Introduction to Hausdorff Measure and Dimension

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Premeasures

A function τ defined on a class $\mathcal{C}$ of subsets of a given set X is called a **premeasure** if

$$\emptyset \in \mathcal{C}$$

$$(\forall C \in \mathcal{C}) \quad 0 \leq \tau(C) \leq +\infty$$

$$\tau(\emptyset) = 0$$

In some examples X will be embedded in a larger set Y and $\mathcal{C}$ will be described in Y with the sets understood to be the intersections of those sets with X.

Method I

For all subsets E of X, let

$$\mu(E) = \inf \sum \tau(C_i)$$

where the C_i are a **finite or countable** collection of sets in $\mathcal{C}$ **covering** E (i.e., each $x \in E$ belongs to at least one E_i).

Then μ is an **outer measure** (defined on the next slide). The definitions make the proof a simple exercise.

Outer Measures

A function μ defined on the class of **all** subsets of a given set X is called an **outer measure** if

$$(\forall E \subseteq X) \quad 0 \leq \tau(E) \leq +\infty$$

$$\mu(\emptyset) = 0$$

$$E_1 \subseteq E_2 \implies \mu(E_1) \leq \mu(E_2)$$

$$\mu\Big(\bigcup E_i\Big) \leq \sum \mu(E_i)$$

Measurable sets

A set E is called **measurable** if for all sets A and B with $A \subseteq E$ and $B \cap E = \emptyset$, we have

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

Definitions assure that $\emptyset$ is measurable, the complement of a measurable set is measurable, and countable unions of measurable sets are measurable.

Examples

1. Counting measure: the number of points in a finite set and $+\infty$ for infinite sets.

2. Lebesgue measure in $\mathbb{R}^n$: start with a premeasure on "rect-angles" equal to the product of the side lengths and use Method I.

The Cantor middle third set

Start from the closed unit interval.

Remove the open middle third.

Repeat on each remaining interval.

The set remaining after countably many steps is the Cantor set.

After n such steps, there are 2^n intervals, each of length 3^{-n}. The measure approaches zero, but **uncountably many** points remain.

We seek a finer description of the size of the set.

The construction premeasure

Let $\mathcal{C}$ be the collection of all intervals appearing in the construction of the Cantor set and no other intervals. For an interval I appearing at step n, let $\tau(I) = 2^{-n}$. For the induced measure, the Cantor set is measurable and has measure 1.

Note that the measure of each interval in $\mathcal{C}$ is its length to the power $\log 2/\log 3$. The description we seek will has this as the **dimension** of the Cantor set.

An arithmetic description

The Cantor set construction should remind you of expansions of numbers base 3. The numbers in $[0, 1]$ removed at step n are those numbers whose n-th ternary digit **must be** 1. That is, $1/3$ is **not** removed because it can be written as $0.0222\ldots$ as well as 0.1. This suggests that the set that uses only k of the n available digits base n should have dimension $\log k / \log n$ based on a premeasure on the collection of intervals selected by initial digits in the base n expansion.

A temporary fix

For now, there will be a fixed construction and the premeasure will be defined only on the result of that construction. In the final theory, there will be a uniform definition of the measure of dimension d and a set will be said to have dimension d if it is measurable for that measure with a measure that is both positive and finite.

More general sets

More generally, substrings—rather than single digits—may be excluded. For example, suppose that we wish to exclude the string 11 in the binary expansion of numbers in [0, 1]. Intervals with the excluded string should again have $\tau = 0$, but there are two types of intervals contributing to the construction: those for which the previous digit was 0 and those for which it was 1. The first type may be followed by 0 or 1; the second, only by 0.

A Goldilocks principle

After n steps, the construction intervals still have length 2^{-n}, and we should let τ be some power of the length of the interval, but how do we choose the power? If the exponent is too small, the measures of parts of construction intervals may add to more than the measure of the interval, so we get no interesting measurable sets. If the exponent is too large, all sets get measure zero.

Graph-based constructions

The construction just described may be represented by a directed graph with nodes 0 and 1 and edges from 0 to both 0 and 1 and from 1 only to 0. A premeasure could be defined on more general sets that are unions of disjoint construction intervals: m_0 of type 0 and m_1 of type 1. If such a set is represented by a vector with entries m_0, n_1, then the set obtained by subdividing **all** contributing construction intervals would be represented by the vector

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}$$

The dimension for graph directed constructions

After several steps, the ratio of types of intervals will be given by the entries of the eigenvector of the largest eigenvalue λ and each step will multiply the number of intervals by λ. The exponent giving a useful measure is $\log \lambda / \log 2$. In this case $\lambda = (1 + \sqrt{5})/2$.

Perron's Theorem

More general graphs allow a similar description using a square matrix whose number of rows is the number of nodes with (i, j) entry the number of edges from node i to node j. In particular, all entries are **nonnegative**. It is also reasonable to expect that the graph be **strongly connected** (there is a path joining every ordered pair of nodes). Then some power of the matrix has all entries positive. Perron proved that such a matrix has a simple positive eigenvalue that is strictly larger than the absolute value of any other eigenvalue.

The Perron-Frobenius Theorem

Frobenius extended this theorem to cover all nonnegative matrices. The dominant eigenvalue need no longer be simple, and no longer strictly dominant, but the structure is completely described.

Others have extended the theory to apply to operators in infinite dimensional spaces. New settings are still usually referred to without additional names.

Metric considerations

This suggests that a d-dimensional measure on $\mathbb{R}$ could be defined by defining τ of an interval to be the d-th power of the length of the interval. However, the "Method I" construction fails to account for our feeling that covers by short intervals are expected to be more accurate than covers including long ones. To account for this, we may restrict our original family to intervals of length no more than some give δ. This gives different measures for each δ. Decreasing δ increases the outer measure of every set since the infimum in the Method I construction is taken over a smaller family of covers.

Metric considerations

In the general theory, the sets in $\mathcal{C}$ will be described in an arbitrary metric space, and the premeasure τ will be taken to be the d-th power of the diameter of the set.

A set of dimension d will be measurable and have finite measure for all such measures. Sets of smaller dimension will have measure zero and sets of larger dimension will generally fail to be measurable.

Method II

The individual measures constructed from restricting the diameter of the sets in $\mathcal{C}$ have no special significance. We need to take a **limit** as $\delta \to 0$. Since the outer measure of a set increases as δ decreases, the limit exist (it may be $+\infty$) and is the supremum of Method I measures defined by sets of bounded diameter.

Hausdorff measure

The Hausdorff d dimensional measure in any metric space is the Method II measure defined with $\mathcal{C}$ being **all** bounded subsets with premeasure the d-th power of the diameter. Other functions of the diameter have also been used, but powers are most appropriate for examples considered here. Measuring a set with a dimension that is too small now makes the set measurable with infinite measure, and a dimension that is too large gives zero measure.

Hausdorff dimension

It is easily seen that there is at most one dimension for which a given set can have a measure that is finite and positive. This is the Hausdorff dimension of the set.

To show that our "missing digit" examples have the dimension announced earlier, we need to show that enlarging the family $\mathcal{C}$ has only a minor effect on the measure.

Such arguments are easy for subsets of $\mathbb{R}$.

Continued fractions

A more interesting example is the set of numbers whose continue fraction example is restricted. In particular, one may allow only 1 and 2 as partial quotients. The construction still looks like a Cantor set, but each removed interval depends on the steps leading to that point in the construction.

In this case, the dimension is found by studying an operator on the space of functions on $[0, 1]$ built from mapping $f(x)$ to $f(c + 1/x)$ for $c = 1$ and $c = 2$.