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$29 \neq 29$

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1. Introduction When two combinatorial problems have the same numerical answer, it is usual to expect that there is a combinatorial relation between the problems. Coincidences do occur, but they are rare. It would be of interest to know how to recognize a coincidence. In this note we shall present an argument to support a claim that the numbers 29 arising from two particular combinatorial problems are combinatorially distinct. We begin by formulating both problems. Then various evidence for the combinatorial difference between the problems is presented.

2. The first problem. Consider the set $\{X,Y,Z\}$. How many topologies are there on this set? The topologies will be given by listing all open sets. In a finite space any union or intersection of open sets will be open. It is then fairly easy to systematically enumerate the topologies. There are 29. (We have tabulated them in Table 1, but please do not look at that list now.)

3. The second problem. Consider the vector space R^3, i.e. the space of ordered triples of real numbers. In this space consider the 8 vectors whose coordinates are all either 0 or $+1$. How many ways are there of choosing 3 vectors from these 8 so that these vectors will be a basis for R^3?

The somewhat similar problem of enumerating the bases for a three dimensional space over the field with two elements is easily solved. Picking vectors one at a time so that each is independent of the previously chosen vectors gives $(8-1)(8-2)(8-4)$ ordered triple of independent vectors. Each basis is counted exactly 6 times, so there are 28 bases.

The vectors $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ are independent over the reals, but dependent if interpreted as vectors over the field with two elements. This gives a twenty-ninth basis for R^3 of the required type. (You will find these results summarized when you get to Table 2).

4. Do these results generalize? The combinatorial equivalence of these questions should mean that the relation between the "29" in the answer and the "3" in the question is somehow the same.

As a first attempt to study this, we propose that the questions be regarded as the result of setting $n = 3$ in:

(a) How many topologies are there on a set of n points?

(b) How many bases for R^n can be found using only vectors whose coordinates are 0 or 1?

Question (b) appears to be less natural. The author knows of no discussion of it in the literature. Of course, the related question about bases over the field of two elements is well known and can be used to get some estimate on the behavior of this function for large n.

The reader is encouraged to verify that for $n = 2$, the problems have different answers (4 for (a) and 3 for (b)) and to consider the case $n = 4$.

It is then plausible that the common answer to these questions, when $n = 3$, is accidental. However, there remains the possibility that we have not abstracted the right generalizations. Although this approach is attractive it leads only to the conclusion that either the coincidence is accidental or we do not understand the role of "3" in each problem.

5. Symmetry. In the first problem, we can permute the three points X, Y, Z. Each permutation induces a permutation of the topologies on $\{X,Y,Z\}$. We call two topologies equivalent if they are related by such a permutation. The equivalence classes give the distinct three-point topological spaces.

In the second problem, we can permute the given coordinates of R^3. This induces an action of the group S_3 of all permutations of three objects on R^3. This action takes a basis into a basis, and a vector having coordinates 0 or 1 into another vector of the same type.

Thus the groups S_3 acts on the data of both problems. A consideration of the role of "3" in the problems *suggests that this action is a fundamental part of the combinatorial structure of the problems.* Thus, to show that the problems are not the same we tabulate the equivalence classes of solutions to both problems showing the number of solutions in each equivalence class.

6. Tables. For the first problem we will list the open sets other than ϕ and $\{X,Y,Z\}$. We give one representative of each equivalence class and the number of topologies in this class.

Table 1

open sets	number of equivalent topologies
no other open sets	1
$\{X\},\{Y\},\{Z\},\{X,Y\},\{X,Z\},\{Y,Z\}$	1
$\{X,Y\}$	3
$\{X,Y\},\{Z\}$	3
$\{X\}$	3
$\{X\},\{X,Y\},\{X,Z\}$	3
$\{X\},\{Y\},\{X,Y\}$	3
$\{X\},\{X,Y\}$	6
$\{X\},\{Y\},\{X,Y\},\{X,Z\}$	[illegible]

Table 2

basis			number of equivalent bases
(1,0,0)	(0,1,0)	(0,0,1)	1
(1,1,0)	(1,0,1)	(0,1,1)	1
(1,1,1)	(0,1,0)	(0,0,1)	3
(1,0,1)	(0,1,1)	(0,0,1)	3
(1,1,1)	(1,1,0)	(0,1,1)	3
(1,0,1)	(0,1,0)	(0,0,1)	6
(1,1,0)	(0,1,1)	(0,0,1)	6
(1,1,1)	(0,1,1)	(0,0,1)	6

If we require identical structures under the action of S_3 for "equality", then we have distinct numbers, namely

$$1 + 1 + 3 + 3 + 3 + 3 + 3 + 6 + 6 \neq 1 + 1 + 3 + 3 + 3 + 6 + 6 + 6.$$

The usual rules of arithmetic then give the assertion of our title.

Postcript. The second problem was actually an exercise used by P.R. Halmos in a finite dimentional vector space course. By chance Halmos overheard some people discussing the first problem while he was working on that section of the course. He then assigned the author the task of determining the connection between the problems. All of this took place at the University of Michigan 1964-65.

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