HW #3 Math 316 Key

4 pts each, 12 pts total

5-3

1 (b) Solve the system

$$x \equiv 1 \pmod 3$$
$$x \equiv 3 \pmod 5 \qquad M = 3 \cdot 5 \cdot 7 = 105$$
$$x \equiv 5 \pmod 7$$

$$n_1 = \frac{105}{3} \quad 35 \equiv 2 \pmod 3 \Rightarrow n_1^{-1} = 2$$

$$n_2 = \frac{105}{5} = 21 \equiv 1 \pmod 5 \Rightarrow n_2^{-1} = 1$$

$$n_3 = \frac{105}{7} = 15 \equiv 1 \pmod 7 \Rightarrow n_3^{-1} = 1$$

$$\therefore \quad x_0 = (1)(35)(2) + (3)(21)(1) + (5)(15)(1)$$
$$= 70 + 63 + 75$$
$$= 208$$
$$\equiv 103 \pmod{105}$$

$$\begin{array}{r} 75 \\ 70 \\ 63 \\ \hline 208 \end{array}$$

$\therefore$ General soln is any number of the form $105k + 103$, $k \in \mathbb{Z}$.

6-1 4. Prove that $\phi(m)$ is even if $m > 2$

Case 1: $m = 2^n$ for $n > 1$. then $\phi(m) = \phi(2^n) = 2^n - 2^{n-1}$

$= 2^{n-1}$, which is clearly even for $n > 1$.

Case 2: $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where at least one of the $p_i \neq 2$,
and $p_i \neq p_j$ if $i \neq j$.
We may assume $p_1 \neq 2$.

$$\phi(m) = \phi\left(p_1^{\alpha_1} \cdot \frac{m}{p_1^{\alpha_1}}\right) = \phi(p_1^{\alpha_1})\, \phi\left(\frac{m}{p_1^{\alpha_1}}\right) = \left(p_1^{\alpha_1} - p_1^{\alpha_1 - 1}\right) \phi\left(\frac{m}{p_1^{\alpha_1}}\right)$$

$$= p_1^{\alpha_1 - 1}(p-1)\, \phi\left(\frac{m}{p_1^{\alpha_1}}\right).$$ Since p is odd, $p-1$ is even

$\therefore \phi(m)$ is even

6.1 (6) Find all integers such that $\phi(n) = 12$

$\phi(13) = 12$

$\phi(26) = \phi(2)\phi(13) = 1 \cdot 12 = 12$

$\phi(21) = \phi(3)\phi(7) = 2 \cdot 6 = 12$

$\phi(42) = \phi(2)\phi(3)\phi(7) = 1 \cdot 2 \cdot 6 = 12$

$\phi(28) = \phi(4)\phi(7) = 2 \cdot 6 = 12$

$\phi(36) = \phi(4)\phi(9) = 2 \cdot 6 = 12$

6-2 $\boxed{\begin{matrix}4\text{ pts}\\ \#2\end{matrix}}$

HW #4 Key

Total 12 pts

Prove that $\prod_{d|n} d = n^{d(n)/2}$

Pf Let the prime factorization of n be given by $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$.

Then $d|n \Rightarrow d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ where $0 \leq \beta_i \leq \alpha_i$ for each $i = 1, \ldots, k$

Thus, by the general combinatorial principle,

$$\prod_{d|n} d = \prod_{i=1}^{k} p_i^{(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_{i-1}+1)(\alpha_{i+1}+1)\cdots(\alpha_k+1)(0+1+2+\cdots+\alpha_i)}$$

$$= \prod_{i=1}^{k} p_i^{\frac{(\alpha_1+1)\cdots(\alpha_k+1)}{(\alpha_i+1)} \cdot \frac{\alpha_i(\alpha_i+1)}{2}}$$

$$= \prod_{i=1}^{k} p_i^{\frac{d(n)\,\alpha_i}{2}} = \prod_{i=1}^{k} \left(p_i^{\alpha_i}\right)^{d(n)/2} = \left(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}\right)^{d(n)/2} = n^{d(n)/2} \quad \square$$

(See end for an alternate proof)

6.2 $\boxed{\begin{matrix}1\text{ pt}\\ \#10\end{matrix}}$

$\sigma(210) = \sigma(2)\sigma(3)\sigma(5)\sigma(7) = 3 \cdot 4 \cdot 6 \cdot 8 = 576$

$\sigma(100) = \sigma(2^2)\sigma(5^2) = (1+2+4)(1+5+25) = 7 \cdot 31 = 217$

$\sigma(999) = \sigma(3^3)\sigma(37) = (1+3+9+27)(1+37) = 40 \cdot 38 = 1520$

6-2 $\boxed{\begin{matrix}1\text{ pt}\\ \#11\end{matrix}}$

$d(47) = 2$ (since 47 is prime)

$d(63) = d(7)d(9) = 2 \cdot 3 = 6$

$d(150) = d(2)\,d(3)\,d(5^2) = 2 \cdot 2 \cdot 3 = 12$

T4,0 in book? they ask for $\phi(100)$. Please give credit for either calculation

6.3. $\boxed{\begin{matrix}3\text{ pts}\\ \#1\end{matrix}}$ Suppose f + g multiplicative and $f(p^r) = g(p^r)$ for all primes p and any $r \in \mathbb{Z}_+$. Prove $f(n) = g(n)$ for all $n \in \mathbb{Z}_+$

Pf: Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ with p_i's all distinct. $f(n) = f(p_1^{\alpha_1} \cdots p_k^{\alpha_k})$

$= f(p_1^{\alpha_1}) \cdots f(p_k^{\alpha_k}) = g(p_1^{\alpha_1}) \cdots g(p_k^{\alpha_k}) = g(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = g(n). \quad \square$

6-4 $\boxed{\begin{matrix}3\text{ pts}\\ \#11\end{matrix}}$ Prove that if $f(n) = \prod_{d|n} g(d)$, then $g(n) = \prod_{d|n} f(d)^{\mu(n/d)}$

Pf. Suppose $f(n) = \prod_{d|n} g(d)$. Thus $\log f(n) = \log \prod_{d|n} g(d)$

$$= \sum_{d|n} \log g(d).$$

Thus
$$\log g(n) = \sum_{d|n} \mu(d) \log f(n/d) \quad \text{(by Thm 6-6)}$$
$$= \sum_{d|n} \log f(n/d)^{\mu(d)} \quad \text{(by law of logs)}$$
$$= \log \prod_{d|n} f(n/d)^{\mu(d)} \quad \text{(by law of logs)}$$
$$= \log \prod_{d|n} f(d)^{\mu(n/d)} \quad \text{since } \{d : d|n\} = \{\tfrac{n}{d} : \tfrac{n}{d}|n\}.$$

$$\therefore \quad \log g(n) = \log \prod_{d|n} f(d)^{\mu(n/d)}$$

$$\therefore \quad g(n) = \prod_{d|n} f(d)^{\mu(n/d)}. \quad \square$$

$\boxed{6\text{-}2}$ #2 alternate proof of $\prod_{d|n} d = n^{d(n)/2}$

Case 1 Suppose $d(n)$ is even, so $d(n) = 2k$ for some k

Let $d_1, d_2, \ldots, d_k$ be the k smallest positive divisors of n

then the set of all positive divisors is $\{d_1, d_2, \ldots, d_k, n/d_k, \ldots, n/d_1\}$

and their product is $n^k = n^{d(n)/2}$

Case 2 Suppose $d(n)$ is odd, so $d(n) = 2k+1$. By 6-2 #1, n is a perfect square

Let $d_1, \ldots, d_k$ be the k smallest positive divisors. The set of all positive divisors is $\{d_1, d_2, \ldots, d_k\} \cup \{\sqrt{n}\} \cup \{n/d_k, \ldots, n/d_1\}$.

So $\prod_{d|n} d = n^k \sqrt{n} = n^{\frac{d(n)-1}{2}} n^{1/2} = n^{d(n)/2}$. $\square$