MATH 356 Homework Assignments

Fall 2006

- HW 1, Due Thursday, Sept. 21.
  - 1-1: 5
  - 1-2: 6
  - 2-1: 7
  - 3-1: 8
- HW 2, Due Thursday, Sept. 28.
  - 3-4: 2
  - 4-1: 6
  - The Pell sequence $\{P_n\}_{n=0}^{\infty}$ is given by $P_0 = 0$, $P_1 = 1$, and

    $$P_{n+1} = 2P_n + P_{n-1},$$

    for $n \geq 1$.

    1. Express the generating function of $\{P_n\}_{n=0}^{\infty}$ as a rational function.
    2. Prove that

       $$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$$
- HW 3, Due Thursday, Oct. 19.
  - 5-3: 1(b)
  - 6-1: 4, 6.

- HW 4, Due Thursday, Oct. 26.
  - 6-2: 2, 10, 11
  - 6-3: 1
  - 6-4: 11
- HW 5, Due Thursday, Nov. 2.
  1. For each of the following partitions, draw the Ferrers graph and find the conjugate partition:
     (a) $5 + 3 + 2 + 1$
     (b) $6 + 3 + 1$
     (c) $7 + 6 + 4 + 3$
  2. Show that for all positive integers n, the number of partitions of n into m distinct parts equals the number of partitions of n wherein $1, 2, 3, \ldots, m$ all appear at least once as a part, and no part is greater than m. *Hint: consider the Ferrers graph.*
  3. Consider the following claim: the number of partitions of n into nonmultiples of three equals the number of partitions of n where no part may appear more than twice. Prove the claim
     (a) bijectively, and
     (b) using generating functions.
  4. Prove that the number of partitions of n into distinct parts congruent to 0, 2, or 3 modulo 4 equals the number of partitions of n into parts congruent to 2, 3, or 7 modulo 8. *Hint: use generating functions.*
- HW 6, due Thursday, Nov. 30.
  1. (a) Prove that the generating function for partitions with exactly j parts is
     $$\frac{q^j}{(1-q)(1-q^2)\cdots(1-q^j)}.$$
     (b) Give a combinatorial proof of the following series-product identity of Euler:
     $$\sum_{j=0}^{\infty} \frac{q^j}{(1-q)(1-q^2)\cdots(1-q^j)} = \prod_{k=1}^{\infty} \frac{1}{1-q^k}.$$

2. The first Rogers-Ramanujan identity is given by

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}. \quad (1)$$

Show that (1) is equivalent to the following partition theorem:

Let $R(n)$ denote the number of partitions of n into parts which are distinct, nonconsecutive integers. Let $S(n)$ denote the number of partitions of n into parts congruent to 1 or 4 modulo 5. Then $R(n) = S(n)$ for all integers n.

Suggested way to proceed:

(a) Show that

$$\sum_{n=0}^{\infty} S(n)q^n = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}.$$

(b) Show that

$$\frac{q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)}$$

is the generating function for partitions of the type counted by $R(n)$ which have exactly j parts.

(c) Use part (b) to show that

$$\sum_{n=0}^{\infty} R(n)q^n = \sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)}.$$

(d) Equate the generating functions and conclude that $R(n) = S(n)$

Note: You are **not** being asked to prove (1).