Math 356
HW #5 Key

#	pts
1	3
2	3
3(a)	4
3(b)	2
4	3

15 pts total

1. For each partition, draw the Ferrers graph & find the conjugate partition.
3pts

	Partition	Ferrers graph	Conjugate
(a)	5+3+2+1	• • • • • • • • • • •	4+3+2+1+1
(b)	6+3+1	• • • • • • • • • •	3+2+2+1+1+1
(c)	7+6+4+3	• • • • • • • • • • • • • • • • • • • •	4+4+4+3+2+2+1

2. Show that the number of partitions of n into m distinct parts equals the number of partitions of n where $1, 2, 3, \ldots, m$ all appear at least once each.
3pts

Pf: ~~Then~~ Consider a partition π of n into m parts, i.e. let π be the partition $\pi_1 + \pi_2 + \cdots + \pi_m$ where $\pi_1 + \cdots + \pi_m = n$ and $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_m$. ~~Notice that~~ With the aid of the Ferrers graph, we see that the number of times j appears as a part in the conjugate partition π' equals $\pi_j - \pi_{j+1}$ for $1 \leq j \leq m-1$ and the number of times m appears in π' is π_m (the smallest part of π).
By hypothesis $\pi_j - \pi_{j+1} \geq 1$ for $1 \leq j \leq m-1$ (since the parts of π are distinct). Thus each integer $1, 2, \ldots, m-1$ appears in π' at least once.
Finally, the largest part of π' equals the number of parts in π, i.e. m. Thus m also appears in π'! $\square$

3. Thm: the number of ptns of n into nonmultiples of 3 equals
" " " " " " in which no part appears more than twice.

(a) Bijective proof 4 pts

$$\text{Let } n = \overbrace{1+1+1+\cdots+1}^{m_1} + \overbrace{2+2+\cdots+2}^{m_2} + \overbrace{4+4+\cdots+4}^{m_4} + \overbrace{5+5+\cdots+5}^{m_5} + \cdots$$

$$= m_1\cdot 1 + m_2\cdot 2 + m_4\cdot 4 + m_5\cdot 5 + m_7\cdot 7 + m_8\cdot 8 + \cdots$$

$$= (a_{10} + a_{11}\cdot 3 + a_{12}\cdot 3^2 + a_{13}\cdot 3^3 + \cdots)\cdot 1$$
$$+ (a_{20} + a_{21}\cdot 3 + a_{22}\cdot 3^2 + a_{23}\cdot 3^3 + \cdots)\cdot 2$$
$$+ (a_{40} + a_{41}\cdot 3 + a_{42}\cdot 3^2 + a_{43}\cdot 3^3 + \cdots)\cdot 4$$
$$+ (a_{50} + a_{51}\cdot 3 + a_{52}\cdot 3^2 + a_{53}\cdot 3^3 + \cdots)\cdot 5$$
$$+ \vdots$$

where $a_{i0} + a_{i1}3 + a_{i2}3^2 + \cdots$ is the base 3 expansion of m_i; $a_{ij} \in \{0,1,2\}$

$$= a_{10} + 3a_{11} + 9a_{12} + 27a_{13} + \cdots$$
$$+ 2a_{20} + 6a_{21} + 18a_{22} + 54a_{23} + \cdots$$
$$+ 4a_{40} + 12a_{41} + 36a_{42} + 108a_{43} + \cdots$$
$$+ 5a_{50} + 15a_{51} + 45a_{52} + 135a_{53} + \cdots$$
$$+ \vdots$$

Each coefficient of a_{ij} is unique since the coefficient in column j contains 3^j in its prime factorization and coefficients in row i are multiples of i where $3 \nmid i$.

This last expression is a partition of n where each part appears at most twice since each $a_{ij} \in \{0,1,2\}$.

(b) Gen. fn. proof: 2 pts

$$\prod_{j=1}^{\infty}(1+q^j+q^{2j}) = \prod_{j=1}^{\infty}\frac{1-q^{3j}}{1-q^j} = \prod_{j=1}^{\infty}\frac{\cancel{1-q^{3j}}}{(1-q^{3j-1})(1-q^{3j-2})\cancel{(1-q^{3j})}} = \prod_{j=1}^{\infty}\frac{1}{(1-q^{3j-1})(1-q^{3j-2})} \quad \square$$

4. Prove that the number of partitions of n into distinct parts $\equiv 0,2,3 \pmod 4$
3 pts
equals " " " " " " parts $\equiv 2,3,$ or $7 \pmod 8$

Pf

$$\prod_{j=0}^{\infty}(1+q^{4j+2})(1+q^{4j+3})(1+q^{4j+4}) = \prod_{j=0}^{\infty}\left(\frac{1-q^{8j+4}}{1-q^{4j+2}}\right)\left(\frac{1-q^{8j+6}}{1-q^{4j+3}}\right)\left(\frac{1-q^{8j+8}}{1-q^{4j+4}}\right)$$

$$= \prod_{j=0}^{\infty}\frac{(1-q^{8j+4})(1-q^{8j+6})(1-q^{8j+8})}{(1-q^{8j+2})(1-q^{8j+6})(1-q^{8j+3})(1-q^{8j+7})(1-q^{8j+4})(1-q^{8j+8})}$$

$$= \prod_{j=0}^{\infty}\frac{1}{(1-q^{8j+2})(1-q^{8j+3})(1-q^{8j+7})} . \quad \square$$