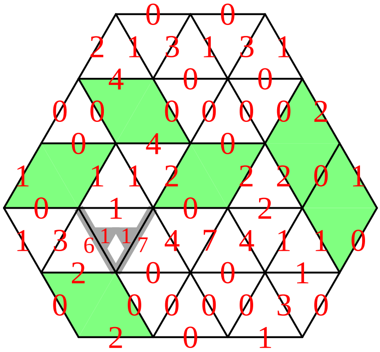
The mutation algorithm



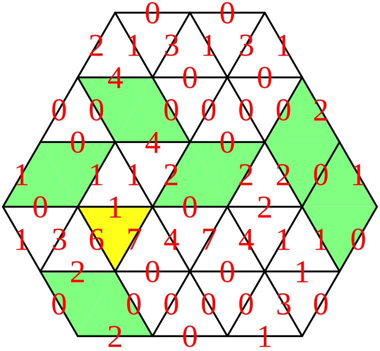
The mutation algorithm



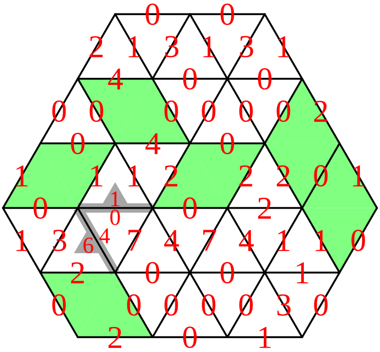
The mutation algorithm



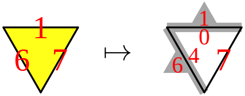
The mutation algorithm



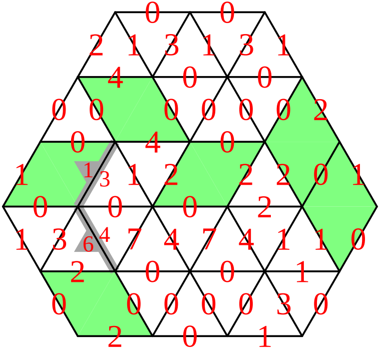
The mutation algorithm



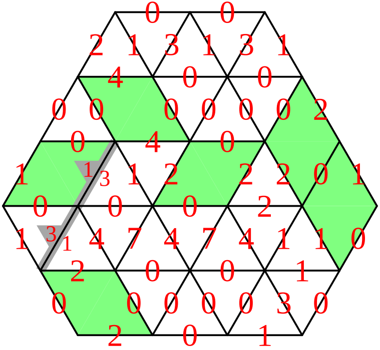
Resolution:



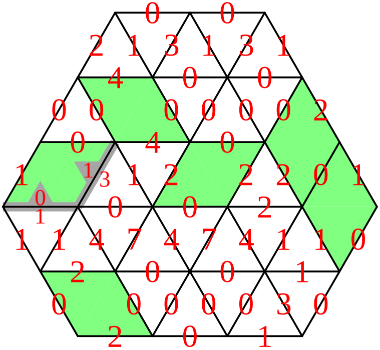
The mutation algorithm



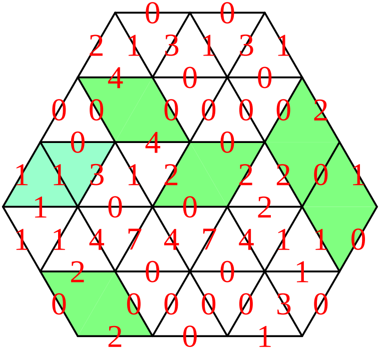
The mutation algorithm



The mutation algorithm

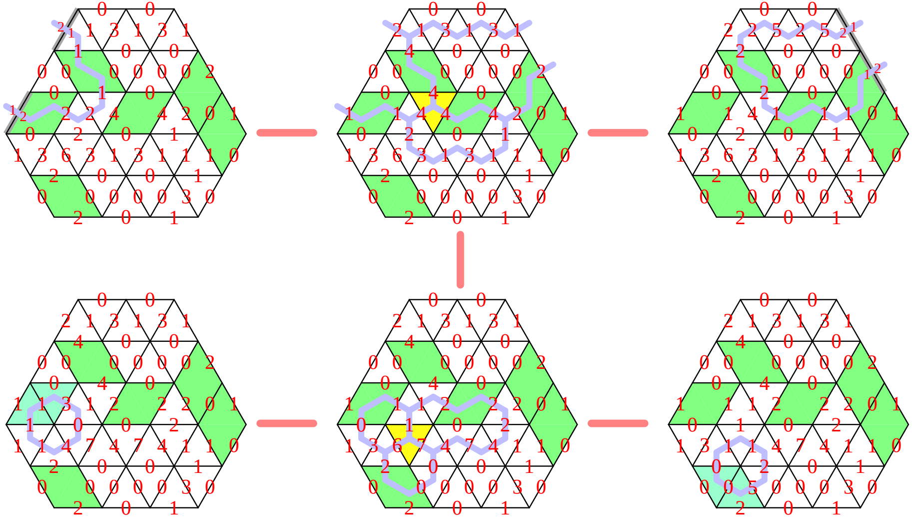


The mutation algorithm

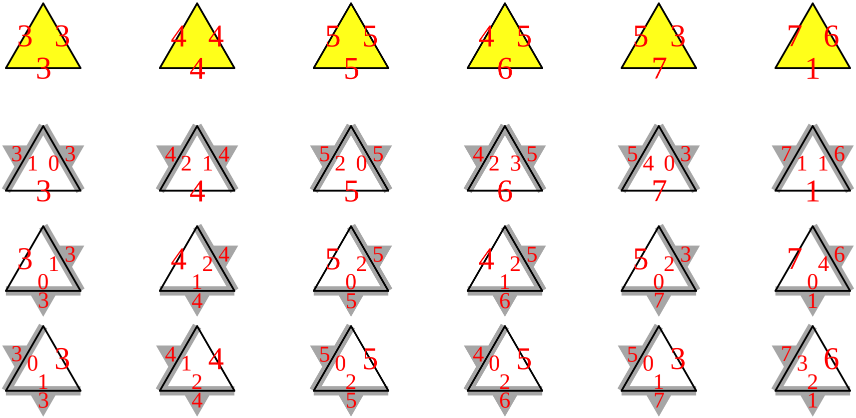


Flawed puzzle containing a marked scab.

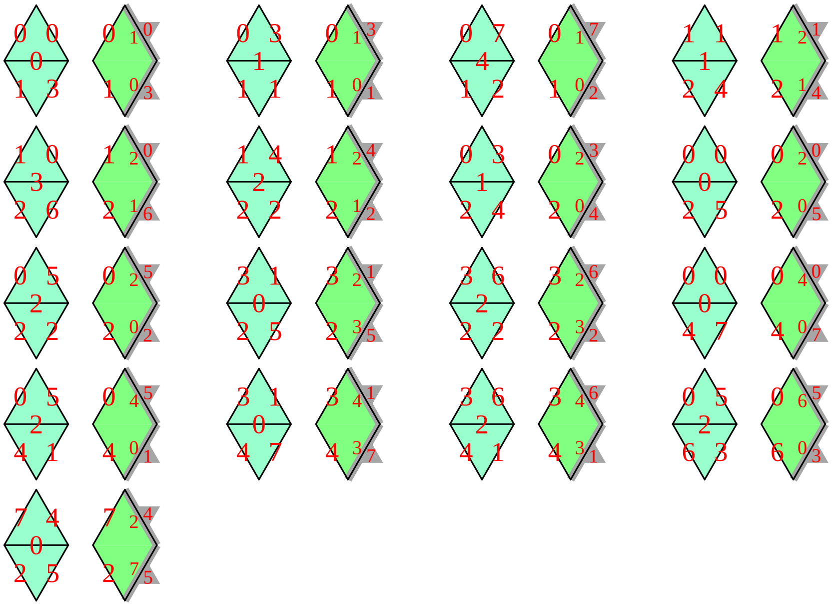
Component of the mutation graph:



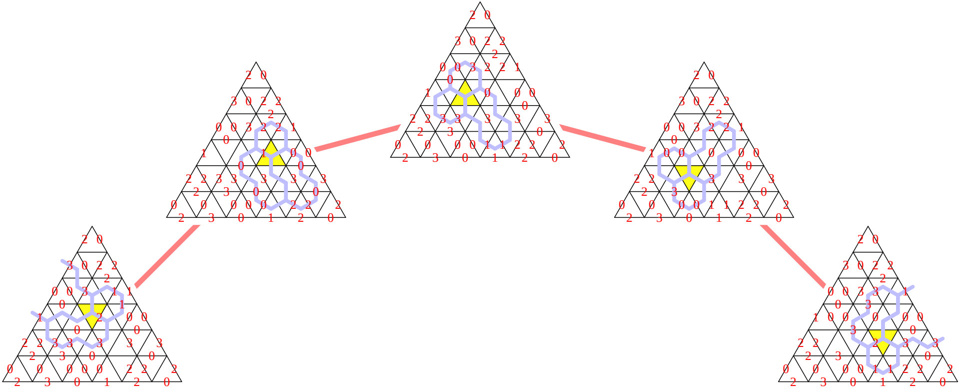
Resolutions of illegal puzzle pieces:

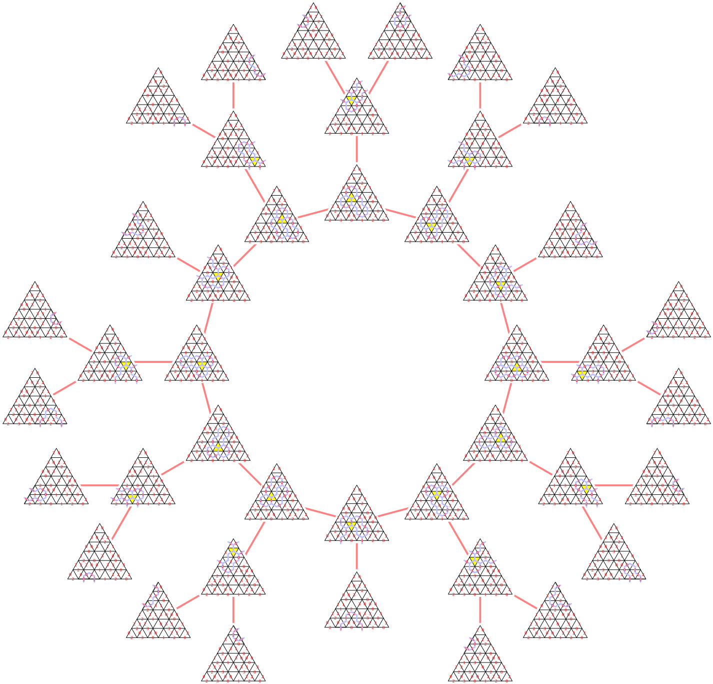


Resolutions of marked scabs:

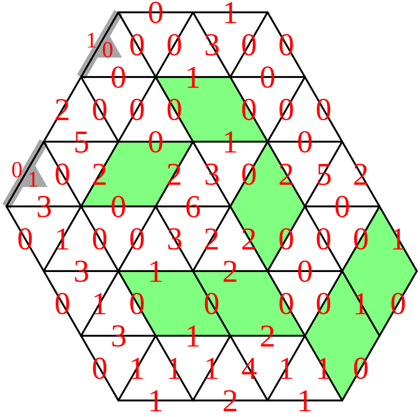


Example:

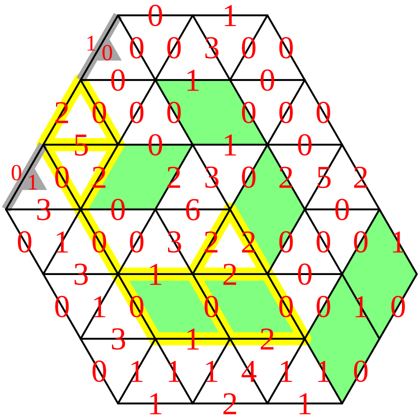




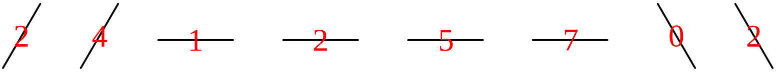
Proof that mutation algorithm works:



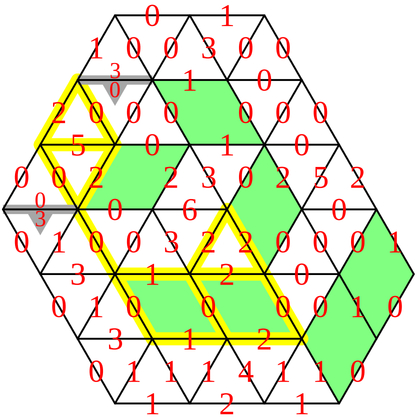
Proof that mutation algorithm works:



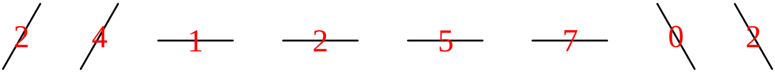
Consider connected component of the edges:



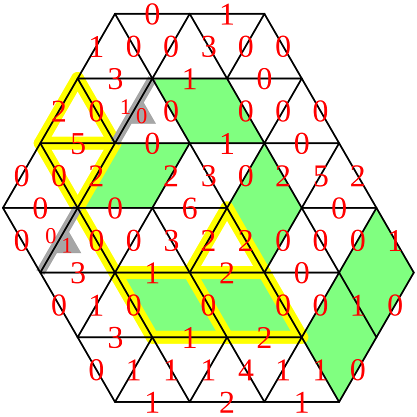
Proof that mutation algorithm works:



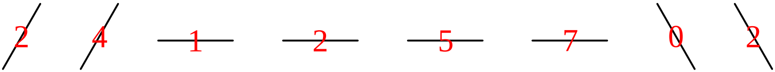
Consider connected component of the edges:



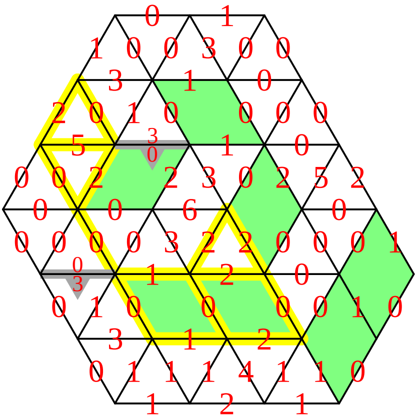
Proof that mutation algorithm works:



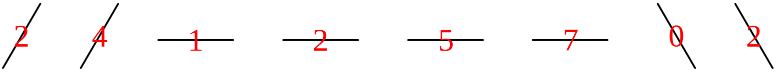
Consider connected component of the edges:



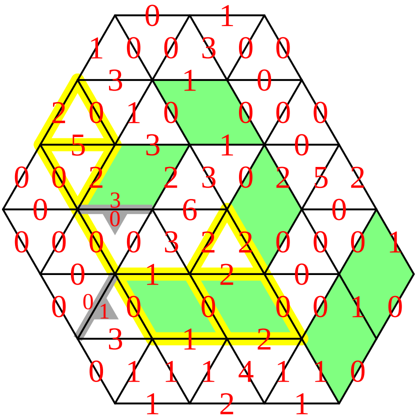
Proof that mutation algorithm works:



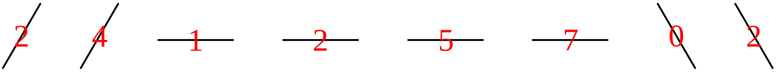
Consider connected component of the edges:



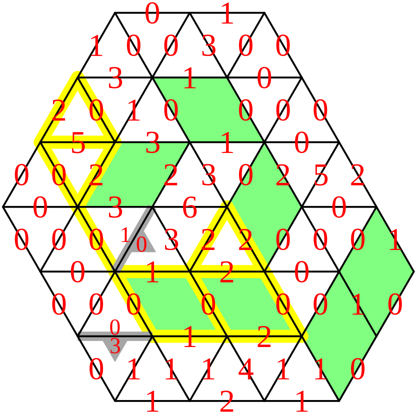
Proof that mutation algorithm works:



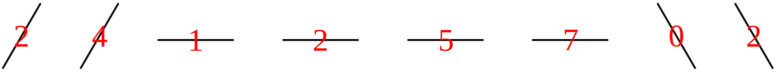
Consider connected component of the edges:



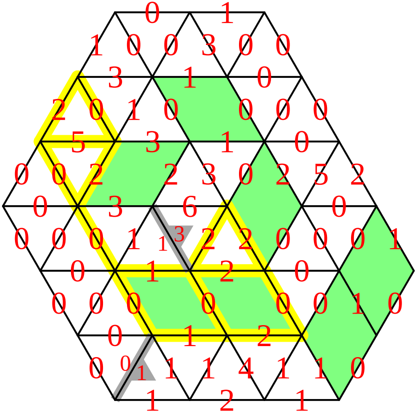
Proof that mutation algorithm works:



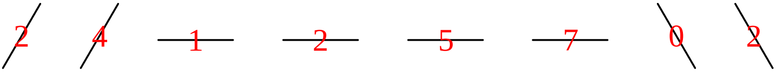
Consider connected component of the edges:



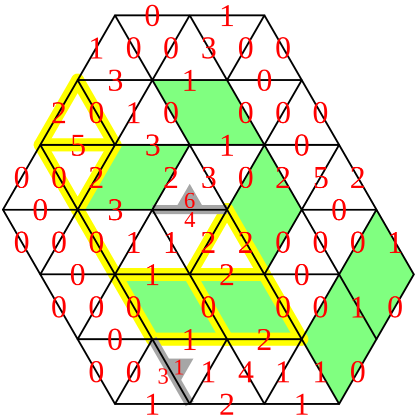
Proof that mutation algorithm works:



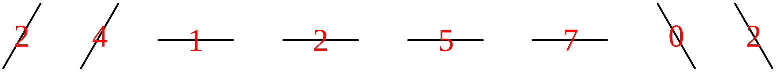
Consider connected component of the edges:



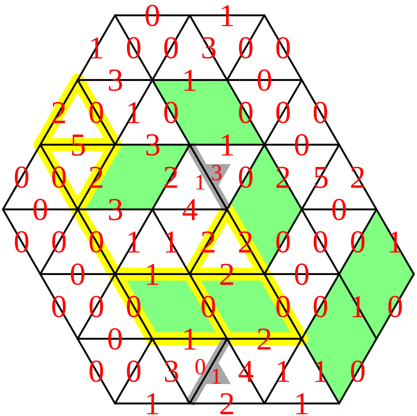
Proof that mutation algorithm works:



Consider connected component of the edges:



Proof that mutation algorithm works:



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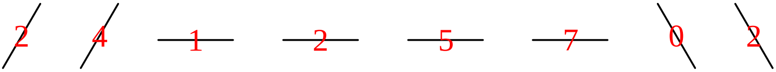

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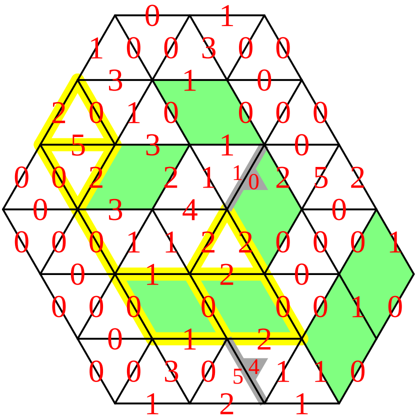

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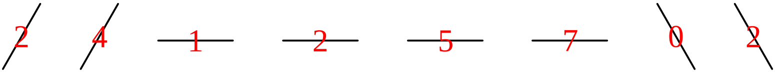

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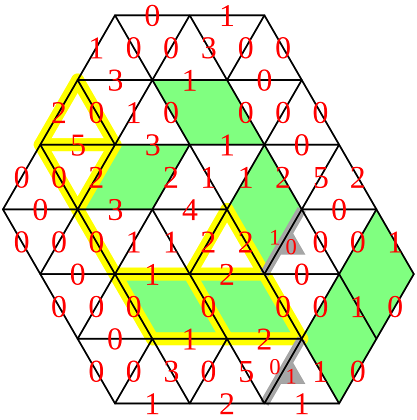

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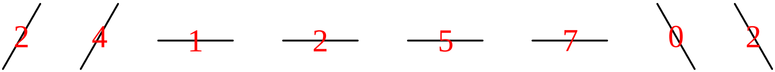

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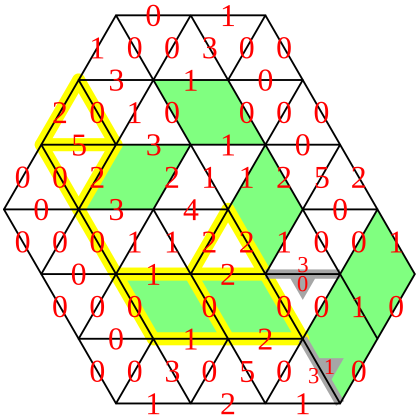

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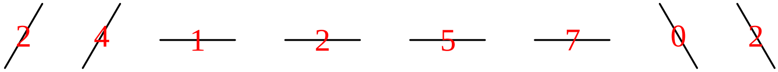

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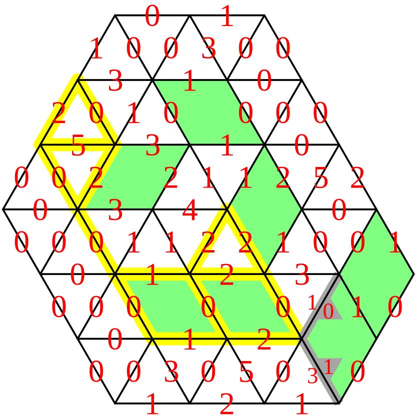

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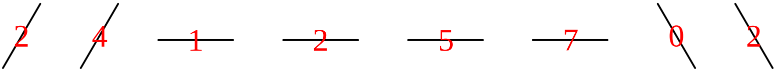

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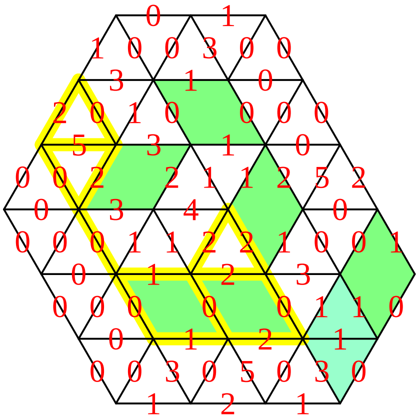

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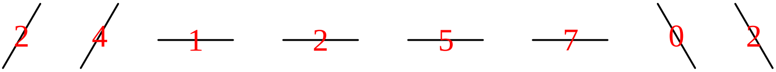

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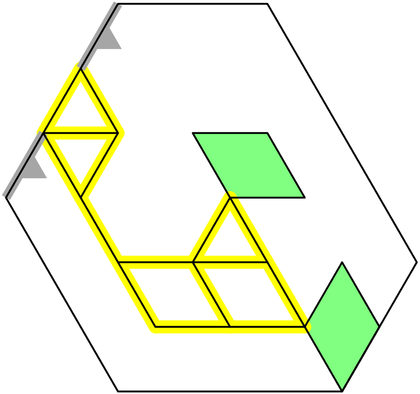

Technical result: The two gashes will propagate to the same location.

In particular, the above situation is impossible !!

Aura of semi-labeled edges

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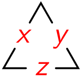
Def: $\mathcal{A}(\overset{0}{\text{—}}) = \delta_0 \uparrow$ $\qquad \mathcal{A}(\overset{1}{\text{—}}) = \delta_1 \uparrow$ $\qquad \mathcal{A}(\overset{2}{\text{—}}) = \delta_2 \uparrow$

If 

is a puzzle piece, then $\mathcal{A}(/_{x}) + \mathcal{A}({}_{y}\backslash) + \mathcal{A}(\overset{z}{\text{—}}) = 0$.

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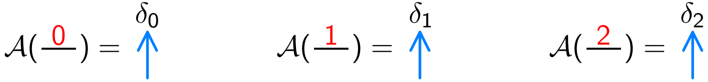

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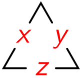

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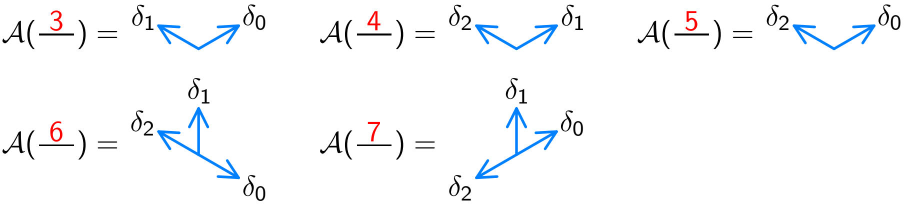
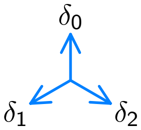


Aura of gashes

Definition: $\mathcal{A}\left(\frac{x}{y}\right) = \mathcal{A}\left(\frac{x}{\ }\right) + \mathcal{A}\left(\frac{\ }{y}\right)$

Example: $\mathcal{A}\left(\frac{0}{4}\right) = \mathcal{A}\left(\frac{0}{\ }\right) + \mathcal{A}\left(\frac{\ }{4}\right) =$



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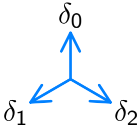
Example: $\mathcal{A}(\frac{0}{4}) = \mathcal{A}(\frac{0}{\ }) + \mathcal{A}(\frac{\ }{4}) =$ 

Properties:

- The aura of a gash is invariant under propagations.

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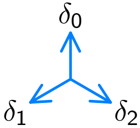
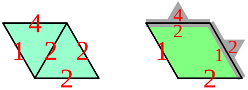


Properties:

- The aura of a gash is invariant under propagations.
- Sum of auras of gashes of any resolution is zero.



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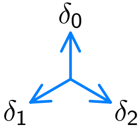

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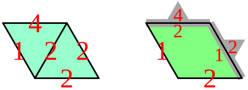
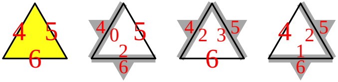


$$\mathcal{A}(\frac{4}{2}) + \mathcal{A}({}_1\backslash{}^2) = 0$$

- Sum of auras of right gashes of resolutions of illegal puzzle piece is zero.

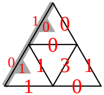


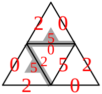
$$\mathcal{A}({}^4/{}_0) + \mathcal{A}({}_3\backslash{}^5) + \mathcal{A}(\frac{1}{6}) = 0$$

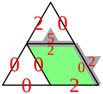
Aura of puzzles

Let $\widetilde{P}$ be a resolution of a flawed puzzle P.

Def: $\mathcal{A}(\widetilde{P}) = \mathcal{A}($ right gash in $\widetilde{P}$ $)$

$\mathcal{A}($  $) = \mathcal{A}($  $)$

$\mathcal{A}($  $) = \mathcal{A}($  $)$

$\mathcal{A}($  $) = \mathcal{A}($  $)$

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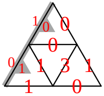
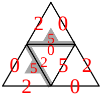
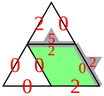
$\mathcal{A}($  $) = \mathcal{A}($ 

$)$ $\quad$ $\mathcal{A}($  $) = \mathcal{A}($ 

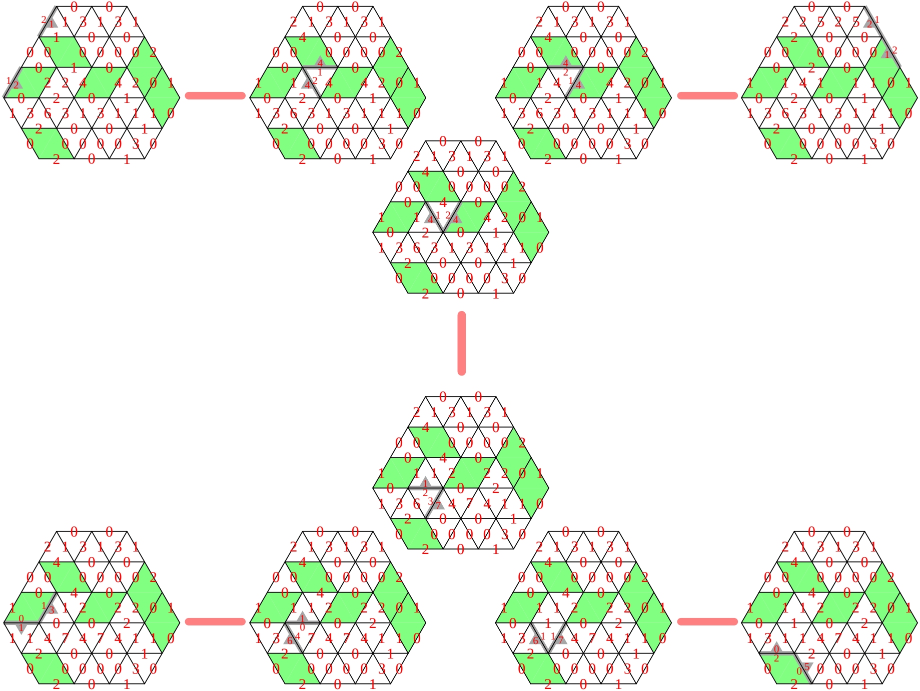
$)$ $\quad$ $\mathcal{A}($  $) = \mathcal{A}($ 

$)$

If $\widetilde{P}$ is the only resolution of P, then set $\mathcal{A}(P) = \mathcal{A}(\widetilde{P})$.

Key identity: Let S be any finite set of flawed puzzles that is closed under mutations. Then

$$\sum_{P \in S_{\mathrm{scab}}} \mathcal{A}(P) \ + \sum_{P \in S_{\mathrm{gash}}} \mathcal{A}(P) \ = \ 0$$



From now on:

- All puzzles are triangles.
- All equivariant puzzle pieces and scabs are vertical.

Def: For any 012-string $u = (u_1, u_2, \ldots, u_n)$ we set

$$C_u := \sum_{i=1}^{n} \delta_{u_i}\, y_i \in R[y_1, \ldots, y_n]$$

Exercise: $\partial P = \triangle_w^{u,v} \Rightarrow$

$$\sum_{s \text{ scab in } P} -\operatorname{weight}(s)\, \mathcal{A}(s) = C_u \cdot \searrow + C_v \cdot \swarrow + C_w \cdot \uparrow$$

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Write $u \to u'$ if $u \le u'$ in Bruhat order and $\ell(u) + 1 = \ell(u')$.

Examples: 022221 $\to$ 122220 ; 02 $\to$ 20 ; 100002 $\to$ 200001

Set $\delta(\frac{u}{u'}) = \delta_{u_i} - \delta_{u'_i}$ where i is minimal with $u_i \neq u'_i$.

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Molev–Sagan type recursion:

$$\left(C_u \cdot \searrow \; + \; C_v \cdot \swarrow \; + \; C_w \cdot \uparrow \right) \cdot \widehat{C}^{w}_{u,v}$$

$$= \sum_{\partial P = \triangle^{u,v}_{w}} \; \sum_{s \text{ scab in } P} -\mathcal{A}(s) \, \mathrm{weight}(s) \prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$$

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$$= \nwarrow \cdot \sum_{u \to u'} \delta\left(\frac{u}{u'}\right) \widehat{C}_{u',v}^{w} \; + \; \nearrow \cdot \sum_{v \to v'} \delta\left(\frac{v}{v'}\right) \widehat{C}_{u,v'}^{w} \; + \; \downarrow \cdot \sum_{w' \to w} \delta\left(\frac{w'}{w}\right) \widehat{C}_{u,v}^{w'}$$

Theorem (Method first applied by Molev and Sagan.)

The equivariant Schubert structure constants $C_{u,v}^{w} \in \mathbb{Z}[y_1, \ldots, y_n]$ of $X = \mathrm{Fl}(a, b; n)$ are uniquely determined by

(1) $C_{w,w}^{w} = \prod_{i<j \,:\, w_i > w_j} (y_j - y_i)$ (Kostant-Kumar)

(2) $$(C_u \cdot \searrow + C_v \cdot \swarrow + C_w \cdot \uparrow) \cdot C_{u,v}^{w}$$

$$= \nwarrow \cdot \sum_{u \to u'} \delta(\frac{u}{u'})\, C_{u',v}^{w} + \nearrow \cdot \sum_{v \to v'} \delta(\frac{v}{v'})\, C_{u,v'}^{w} + \downarrow \cdot \sum_{w' \to w} \delta(\frac{w'}{w})\, C_{u,v}^{w'}$$

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Consequence:

$$C_{u,v}^{w} \;=\; \widehat{C}_{u,v}^{w} \;=\; \sum_{\partial P = \triangle_{w}^{u,v}} \; \prod_{\Diamond \in P} \mathrm{weight}(\Diamond)$$