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A PROBLEM WITH TELEPHONES*

RICHARD T. BUMBY†

Abstract. This paper deals with the "telephone problem," also known as the "gossip problem". Suppose n persons each have a piece of information. Pairs of them can share whatever information they possess by making a telephone call. The question arises, what minimum number of calls allows all n persons to obtain all n pieces of information. The answer is $2n-4$. One can then ask about properties of such minimal sets of calls. In particular, we prove that the graph whose edges are the calls must contain a four-cycle.

1. Introduction. The "telephone problem" has often been solved in the literature [1], [3], [7], and various extensions of the problem have been proposed [2], [4], [5]. This paper is devoted to the proof of the "four-cycle conjecture" introduced in [4] with the words: "We are so convinced of the next statement that even though it is by definition a conjecture, we shall call it a *True conjecture*: $\cdots$" (italics theirs). The rumor of its solution, hastily added in proof in [4], proved to be premature.

We now establish the notation for the proof. We assume that there are n persons and k calls between them. The persons form a set U and the calls form an ordered set $T=\langle t_1, \cdots, t_k\rangle$ of (unordered) pairs of distinct elements of U. T is called a "system of calls."

It is natural to think of T as determining a graph $G(T)$ whose verticles are U and whose edges are the elements of T. Thus, the elements of U are also called "vertices" or "nodes."

The ordering on T can be used to introduce a relation $a \to b$ which holds iff there is a path $a = x_0, \cdots, x_m = b$ such that there is an increasing subsequence $\langle s_i: 1 \leqq i \leqq m\rangle$ in T and $s_i = \{x_{i-1}, x_i\}$. The relation $a \to b$ means that b learns a's information in the system of calls T. If $a \to b$, we then have a path between a and b in $G(T)$, so these points lie in the same component of $G(T)$.

DEFINITION. A system T is called *pooling* if $a \to b$ for each $a, b \in U$.

This paper proves:

THEOREM 1. *If T is pooling, then $k \geqq 2n-4$.*

THEOREM 2. *If T is pooling and $k = 2n-4$, then $G(T)$ contains a four-cycle.*

2. The minimal partial ordering of T. We have described T as a *sequence* of calls; that is, the t_i are totally ordered in time. However, note that whether $a \to b$ holds depends only on the following partial order on T:

DEFINITION. The *minimal order* on T is the transitive closure of the relation $\{(t_i, t_j): i \leqq j \text{ and } t_i \cap t_j \neq \varnothing\}$.

Thus, two calls are ordered in the minimal order only if information can flow through one call into another, or equivalently, if it is the case that their order in time could not be reversed without changing information paths. If we consider all t_i containing a fixed node, the minimal order gives a linear order on them. Since all the essential properties of T are given by the minimal order, we will henceforth ignore the original linear ordering.

If we consider any linear ordering (time sequencing) of T that is compatible with the minimal ordering, and select a time between two of the calls, we partition T into the calls I before that time, and the calls F after that time. The pair (I, F) has the following nice properties:

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† Department of Mathematics, Hill Center for the Mathematical Sciences, Rutgers University, New Brunswick, New Jersey 08903.

(i) $T = I \cup F$,

(ii) $I \cap F = \varnothing$,

(iii) $i \in I$ and $t \leqq i$ imply $t \in I$,

(iv) $f \in F$ and $t \geqq f$ imply $t \in F$.

That is, I and F are complementary lower and upper ideals of the partially ordered set T. Conversely, if (I, F) has the properties (i)–(iv), then there is a time ordering of T that performs the calls I before the calls F.

DEFINITION. (I, F) satisfying (i)–(iv) above is a *splitting* of T.

If we give a splitting (I, F) and we have defined only one component, the other component is its complement in T.

3. Components and closures. If (I, F) is any splitting and u is a node that is involved in some call in F, we define $\min(u)$ to be the first call in F involving u. Clearly, if $u \in t \in F$, then $\min(u) \leqq t$. If $t = (u_1, u_2) \in F$, then $t \geqq \min(u_1), \min(u_2)$. Unless $t = \min(u_1) = \min(u_2)$, there is a call in F strictly smaller (earlier) than t. Also, if $t > t_0 \in F$, then information can flow from t_0 through an increasing sequence of calls to t. The last call of this chain before t must share a node u with t, and so $t \neq \min(u)$. This proves:

PROPOSITION 1. *For any splitting (I, F), a call $t = (u, v)$ of F is minimal in F iff $t = \min(u) = \min(v)$.*

DEFINITION. Given a splitting (I, F) with $I \geqq n-2$ and $G(I)$ not connected, any component of $G(I)$ is a *component of T*.

Starting from the splitting $(\varnothing, T)$, we can successively "bring down" minimal elements of F from F into I. Thus, we can construct splittings (I, F) in which I (or F) has any size from 0 to k. Clearly, any pooling must have $k \geqq n-1$, so selecting $|I| = n-2$ shows that all poolings have components.

The basis of our proofs will be constructions involving the components of T. The first construction will be closure.

Suppose that X is a component of T defined by the splitting (I_0, F_0). Define A_X to be the splitting (I, F) with the largest $I \subseteq I_0$ such that X is a component of $G(I)$. Define B_X to be the splitting (I, F) with the largest $I \subseteq I_0$ that has a component with the same vertices as X.

DEFINITION. If B_X is (I, F), then the component of $G(I)$ with the same vertices as X is the *closure* of X, denoted $\bar{X}$. If $\bar{X} = X$, $B_X = A_X$ and X is *closed*, and (I, F) is called the *canonical splitting* for X.

PROPOSITION 2. *Given a splitting (I, F), if no minimal element of F joins two nodes of X, then $A_X = B_X$, and hence, X is closed. In addition, if every minimal element of F joins a node of X to a node not in X, then $A_X = B_X = (I, F)$. Conversely, if (I, F) is the canonical splitting of a closed X, every minimal element of F joins a node of X to a node not in X.*

Proof. Let $A_X = (I_A, F_A)$ and $B_X = (I_B, F_B)$. I_B is I_A with some additional calls from F_A that connect nodes of X. But, if no minimal element of F joins two nodes of X, no such call can be moved into I_B without connecting X to some other component. If, in addition, every minimal element of F joins a node of X to a node not in X, then no call in F can be moved to I_A without connecting X to some other component. Conversely, if a minimal element of F connects two nodes of X, X is not closed, and if a minimal element of F connects two nodes not in X, then (I, F) is not the canonical splitting of X.

COROLLARY. *If X consists of a single point, then X is closed.*

We use the convention that all "lemmas" have the standing hypothesis that T is a pooling system on U. "Propositions" deal with general call systems.

LEMMA 1. *If any component X consists of a single point x, then $k \geqq 2n-3$. Hence, Theorems 1 and 2 hold in this case.*

Proof. From the corollary to Proposition 2, X is closed. Let (I, F) be its canonical splitting. Except in the trivial case $n = 1$, no node of U has received all the information yet, so each node u has a min (u). From Proposition 2, {min (u): $u \neq x$} are distinct, and so F has $\geqq n-1$ elements. Since $|I| \geqq n-2$, the result follows.

From Lemma 1, if $k = 2n-4$, then T can have no component consisting of a single point. Let $\min_0(u)$ be the first call involving u in T; i.e., $\min_0(u)$ is min (u) for the splitting $(\emptyset, T)$. Then for any splitting (I, F) with $|I| \geqq n-2$, $\min_0(u) \in I$ for any node u.

4. Minimal trees. Suppose we have a closed component X which is a tree. After Lemma 1, we may assume that $|X| > 1$. Let (I, F) be its canonical splitting. As X is a component of $G(I)$, I is a disjoint union $I_X \cup I_Y$, where I_X is all calls between nodes of X and I_Y is the rest. Let $t_0 = (x_0, x_1)$ be a maximal element of I_X. In $G(I - t_0)$, X falls into two parts X_0, X_1 with $x_i \in X_i (i = 0, 1)$.

Suppose F has a minimal element $t_1 = (x_2, y)$ with $y \notin X$ and $x_2 \neq x_0, x_1$. Then t_0 and t_1 are incomparable in T. If we bring t_1 down into I and raise t_0 up into F, we get a new splitting (I', F') with $|I'| = |I|$. What does $G(I')$ look like?

Say $x_2 \in X_0$; then the addition of t_1 to $I - t_0$ causes X_0 to be connected to the component containing y. As y must be outside X_1, this leaves X_1 as a component of $G(I')$. The minimal elements of F' are t_0, some minimal elements of F, and some elements of F having nodes in common with t_1. None of these can have both nodes in X_1. From Proposition 2, X_1 is closed, with canonical splitting (I'', F''), $I' \subseteq I''$.

Induction on this construction proves:

PROPOSITION 3. *If X is a closed tree component of T corresponding to a splitting (I, F), then we can find a closed component $X_0 \subseteq X$ (so X_0 is a tree) corresponding to a splitting (I_0, F_0) with $|I_0| \geqq |I|$ and, if $|X_0| > 1$, a maximal element (x', x'') in I_0 with $x', x'' \in X_0$ such that the only possible minimal elements of F_0 are* min (x') *and* min (x'').

DEFINITION. The component X_0 above is a *minimal tree.*

LEMMA 2. *If some closed component of T is a tree, then $k \geqq 2n-4$.*

Proof. From Proposition 3, we have a minimal tree X with canonical splitting (I, F) such that $|I| \geqq n-2$. Either $|X| = 1$ and Lemma 1 applies, or we have elements $x', x'' \in X$ such that min (x') and min (x'') are the only possible minimal elements of F. Since no node except x' and x'' can have received information from both x' and x'' during the calls of I, all nodes other than x' and x'' are members of some call in F. By Proposition 3, {min (u): $u \neq x', x''$} are distinct elements of F, so $|F| \geqq n-2$.

LEMMA 3. *If $k = 2n-4$, and some closed component X corresponding to a splitting (I, F) is a tree, then $|I| = |F| = n-2$. If X is a minimal tree, then F has precisely two minimal elements, and every element of F is* min (u) *for some u.*

Proof. By Proposition 3, construct a minimal tree X_0 in X and its canonical splitting (I', F'). However, while proving Lemma 2, we showed that $|I'|, |F'| \geqq n-2$, so we must have $|I'| = |F'| = n-2$. By construction, $|I| \leqq |I'|$, and $|I| \geqq n-2$ since (I, F) generates components, so $|I| = |F| = n-2$. If X is a minimal tree, {min (u): $u \neq x', x''$} already gives all $n-2$ elements of F. Because $|I| = n-2$, $X \neq U$, and x', x'' must receive future calls, so min (x') and min (x'') must exist and be min (u) for some $u \neq x', x''$. This implies that they are distinct, and Proposition 1 shows they are exactly the minimal elements of F.

If $k = 2n-4$ and X is a minimal tree with canonical splitting (I, F), Lemma 3 shows that X and F determine x' and x''. Note that $G(F)$ must consist of two trees with all information flowing outward from x' and x''.

This lemma demonstrates why minimal trees are called "minimal." Suppose that a minimal tree X contained a smaller tree component. This could only be if X had a maximal element other than $\{x', x''\}$. But, this would allow us to construct a splitting that contradicts Lemma 3: Raise this other maximal element into F and lower min (x') and min (x'') into I. Since X is closed, all these calls are incomparable, and the part of X left without $\{x', x''\}$ is a closed tree. But, we have now made $|I| = n-1$, so the canonical splitting for this tree must have $|I| \geqq n-1$. Thus, in a minimal tree, the calls of I transfer information inward along the tree into x' and x''. This fact about minimal trees is not needed for our proofs, but is a useful tool; see, e.g., [6].

5. Completion of the proof of Theorem 1. We begin with a construction.

PROPOSITION 4. *Suppose (I, F) is a splitting with $|I| \geqq n-2$ such that at least two components of $G(I)$ are trees. Then one of these components is closed or both contain closed components (which must be trees).*

Proof. Call the components X and Y. If either is closed we are done, so we may assume by Proposition 2 that F has a minimal element y which joins two nodes of Y. X is not a point, so I contains a maximal element $x \in X$.

Interchange x and y, that is, raise x into F and lower y into I, to form a splitting (I', F'). Thus, $|I'| = |I|$, and in $G(I')$ the points of X fall into two components X_0 and X_1, while the nodes of Y no longer span a tree in $G(I)$.

Each X_i is a tree, so we may apply this construction inductively to produce a closed component inside X. Reversing the roles of X and Y produces a closed component inside Y.

LEMMA 4. *There is a closed component of T which is a tree.*

Proof. Let (I, F) be any splitting with $|I| = n-2$. Counting edges of $G(I)$ shows that at least two components of $G(I)$ are trees. Proposition 4 gives the desired result.

Proof of Theorem 1. Lemma 2 and Lemma 4 prove Theorem 1.

Proposition 4, together with Lemma 3, gives additional insight into the structure of pooling systems with $k = 2n-4$. The remainder of this section is devoted to such results which are not needed for the proof of Theorem 2, but are of independent interest. So we now concentrate on T which are pooling with $k = 2n-4$. We let (I, F) be a splitting satisfying the hypothesis of Proposition 4.

If one of the tree components, say X, is closed, then Lemma 3 implies that $|I| = n-2$ and every minimal element of F has one end in X. Thus, all components are closed. Now applying the same analysis to Y, we find that the minimal elements of F must link X and Y. There can be no further tree components, so every other component must have the same number of edges as nodes. If u is a minimal element of F, then one component of $G(I \cup u)$ is a tree containing X and Y. Again, Lemma 3 tells us that this component is not closed, so adding an element of its closure gives a splitting (I', F') with $|I'| = n$ and all components of $G(I')$ having the same number of edges as nodes. Call such a splitting "balanced."

Now suppose that the tree components of $G(I)$ are not closed. The construction of Proposition 4 does not change $|I|$, nor does it alter any component disjoint from the trees X and Y. When we are finished, we have a closed tree component. Thus, $|I| = n-2$ and any component other than X or Y is closed and has the same number of edges as nodes. If we add a minimal element of F joining two nodes of X and a minimal element of F joining two nodes of Y to the given I, we will obtain a balanced splitting (I', F').

We now show that a balanced splitting has at most two components and that these components are closed. A single component arises only from a splitting (I, F) for which $G(I)$ has only two components, both closed trees. We may then limit ourselves to the case in which $G(I')$ has more than one component. Remove any maximal element from I'. The resulting graph has one tree component, and the total number of components is either the same or increased by one. By Lemma 3, the tree cannot be closed and hence, by the corollary to Proposition 2, cannot consist of a single point. Now remove a maximal element of this tree to get a splitting satisfying the hypothesis of Proposition 4. As we have already noted, all components other than the trees are closed, so all components of $G(I')$ except the one we dissected are closed. But we could have chosen any component, so all components are closed.

If $G(I')$ has more than two components, add a minimal element of F'. This connects only two components. Dissecting any other components leads to a splitting satisfying the hypothesis of Proposition 4 for which $|I|>n-2$, which we have seen contradicts Lemma 3. Thus, $G(I')$ can have only two components. Also, if the removal of a maximal element of I' were to give a graph with three components, we could add a link to the tree component to give a new balanced splitting with three components. Thus, this possibility is also ruled out. Finally, removing a maximal link from the tree component gives a splitting with $|I|=n-2$ and two tree components. If these trees were not closed, we could construct a balanced splitting with three components by closing them. Thus:

THEOREM 3. *If T is pooling with $k=2n-4$ and (I, F) is a splitting with $|I| \geqq n-2$ with at least two tree components in $G(I)$, then $|I|=n-2$ and one of these cases holds:*

Case I. *The two trees are closed, and they are the only components.*

Case II. *The two trees are closed and there is one other component. If one removes a maximal link from this component, one gets a single tree; and if one removes a maximal link from this, then one gets two closed trees. There is an $I'' \subseteq I$ such that $G(I'')$ is the union of four closed trees.*

Case III. *The trees are not closed. Now there can be no further components. Removal of a maximal link from each of the components again gives closed trees.*

6. Blocks. Let T be a pooling system with $k=2n-4$ and (I, F) a splitting with some minimal tree component X. This gives, from Lemma 3, elements $x', x'' \in X$ such that $F=\{\min(u): u \neq x', x''\}$. This X will be fixed for the rest of the section.

PROPOSITION 5. *With these assumptions, if $u \in U-X$, then there are elements u', u'' such that $u \to u'$ and $u \to u''$ in I and* $\min(u')=\{u', x'\}$ *and* $\min(u'')=\{u'', x''\}$.

Proof. Consider the path $u=u_0, u_1, \cdots, u_{i-1}, u_i=x'$ proving $u \to x'$ in T. Since u and x belong to different components of $G(I)$, some step must be in F, and hence, all steps from that point on must be in F. The last step, $\{u_{i-1}, x'\}$ must be $\min(u_{i-1})$, but then $\{u_{i-2}, u_{i-1}\}$ cannot belong to F. The element u_{i-1} is the desired u'. We find u'' the same way, starting from $u \to x''$.

Note that u' and u'' must lie on the same component of $G(I)$ as u does. From a count of edges of $G(I)$, there is a component Y of $G(I)$, different from X, which is a tree. The component Y is an example of a "block."

DEFINITION. A tree component Y is a *block* if for each $y \in Y$, there are $y', y'' \in Y$ such that $y \to y', y''$ in Y and $\min(y')=\{x', y'\}$ and $\min(y'')=\{x'', y''\}$.

Knowing that blocks exist, we will construct "minimal blocks" by inductively reducing the size of a block until certain properties hold. If $t=\{y_0, y_1\}$ is a maximal element in Y, then any sequence of calls that proves $u \to v$ in Y must prove $u \to v$ in $Y-t$ or else the last step is t. In the latter case, $v=y_0$ and $u \to y_1$ in $Y-t$ or $v=y_1$ and $u \to y_0$ in $Y-t$. $G(Y-t)$ has two components Y_0, Y_1 with $y_i \in Y_i (i=0, 1)$.

If $\min(y_1) \neq \{y_1, x'\}$ or $\{y_1, x''\}$, then y_1 can never be a y' or y''. From this it follows that Y_0 is a block. Similarly, if $\min(y_0) \neq \{y_0, x'\}$ or $\{y_0, x''\}$, then Y_1 is a block. In either case, we get a smaller block.

Now suppose that $\min(y_0)$ and $\min(y_1)$ both involve the same element of X, say x'. If $y \in Y_0$, then $y \to y''$ in $Y-t$ so $y'' \in Y_0$. Either $y' \in Y_0$, in which case we also have $y \to y'$

in $Y-t$; or $y'=y_1$. In the latter case, $y \to y_0$ in $Y-t$, so we could use y_0 for y' instead. In either case, Y_0 is a smaller block than Y. (In fact, Y_1 is also.)

LEMMA 5. *Minimal blocks exist, and if Y is a minimal block with y_0, y_1 a maximal edge, then either* $(x'-x''-y_0-y_1-x')$ *or* $(x'-x''-y_1-y_0-x')$ *is a four-cycle in* $G(T)$.

Proof. Induction on the above construction gives a minimal block. The only minimal blocks are where y_0 and y_1 are adjacent to different elements of $\{x', x''\}$.

This completes the proof of Theorem 2.

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