GROUP FUNCTORS, FIELDS AND TITS GEOMETRIES

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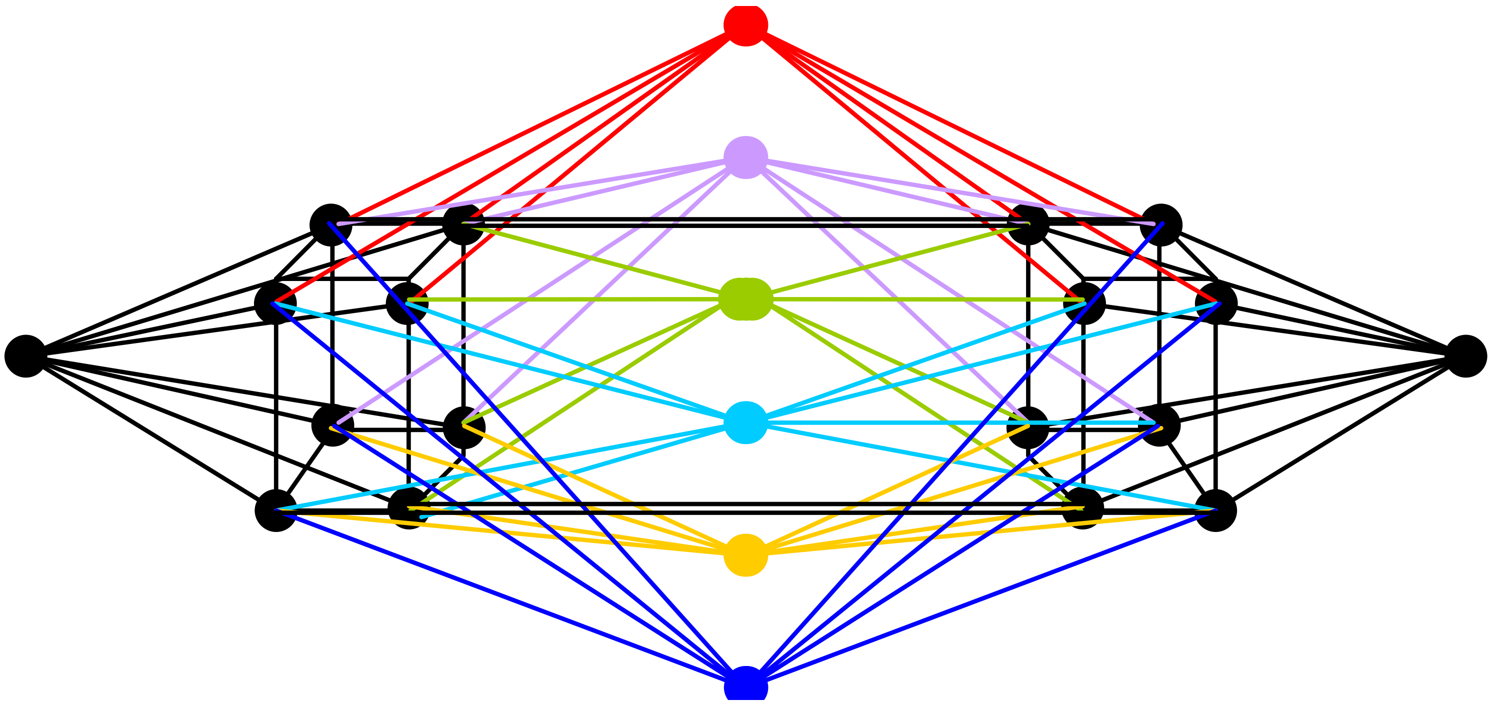
LECTURE I.
TITS DISCOVERS SPHERICAL BUILDINGS



This diagram of Samia Al-Azab, Mohammed El-Atrash, Osama Ahmed Al-Absi shows the geometry of F_4 over $\mathbb{F}_1$ (also called the Coxeter complex).

I. TITS' DISCOVERY OF SPHERICAL BUILDINGS

Tits discovers a geometry X associated to a finite dimensional simple algebraic group G which has G as its automorphism group. Tits' geometry X has the property that for G constructed over a finite field $\mathbb{F}_q$, one could take the limit $q \longrightarrow 1$ in which G tends to a discrete subgroup W, the *Weyl group* of G, and the geometry X tends to the geometry of W.

Tits discovers that Weyl groups of finite dimensional simple algebraic groups are 'Coxeter groups' whose 'Coxeter complexes' correspond to his geometry 'over $\mathbb{F}_1$'.

The notion of 'chamber complex' allows Tits to make several notions rigorous such as the existence of a well defined *retraction* map from the geometry of G to the geometry of W. Tits also obtains an axiomatic description of the geometry of G, called a *building*, as a thick chamber complex.

IS THERE A FIELD WITH 1 ELEMENT?

To construct a field, we need 2 elements.

However the mythical field with 1 element, denoted $\mathbb{F}_1$, has found a place in

◦ Algebraic geometry (Deitmar, Kapranov and Smirnov, Manin, Soulé)
◦ Noncommutative geometry (Connes, Consani and Marcolli)
◦ Arakelov geometry (Durov), and
◦ The geometric interpretation of zeros of zeta and L-functions (Connes, Consani, Marcolli, Manin).

Although the field structure of $\mathbb{F}_1$ *is degenerate, groups and geometries defined over* $\mathbb{F}_1$ *do not degenerate.*

The first appearance of $\mathbb{F}_1$

The first reference in the mathematical literature to the field $\mathbb{F}_1$ of 'characteristic 1' seems to have been the paper 'Sur les analogues algebriques des groupes semi-simples complexes', Colloque d'Algebre superieure [1956, Bruxelles] by Jacques Tits.

Roughly speaking, if G is a group that is defined over a finite field $\mathbb{F}_q$ and G has a Dynkin diagram $\mathcal{D}$, then G has an associated 'Tits geometry' X such that G is 'essentially' the group of automorphisms of X.

It makes sense to talk about G and X in the limit $\mathbb{F}_q \longrightarrow \mathbb{F}_1$ in such a way that G inherits a discrete structure and X inherits the geometry of this discrete structure.

Although the field structure of $\mathbb{F}_1$ is degenerate, groups and geometries defined over $\mathbb{F}_1$ do not degenerate.

These ideas are deeply embedded in the work of Tits concerning finite geometries, buildings and Chevalley and Kac-Moody group functors.

Finite dimensional simple algebraic groups

Let G be a finite dimensional simple algebraic group and let $\mathfrak{g}$ denote its Lie algebra.

The classical types are A_n, B_n, C_n and D_n, and the exceptional types are G_2, F_4, E_6, E_7 and E_8.

How do we define G over a field K such as $\mathbb{F}_q$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, $\mathbb{F}_q((t))$, or a ring such as $\mathbb{Z}$?

Some constructions of classical and exceptional groups over arbitrary fields were known before the 1950s by Jordan (1870), Dickson (1901) and Dieudonné (1948), but a unified method for their construction was missing.

$\mathbb{Z}$-forms and Chevalley groups

By a $\mathbb{Z}$-*form* of a $\mathbb{C}$-algebra, $\mathfrak{g}_{\mathbb{C}}$, we mean a subring $\mathfrak{g}_{\mathbb{Z}}$ of $\mathfrak{g}_{\mathbb{C}}$ such that the canonical map

$$\mathfrak{g}_{\mathbb{Z}} \otimes \mathbb{C} \longrightarrow \mathfrak{g}_{\mathbb{C}}$$

is bijective.

In 1955 Chevalley constructed a $\mathbb{Z}$-form $\mathcal{U}_{\mathbb{Z}}$ of the universal enveloping algebra $\mathcal{U}$ of a complex simple Lie algebra. He then defined a $\mathbb{Z}$-form of $\mathfrak{g}$:

$$\mathfrak{g}_{\mathbb{Z}} = \mathfrak{g}_{\mathbb{C}} \cap \mathcal{U}_{\mathbb{Z}}.$$

For K an arbitrary field, we set

$$\mathcal{U}_K = \mathcal{U}_{\mathbb{Z}} \otimes K$$

$$\mathfrak{g}_K = \mathfrak{g}_{\mathbb{Z}} \otimes K.$$

A *Chevalley group of adjoint type* is a group G_K generated by elements of $Aut(\mathfrak{g}_K)$.

Chevalley groups and Steinberg groups

If K is a field, a *Chevalley group of adjoint type* is a group G_K generated by elements of $Aut(\mathfrak{g}_K)$.

The Chevalley groups G_K give an alternate construction of simple Lie groups. Namely Chevalley gave a unified way of describing a matrix group G by generators which are given in terms of the root system and automorphisms of the Lie algebra.

This gave the first unified construction of classical groups over fields other than $\mathbb{R}$ and $\mathbb{C}$, and also gave groups associated to E_6, E_7, E_8, F_4, and G_2 over finite fields.

Chevalley's construction did not give all of the known classical groups. For example it did not include the unitary groups and the non-split orthogonal groups. Steinberg found a modification of Chevalley's construction that gave these groups and some additional families. Steinberg also gave a complete set of generators and relations for Chevalley's groups, known as the Steinberg presentation.

Tits geometries

Motivated by trying to find a 'geometric' interpretation of a finite dimensional simple Lie group G in contrast to the 'algebraic' version of G proposed by Chevalley the previous year, in 1956 Tits introduced a 'geometry' X which has G as its automorphism group.

Tits' geometry associated to a finite dimensional simple algebraic group G was a precursor to the notion of a spherical building for a Chevalley group over a finite field, leading later to the notion of a BN-pair and Bruhat-Tits affine building of a simple algebraic group over a nonarchimedean local field.

From our current viewpoint, this leads to the following correspondence:

Chevalley groups over $\mathbb{F}_1 \leftrightarrow$ *discrete groups*

Tits geometries over $\mathbb{F}_1 \leftrightarrow$ *apartments of Tits buildings*

Tits in his own words

‘Interview with John G. Thompson and Jacques Tits’, Martin Raussen and Christian Skau, Notices of the AMS, Volume 56, No 4, April 2009, 471–478

‘I studied these objects because I wanted to understand these exceptional Lie groups geometrically. In fact, I came to mathematics through projective geometry; what I knew about was projective geometry. In projective geometry you have points, lines, and so on. When I started studying exceptional groups I sort of looked for objects of the same sort. For instance, I discovered - or somebody else discovered, actually - that the group E_6 *is the collineation group of the octonion projective plane. And a little bit later, I found some automatic way of proving such results, starting from the group to reconstruct the projective plane. I could use this procedure to give geometric interpretations of the other exceptional groups, e.g.,* E_7 *and* E_8*. That was really my starting point. Then I tried to make an abstract construction of these geometries.’*

Weyl group

Let G be a simple algebraic group over a field K and let $\underline{G}$ denote the corresponding Chevalley group scheme.

An n-dimensional torus T is an algebraic group isomorphic to $(K^{\times})^n$ over the algebraic closure $\overline{K}$. The torus T is split over K if the isomorphism $(K^{\times})^n$ is defined over K. A torus is maximal if for any other torus T' with $T \leq T'$ we have $T = T'$.

Let W be the Weyl group of G, defined as $W = N(T)/Z(T)$ where T is a maximal torus in G, $N(T)$ and $Z(T)$ are the normalizer and the centralizer of T in G. Since T is commutative, T is self-centralizing, thus $W = N(T)/T$.

If Φ is the root system of G, then W is a subgroup of the isometry group of Φ. Specifically, it is the subgroup which is generated by reflections in the hyperplanes orthogonal to the roots.

Tits geometries over $\mathbb{C}$ (Tits, 1956)

Let $G = PGL_{n+1}(\mathbb{C})$. Then the Tits' geometry for G is n-dimensional projective geometry $\mathcal{P}_n$ over $\mathbb{C}$.

This consists of subspaces $\mathcal{P}_i \subseteq \mathcal{P}_n,\ i = 0, \ldots, n$, such that

$\mathcal{P}_0$ = 'points' = 1-dim subspaces of $\mathbb{C}^{n+1}$,
$\mathcal{P}_1$ = 'lines' = 2-dim subspaces of $\mathbb{C}^{n+1}$, ...,
$\mathcal{P}_{n-1}$ = 'hyperplanes' = n-dim subspaces of $\mathbb{C}^{n+1}$,

with incidence given by inclusion as subspaces of $\mathbb{C}^{n+1}$. We have subgroups G_i of G

G_0 = stabilizer of a point,
G_1 = stabilizer of a line, ...
G_{n-1} = stabilizer of a hyperplane,

and families $\mathcal{F}_i$

$\mathcal{F}_0 = G/G_0 \leftrightarrow$ points,
$\mathcal{F}_1 = G/G_1 \leftrightarrow$ lines, ...
$\mathcal{F}_{n-1} = G/G_{n-1} \leftrightarrow$ hyperplanes

which inherit the incidence relation.

The group G is then the group of automorphisms of this geometry preserving families $\mathcal{F}_i$ and incidence.

Tits geometries over finite fields (Tits, 1956)

Let $G = PGL_3(\mathbb{F}_2)$. Then G is a simple Lie group of type A_2 and order

$$2^3(2^3 - 1)(2^2 - 1) \ = \ 168$$

and W is the dihedral group of order 6, the group of type preserving automorphisms of a hexagon whose vertices have 2 types.

The Tits geometry X for G is the flag complex of a projective plane over $\mathbb{F}_2$. A projective plane is a 2 dimensional incidence geometry of points $\mathcal{P}_0$ and lines $\mathcal{P}_1$ satisfying the usual axioms:

$$\mathcal{P}_0 = \text{'points'} = \text{1-dim subspaces of } \mathbb{F}_2^3,$$
$$\mathcal{P}_1 = \text{'lines'} = \text{2-dim subspaces of } \mathbb{F}_2^3,$$

with incidence given by inclusion - a point $p \in \mathcal{P}_0$ is incident on a line $L \in \mathcal{P}_1$ if $p \subset L$ as subspaces of $\mathbb{F}_2^3$.

The flag complex is a graph where adjacent vertices correspond to pairwise incident elements.

points $\leftrightarrow \ G/G_0$,

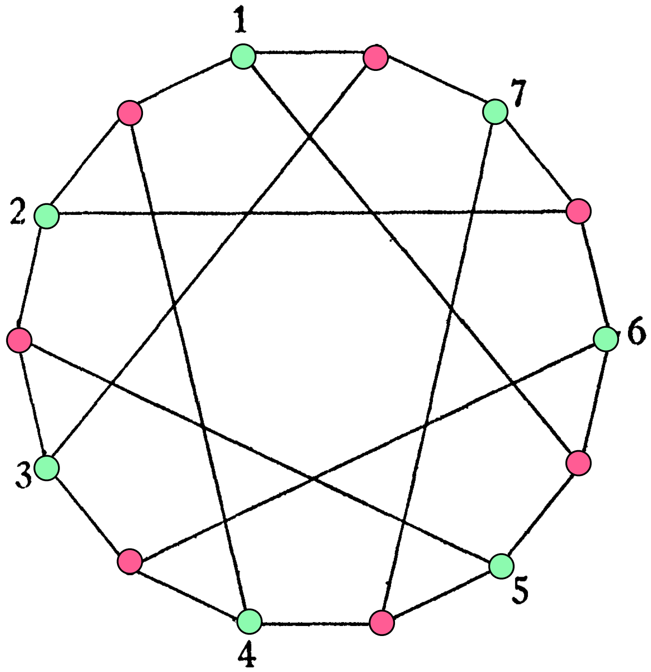
lines $\leftrightarrow \ G/G_1$,

$G_0 =$ stabilizer of a point,
$G_1 =$ stabilizer of a line.

The Tits geometry for $G = PGL_3(\mathbb{F}_2)$

In the flag complex of a projective plane over $\mathbb{F}_2$, there are 7 points, 7 lines, 14 vertices and 21 edges. Each basis of $\mathbb{F}_2^3$ determines a closed path in X. There are 28 possible bases of $\mathbb{F}_2^3$, hence 28 hexagons.

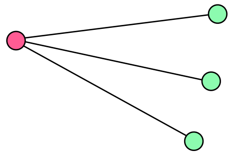
The underlying diagram is taken from Coxeter's paper Self-dual configurations and regular graphs Bull. Amer. Math. Soc. 56 (1950), 413-455, where he names it the '6-cage'.



Local picture in the Tits geometry for $G = PGL_3(\mathbb{F}_2)$

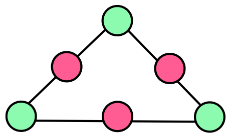
When $q = 2$, we can try to understand the limit $\mathbb{F}_2 \longrightarrow \mathbb{F}_1$ by looking at the local picture in the geometry X of G.

If the vertex v represents a point, then

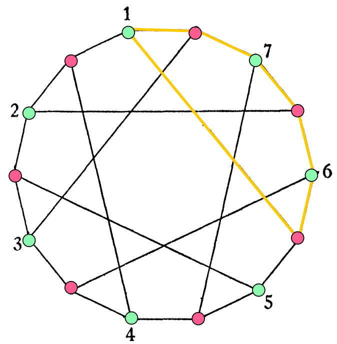
$Star(v) =$ 

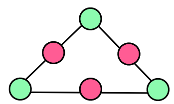
$| \, Star(v) \, | =$ no. of 1 dim subspaces of the 2 dim space $\mathbb{F}_2^2 = | \, \mathbb{P}^1(\mathbb{F}_2) \, | = 3$

As $\mathbb{F}_2 \longrightarrow \mathbb{F}_1$, $| \, Star(v) \, | \longrightarrow | \, \mathbb{P}^1(\mathbb{F}_1) \, | = 2$

Geometry of $PGL_3(\mathbb{F}_2) \longrightarrow$ 

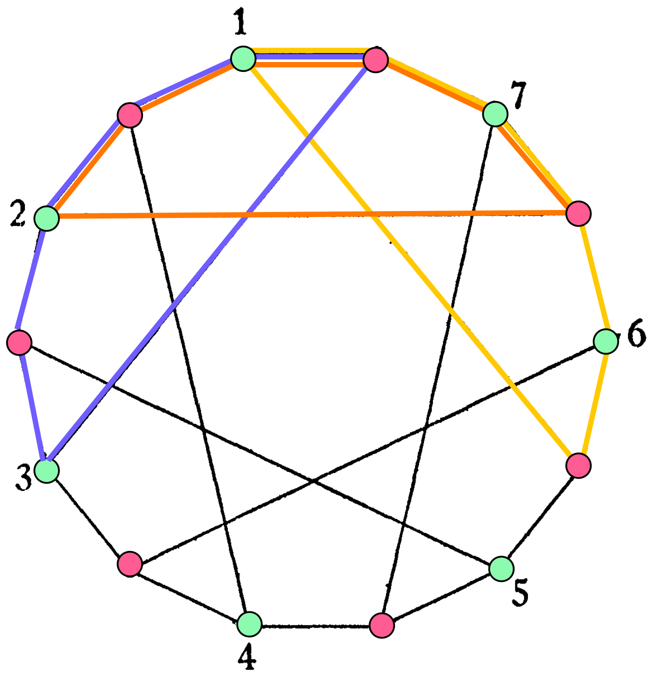
This is the Geometry of W, the dihedral group of order 6.



And $PGL_3(\mathbb{F}_2) \longrightarrow$ group of type preserving automorphisms of 

Thus $PGL_3(\mathbb{F}_2) \longrightarrow W$.

Conversely we can think of $X = \Gamma_{\mathbb{F}_2}(G)$ as gluing together copies of the geometry over $\mathbb{F}_1$: 3 hexagons along each edge



Gluing is done according to the local structure which is isomorphic to $\mathbb{P}^1(\mathbb{F}_q)$ and the action of the Weyl group on the geometry over $\mathbb{F}_1$

Tits geometry of $G \longrightarrow$ Tits geometry of W

This limit is not well-defined. Tits knew that he needed a stronger notion of 're-traction' in order to show that the geometry over $\mathbb{F}_1$ is *intrinsic* and independent of the form of the group or the base field.

Tits achieved this later by axiomatizing the notion of a building as a certain type of simplicial complex called a chamber complex.

Tits also made his ideas more rigorous by connecting his work to Coxeter's work on reflection groups. In what follows, we will find the following correspondence

Weyl group $W \leftrightarrow$ *Coxeter group*

Geometry of W *over* $\mathbb{F}_1 \leftrightarrow$ *Coxeter complex*

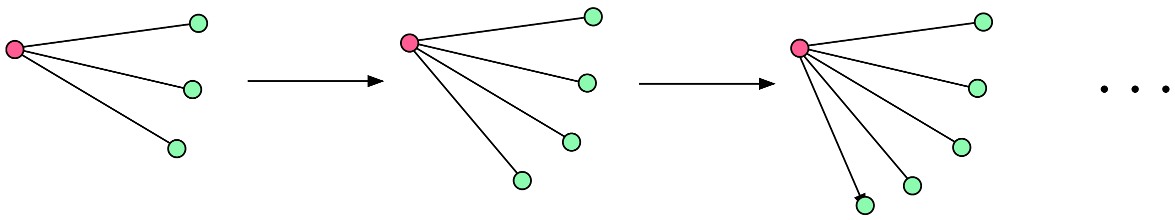
Tits geometries for $GL_3(\mathbb{F}_p)$, p a prime

The Tits geometry $\Gamma_{\mathbb{F}_p}(G)$ for $G = GL_3(\mathbb{F}_p)$ is the flag complex of a projective plane over $\mathbb{F}_p$ and has $1 + p + p^2$ points (similarly $1 + p + p^2$ lines) when p is prime. This means 7 points and 7 lines when $p = 2$, 13 points and 13 lines when $p = 3$, and 31 points and 31 lines when $p = 5$.

In spite of the fact that they each have the same geometry over $\mathbb{F}_1$ *(are each formed by gluing hexagons along edges) it seems that the axioms of incidence geometry prevent embeddings of* $\Gamma_{\mathbb{F}_2}(G)$ *into* $\Gamma_{\mathbb{F}_3}(G)$ *or of* $\Gamma_{\mathbb{F}_3}(G)$ *into* $\Gamma_{\mathbb{F}_5}(G)$*.*

However $\Gamma_{\mathbb{F}_2}(G)$ and $\Gamma_{\mathbb{F}_3}(G)$ have a connected subcomplex in common which includes at least 3 hexagons.

We also have local embeddings of stars



$$\mathbb{P}^1(\mathbb{F}_2) \quad \longrightarrow \quad \mathbb{P}^1(\mathbb{F}_3) \quad \longrightarrow \quad \mathbb{P}^1(\mathbb{F}_5) \quad \longrightarrow \quad \dots$$

and maps

$$\Gamma_{\mathbb{F}_2}(GL_3(\mathbb{F}_2)) \longrightarrow \Gamma_{\mathbb{F}_3}(GL_3(\mathbb{F}_3)) \longrightarrow \Gamma_{\mathbb{F}_5}(GL_3(\mathbb{F}_5)) \longrightarrow \dots$$

And this corresponds on the group level to embeddings of cosets

$$P_i(\mathbb{F}_2)/B(\mathbb{F}_2) \hookrightarrow P_i(\mathbb{F}_3)/B(\mathbb{F}_3) \hookrightarrow P_i(\mathbb{F}_5)/B(\mathbb{F}_5)\dots$$

for $i = 1, 2$.

Coxeter groups and Coxeter complexes

A Coxeter group can be defined as a group with the presentation

$$\langle r_1, r_2, \ldots, r_n \mid (r_i r_j)^{m_{ij}} = 1\rangle$$

where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. Since r_i has order 2, each r_i is an involution.

The Weyl group W of a Chevalley group G is a Coxeter group.

We may also define Coxeter groups by geometric representations in which the group acts discretely on a certain domain and in which the generators are represented by reflections.

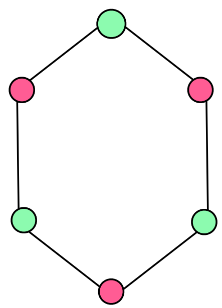
Consider a Coxeter group W and its reflection representation on a metric space. The cells in the cell decomposition associated to the action of W are simplices, and we obtain a simplicial complex $\Sigma(W)$ triangulating the reflection space. This is the *Coxeter complex* associated with W.

The terms 'Coxeter groups' and 'Coxeter complex' were first used by J. Tits, *Groupes et géométries de Coxeter* (1961), unpublished.

Coxeter complexes - examples

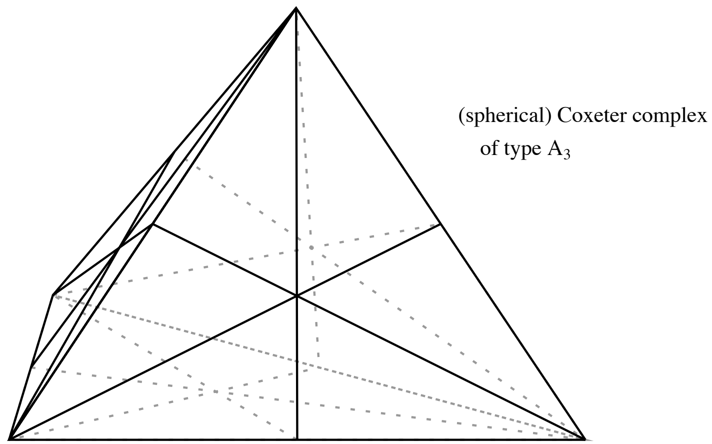
Regular polytopes are higher dimensional analogs of regular polygons. Whenever our Coxeter group is naturally the symmetries of a polytope, we can get the Coxeter complex by 'barycentrically subdividing' the surface of this polytope.

For example the Weyl group of PGL_3 is the symmetry group of the triangle so we can get its Coxeter complex by barycentrically subdividing the sides of the triangle. We get a hexagonal graph with 6 edges.



Coxeter complexes - examples

The Weyl group of PGL_4 is the symmetry group of the tetrahedron, so we can get its Coxeter complex by barycentrically subdividing the surface of the tetrahedron, obtaining a shape with 24 triangles.



The number of maximal (top-dimensional) simplices in the Coxeter complex is the same as the number of elements in the Coxeter group. If we choose any maximal simplex in the Coxeter complex, there always exists a unique element of the Coxeter group that maps it to any other maximal simplex.

The full Tits building is glued together from multiple copies of the Coxeter complex, called its apartments, in a certain regular fashion.

Tits building axioms, (1965)

A building is a simplicial complex X *that can be expressed as the union of subcomplexes* Σ *(called apartments) satisfying the following axioms:*

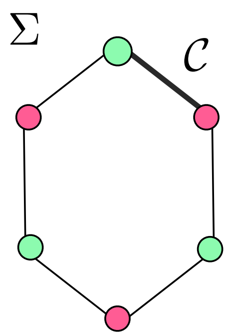
(B0) Each apartment Σ is a Coxeter complex of the same dimension d which also equals $dim(X)$.

(B1) For any two simplices σ and ω there is an apartment Σ containing both of them.

(B2) If Σ and Σ' are two apartments containing σ and ω, then there is an isomorphism $\Sigma \longrightarrow \Sigma'$ fixing σ and ω pointwise.

It follows that all apartments are isomorphic.

The flag complex of the projective plane over $\mathbb{F}_2$ is a building with apartments that look like



These axioms were first stated in 1963 by Tits in terms of incidence geometries instead of simplicial complexes. A reformulation of the axioms in terms of simplicial complexes was then given by Tits in 1965.

Homotopy type

Let G be a finite dimensional simple algebraic group of rank $n \geq 3$ over a field K. Let X be the Tits building of G.

Theorem (Solomon-Tits). *If W is finite, X has the homotopy type of a wedge of $(n-1)$-spheres.*

Thus if W is finite, X is called a *spherical building*.

Affine building

We shall see later on that if K is a field with a discrete valuation

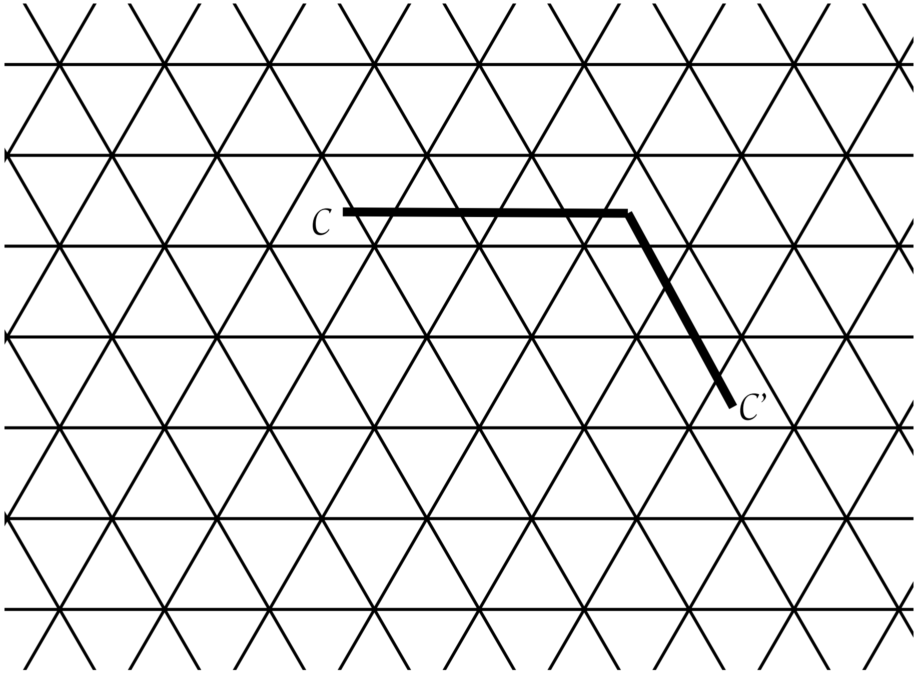
$$v : K \longrightarrow \mathbb{Z} \cup \{\infty\}$$

then G has a second type of building called an affine building corresponding to an infinite Weyl group. The Solomon-Tits theorem also tells us that if W is infinite, X is contractible.

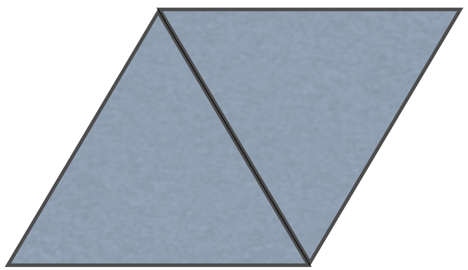
Chamber complexes

Let X be a simplicial complex such that all maximal simplices, called *chambers* have the same dimension d. Two chambers $\mathcal{C}$, $\mathcal{C}'$, are adjacent if $\mathcal{C} \cap \mathcal{C}'$ has codimension 1.

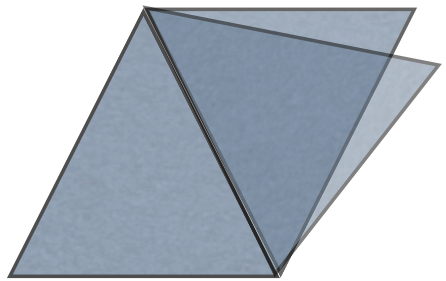
The simplicial complex X is called a *chamber complex* if any two chambers are connected by a *gallery* which is a sequence of adjacent chambers.



A chamber complex is *thin* (respectively *thick*) if each simplex of codimension 1 is a face of exactly 2 (respectively at least 3) chambers.



Thin



Thick

Coxeter complexes are buildings which are thin as simplicial complexes.

A building is a thick chamber complex.

Retractions

The geometry of a simple algebraic group G over $\mathbb{F}_1$ is *intrinsic* and independent of the form of the group or the choice of base field. For example, both SL_3 and PGL_3 have the geometry of a barycentrically subdivided equilateral triangle over $\mathbb{F}_1$.

The following construction of Tits allows us to establish this rigorously, though the notion of geometry over $\mathbb{F}_1$ disappeared from the writings of Tits after his 1956 paper.

Let X be a Tits building. Let Σ be an apartment, let $\mathcal{C} \in \Sigma$ be a chamber (maximal simplex) contained in Σ.

There is a unique chamber map $\rho = \rho(\Sigma, \mathcal{C}) : X \longrightarrow \Sigma$ which fixes $\mathcal{C}$ pointwise and maps every apartment containing $\mathcal{C}$ isomorphically onto Σ.

The retraction map ρ preserves distances from $\mathcal{C}$ and preserves colors of vertices.

GROUP FUNCTORS, FIELDS AND TITS GEOMETRIES

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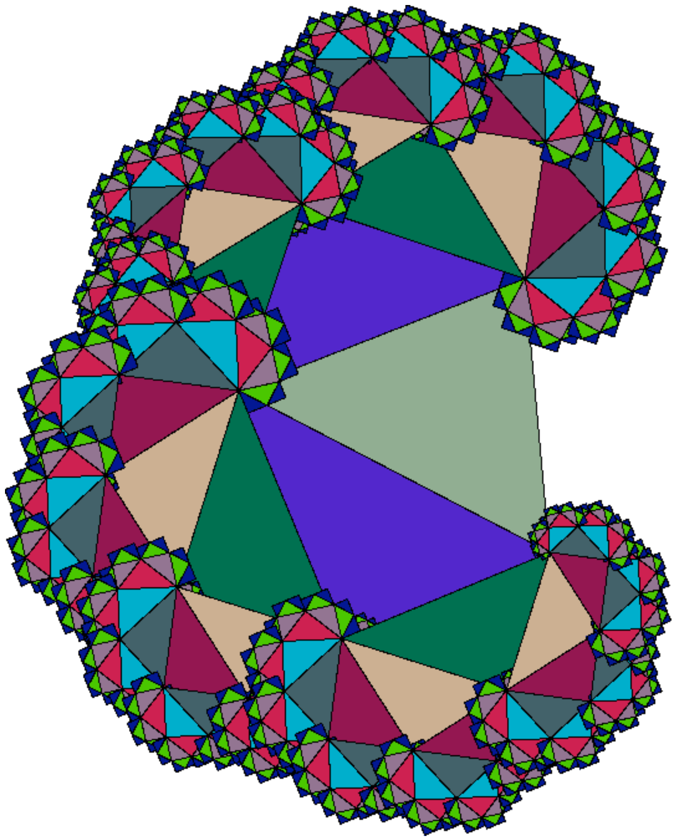
LECTURE II.
BN-PAIRS AND BRUHAT-TITS AFFINE BUILDINGS



This picture by Paul Garrett shows a region of a 2 dimensional affine building.

II. BN-PAIRS AND BRUHAT-TITS AFFINE BUILDINGS

Tits studies the work of Bruhat who decomposed a finite dimensional simple algebraic group G as a union of double cosets indexed over the Weyl group.

Tits revealed a highly structured relationship between groups and geometry, namely BN-pairs and their associated buildings.

Bruhat's work was also used by Iwahori and Matsumoto to give Bruhat decompositions for Chevalley groups over p-adic fields.

This eventually led to Tits' joint work with Bruhat that associated a second type of building, known as an *affine building*, to groups over fields with a discrete valuation.

Last time we constructed a well defined simplicial retraction from a building X onto any of its apartments Σ.

Using Tits' axiomatic description of buildings and the Bruhat decomposition, we obtain a rigorous proof that if G is a group with a BN-pair (B, N), then there is a well defined 'degeneration' map $G \longrightarrow W$.

In joint work with Katia Consani, this will allow us to merge Tits' theory of BN-pairs with Connes and Consani's proof that Chevalley group schemes are varieties over $\mathbb{F}_1$ with group structure defined over $\mathbb{F}_{1^2}$.

Double coset spaces and Bruhat decomposition

Let G be a groups and let H and K be subgroups. The coset space G/K has a left G-action, the coset space $H\backslash G$ has a right G-action. The double coset space $H\backslash G/K$ has no additional structure.

For a finite dimensional simple algebraic group G with Borel subgroup B, in 1954 Bruhat discovered that the double coset space

$$B\backslash G/B,$$

is finite, and there is a natural bijective correspondence between the sets

$$B\backslash G/B \cong W.$$

Thus $G = BWB$ and in fact G has a decomposition

$$G = \bigsqcup_{w \in W} BwB.$$

That is, each $g \in G$ lies in a unique Bruhat cell BwB.

Bruhat's decomposition of finite dimensional simple algebraic groups over $\mathbb{F}_q$ *was used extensively by Chevalley in his construction of new finite simple groups.*

Bruhat's work was also used by Iwahori and Matsumoto to give Bruhat decompositions for Chevalley groups over p-adic fields.

This eventually led to Tits' joint work with Bruhat that revealed a highly structured relationship between groups and geometry, namely BN*-pairs. In particular, Bruhat and Tits showed that any group with a* BN*-pair admits a Bruhat decomposition.*

Motivating example: The BN-pair for $PGL_3(K)$

Let X be the Tits building for $PGL_3(K)$, K a field.

Fix a chamber (maximal simplex) $\mathcal{C}$ in a fixed apartment Σ.

Let W denote the group of type-preserving automorphisms of Σ.

Let S be the set of reflections in the codimension-1 faces of $\mathcal{C}$.

Then W is a Coxeter group generated by S, and Σ is naturally identifiable with the associated Coxeter complex.

The subgroups B and N of G are defined as follows:

$$B = \{g \in G \mid g\mathcal{C} = \mathcal{C}\} = \text{upper triangular subgroup of } PGL_3(K),$$

$$N = \{g \in G \mid g\Sigma = \Sigma\} = \text{monomial subgroup of } PGL_3(K).$$

Also

$$T = B \cap N$$

is a maximal torus.

Axioms for a BN-pair

A collection (G, B, N, S) of data is called a *BN-pair*, or *Tits system* if G is a group, B and N are subgroups, S is a subset of $W = N/(B \cap N)$, and the data satisfies the following axioms:

(T1) $B \cup N$ generates G, and $B \cap N$ is normal in N,
(T2) S generates $W = N/(B \cap N)$, and S consists of elements of order 2, each different from the identity,
(T3) $sBw \subset BwB \cup BswB$, $s \in S$, $w \in W$,
(T4) for each $s \in S$, sBs is not contained in B.

The group W is called the *Weyl group* of the Tits system. The axioms imply a *Bruhat decomposition* of G:

$$G = \bigsqcup_{w \in W} BwB.$$

The building of a BN-pair

Let G be a group with a BN-pair, with Weyl group W and Bruhat decomposition $\sqcup_{w \in W} BwB$. Let X be the Tits building of G. Let $S = \{w_1, w_2, \ldots, w_\ell\}$ be the generating set for W.

The chambers of X are the maximal simplices of X and are in bijective correspondence with G/B. Let

$$P_i = \sqcup_{w \in \langle S \setminus \{w_i\} \rangle} BwB.$$

The P_i are the *maximal standard parabolic subgroups* of G. Then the vertex set of X is $\sqcup_i G/P_i$.

The incidence relation is described as follows. The $r+1$ vertices $Q_1, \ldots, Q_{r+1}$ span an r-simplex if and only if the intersection $Q_1 \cap \cdots \cap Q_{r+1}$ is parabolic, that is, contains a conjugate of B.

Let $\mathcal{C}$ be the *standard simplex*, spanned by the P_i viewed as cosets for the identity element of G. The *standard apartment* Σ then consists of W-translates of $\mathcal{C}$. Then

$$B = \{g \in G \mid g\mathcal{C} = \mathcal{C}\},$$

$$N = \{g \in G \mid g\Sigma = \Sigma\}.$$

If W is finite, X is called a *spherical building*. If W is infinite, X is called an *affine building*.

Action of G on its building

The group G acts on its building X by left translation on cosets and with quotient a maximal simplex. That is, G acts transitively on maximal simplices and has an orbit of vertices for each maximal standard parabolic subgroup.

Stabilizers of chambers then correspond to conjugates of B and stabilizers of apartments correspond to conjugates of N.

It follows that the stabilizers of apartments containing a given chamber correspond to conjugates of T, the maximal tori.

Thus we have a correspondence between maximal tori and stabilizers of apartments containing a given chamber.

Strong transitivity of the action of a BN-pair on its building

Let X be the building of a BN-pair for a group G. Then the action of G on X is *strongly transitive*. That is, G acts transitively on pairs $(\Sigma, \mathcal{C})$, where Σ is an apartment and $\mathcal{C}$ is a chamber contained in Σ.

This means that G acts transitively on the set of apartments and if Σ is an apartment, then the stabilizer of Σ acts transitively on the chambers of Σ. Equivalently G is transitive on the set of chambers and if $\mathcal{C}$ is a chamber, then the stabilizer of $\mathcal{C}$ acts transitively on the apartments containing $\mathcal{C}$.

As before, B is the stabilizer of the standard simplex $\mathcal{C}$ and N is the stabilizer of the standard apartment $\Sigma = W\mathcal{C}$.

Suppose conversely that G acts strongly transitively on a building X. Let $\mathcal{C}$ denote the standard simplex and Σ the standard apartment. Let B denote the stabilizer in G of $\mathcal{C}$ and let N denote a subgroup of G that stabilizes Σ and is transitive on the chambers of Σ. Then (B, N) is a BN-pair for G whose building is canonically isomorphic to X.

Classification of spherical buildings

Theorem (Borel and Tits, 1972). Every reductive algebraic group G over a field K gives rise to a BN-pair with finite Weyl group W from which one obtains a spherical building X for G.

Theorem (Tits, 1974). Every thick spherical building of dimension ≥ 2 comes from a BN-pair of a reductive algebraic group G over a field K.

Retraction of a building onto an apartment and G onto W

Theorem. Let G be a group with a BN-pair, (B, N), with Weyl group W and Bruhat decomposition $\sqcup_{w\in W} BwB$. Let X be the Tits building of G. Let Σ be an apartment of X, let $\mathcal{C} \in \Sigma$ be a chamber contained in Σ. Then

(1) There is a unique chamber map $\rho = \rho(\Sigma, \mathcal{C}) : X \longrightarrow \Sigma$ which is a retraction of X onto Σ, which fixes $\mathcal{C}$ pointwise and maps every apartment containing $\mathcal{C}$ isomorphically onto Σ, preserving distance from $\mathcal{C}$.

(2) There is an induced map

$$\overline{\rho} : G \longrightarrow W$$

with

$$g \mapsto w$$

where w is the unique element of W such that $\rho(g\mathcal{C}) = w\mathcal{C}$, and

$$BwB \mapsto w.$$

(3) If n is a chosen lift of w to N, then there is a well defined map

$$\begin{array}{ccc} BnB & \longrightarrow & n \in N \\ & \searrow & \downarrow \\ & & w \in W \end{array}$$

where w is the unique element of W such that $\rho(g\mathcal{C}) = w\mathcal{C}$.

GROUP FUNCTORS, FIELDS AND TITS GEOMETRIES

Lisa Carbone, Rutgers University

LECTURE III.
APPLICATIONS OF TITS' THEORY OF BUILDINGS

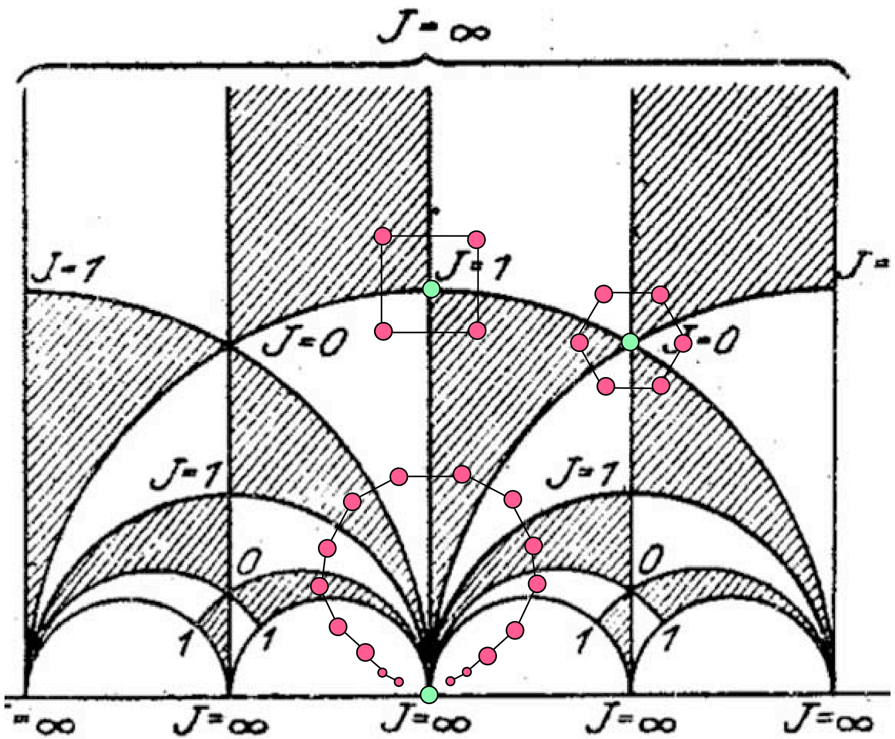


Fig. 7.

The underlying diagram is Figure 7 in Felix Klein's paper 'Uber die Transformation der elliptischen Funktionen und die Auflosung der Gleichungen funften Grades' which appeared in May 1878 in Mathematische Annalen. This shows the standard apartment of the Tits building of a hyperbolic Kac-Moody group whose Weyl group is $PGL_2(\mathbb{Z})$.

III. APPLICATIONS OF TITS' THEORY OF BUILDINGS

(1) The Connes-Consani graded functor for Chevalley groups over $\mathbb{F}_{1^2}$

(2) Affine buildings for groups over fields with a discrete valuation

(3) Tits functor for Kac-Moody groups and hyperbolic buildings

(4) Fields: discrete to continuous

Retraction of a building onto an apartment and G onto W

Theorem. Let G be a group with a BN-pair, (B, N), with Weyl group W and Bruhat decomposition $\sqcup_{w \in W} BwB$. Let X be the Tits building of G. Let Σ be an apartment of X, let $\mathcal{C} \in \Sigma$ be a chamber contained in Σ. Then

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where w is the unique element of W such that $\rho(g\mathcal{C}) = w\mathcal{C}$.

The Connes - Consani graded functor for Chevalley group schemes

We recall that Tits first discovered the notion of a building as a 'geometry' X associated to a finite dimensional simple algebraic group G which has G as its automorphism group. For G a Chevalley group over a finite field $\mathbb{F}_q$, Tits indicated that one could take the limit $q \longrightarrow 1$ in which G tends to the Weyl group W of G and the geometry X tends to the geometry of W.

In order to answer a question of Soulé, Connes and Consani aimed to find explicit algebro-geometric descriptions of Chevalley group schemes over $\mathbb{F}_1$ compatible with Tits' geometries over $\mathbb{F}_1$. They showed that Chevalley group schemes $\underline{G}$ have a model G which is a variety over $\mathbb{F}_{1^2}$ such that

$$G \otimes_{\mathbb{F}_{1^2}} \mathbb{Z} \cong \underline{G},$$

and such that the normalizer N (as a sub-group-scheme) of a maximal torus T has a model $\mathcal{N}$ which is an algebraic group over $\mathbb{F}_{1^2}$. This means that given the multiplication map $m : N \times N \longrightarrow N$, there exists a multiplication map $\mu : \mathcal{N} \times \mathcal{N} \longrightarrow \mathcal{N}$ and an isomorphism $\mathcal{N}_{\mathbb{Z}} \cong N$, where

$$\mathcal{N}_{\mathbb{Z}} = \mathcal{N} \otimes_{\mathbb{F}_{1^2}} \mathbb{Z},$$

such that the diagram

$$\begin{array}{ccccc} \mathcal{N}_{\mathbb{Z}} & \times & \mathcal{N}_{\mathbb{Z}} & \xrightarrow{\mu_{\mathbb{Z}}} & \mathcal{N}_{\mathbb{Z}} \\ \downarrow & & \downarrow & & \downarrow \\ N & \times & N & \xrightarrow{m} & N \end{array}$$

commutes.

Let $\mathcal{G}$ denote the 'gadget' over $\mathbb{F}_{1^2}$ constructed by Connes and Consani associated to a Chevalley group G. This is a graded set which is empty in all grades less than $\ell = rank(G)$, and in grade ℓ is a group; namely a quadratic extension of W known as the *extended Weyl group* of Tits.

Moreover $\mathcal{G}$ defines a variety over $\mathbb{F}_{1^2}$. We write $\mathcal{G} = \mathcal{G}(\mathbb{F}_{1^2})$. Then there is a natural map

$$\mathcal{N}_{\mathbb{Z}} \longrightarrow \mathcal{G}(\mathbb{F}_{1^2}).$$

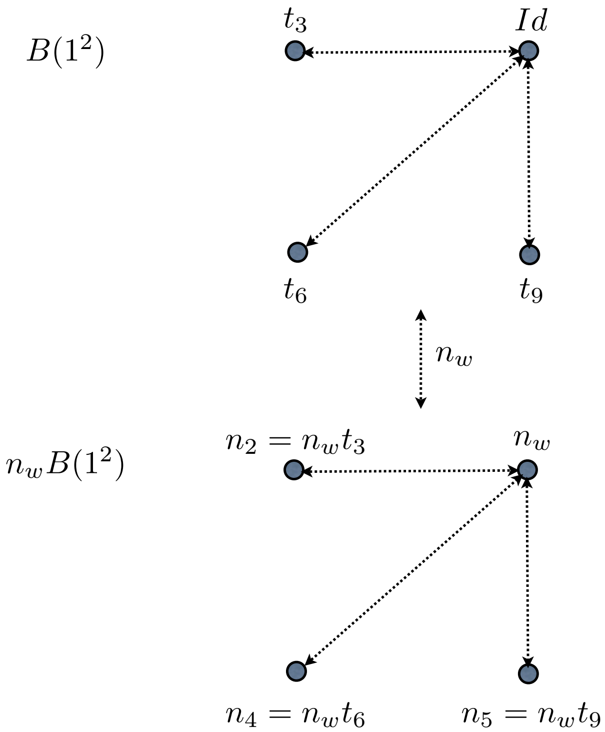
Example - Chevalley groups over $\mathbb{F}_{1^2}$

Recall that Tits showed that $G = PGL_3(\mathbb{F}_1)$ has the finite geometry of a triangle with an action of the symmetric group on 3 letters.

Connes and Consani showed the Chevalley group scheme over $\mathbb{F}_{1^2}$ associated to PGL_3 is described a graded functor which is empty in all grades less than $\ell = rank(PGL_3)$, and in grade ℓ is a group, namely a non trivial functorial extension of the Weyl group S_3 by $Hom(L, -)$, where L is a lattice (the character group of a maximal torus). The description of this functorial extension over $\mathbb{Z}$ gives an extension of S_3 by $(\mathbb{Z}/2\mathbb{Z})^{\ell}$, $\ell = rk(G) = 2$.

Example - Chevalley groups over $\mathbb{F}_{1^2}$

The notion of a Tits building can be associated to $G = GL_2(\mathbb{F}_{1^2})$. This geometry is a degenerate configuration of points, which may be depicted in the following way:



Bruhat-Tits affine buildings

We have seen that every reductive algebraic group G over a field K gives rise to a BN-pair with finite Weyl group W from which one obtains a spherical building X for G.

In the case that K is a field with a discrete valuation, one can associate to G a second type of building called an affine building whose apartments are Euclidean planes and whose Weyl group is infinite.

This was first observed by Iwahori and Matsumoto in the mid 1960s who gave Bruhat decompositions for Chevalley groups over p-adic fields and developed the BN-pairs for such groups.

The general theory of affine buildings was developed by Bruhat and Tits in 1972. They developed 'affine' BN-pairs for reductive algebraic groups over nonarchimedean local fields, as well as the simplicial structure of their affine buildings and interconnections with the spherical BN-pairs and buildings.

Fields with a discrete valuation

Let K be a field. A *discrete valuation* on K is a surjective homomorphism $\nu : K^\times \longrightarrow \mathbb{Z}$ satisfying

$$\nu(x+y) \geq \min\{\nu(x), \nu(y)\},$$

for $x, y \in K$, with the convention that $\nu(0) = \infty$.

Let $\mathcal{O}$ denote the ring of integers of K, that is

$$\mathcal{O} = \{x \in K \mid v(x) \geq 0\}.$$

Let $\pi \in K$ be such that $v(\pi) = 1$ and let k denote the quotient ring $\mathcal{O}/\pi\mathcal{O}$ which is a field, called the *residue class field.*

A *non-archimedean local field* is a field that is complete with respect to a discrete valuation and whose residue class field is finite.

The nonarchimedean local fields have been classified. They are the p-adic numbers $\mathbb{Q}_p$, the finite extensions of $\mathbb{Q}_p$, and the fields of formal Laurent series $\mathbb{F}_q((t))$.

‘Lifting’ a BN-pair

Let K be a field with a discrete valuation ν, let $\mathcal{O}$ denote the ring of integers of K and let $k = \mathcal{O}/\pi\mathcal{O}$ denote the residue class field.

The ring of integers $\mathcal{O}$ serves as an intermediary between K and k by virtue of the inclusion and quotient maps:

$$\begin{array}{ccc} \mathcal{O} & \hookrightarrow & K \\ \downarrow & & \\ k & & \end{array}$$

We will construct an affine building for a Chevalley group scheme $\underline{G}$ over K by lifting a spherical BN-pair for $\underline{G}$ over k to an affine BN-pair for $\underline{G}$ over K through $\mathcal{O}$.

Relationship between the spherical and affine buildings

Let $\underline{G}$ be a Chevalley group scheme with Dynkin diagram $\mathcal{D}$. If $k = \mathbb{F}_q$, then $\underline{G}(k) = \underline{G}(\mathbb{F}_q)$ is a finite Chevalley group. Moreover $\underline{G}(\mathbb{F}_q)$ has an associated spherical building $X(\underline{G}(\mathbb{F}_q))$ corresponding to a BN-pair $(\underline{G}(\mathbb{F}_q), \mathcal{B}, \mathcal{N})$.

We have maps

$$\begin{array}{ccc} \underline{G}(\mathcal{O}) & \hookrightarrow & \underline{G}(K) \\ \downarrow & & \\ \underline{G}(\mathbb{F}_q) & & \end{array}$$

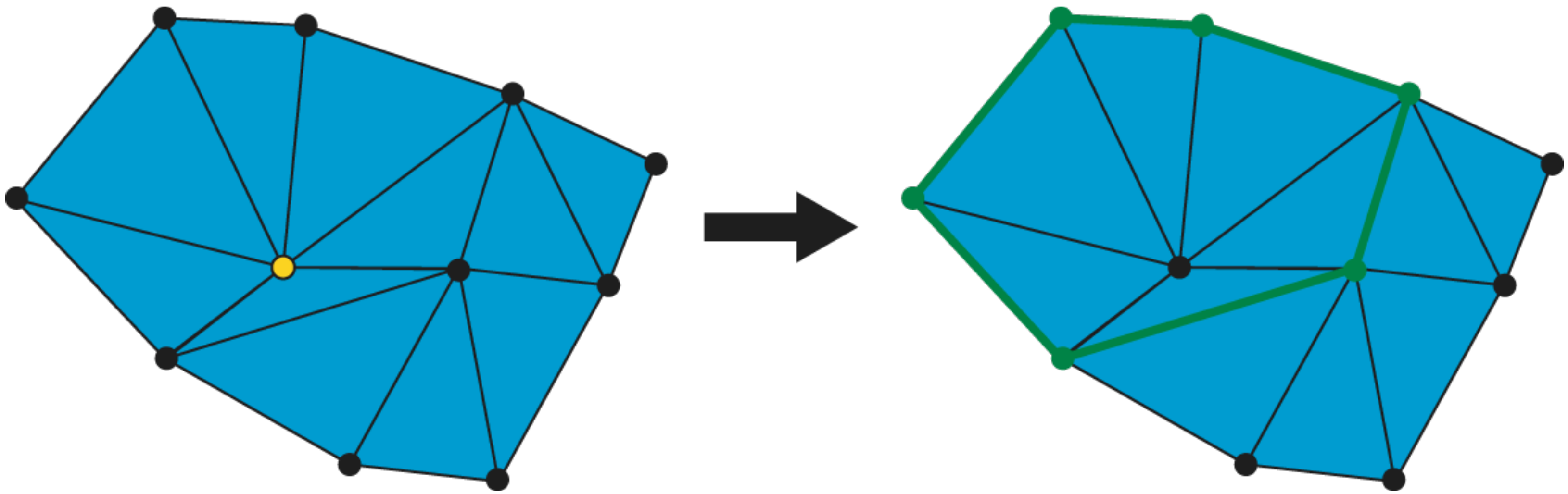
We may then 'lift' the BN-pair $(\underline{G}(\mathbb{F}_q), \mathcal{B}, \mathcal{N})$ to the group $\underline{G}(K)$. That is, we construct a BN-pair $(\underline{G}(K), B, N)$ by taking $N = \mathcal{N}$ and taking B to be the inverse image in $\underline{G}(\mathcal{O})$ of $\mathcal{B} \subseteq \underline{G}(\mathbb{F}_q)$.

The building $X(\underline{G}(K))$ of the BN-pair $(\underline{G}(K), B, N)$ is an *affine* building.

The Weyl group of $\underline{G}(K)$ is the *affine Weyl group* of the extended Dynkin diagram corresponding to $\mathcal{D}$.

Local structure in the affine building

The link of a vertex v in a building X is the boundary of the closure of the star of v.



The link of each vertex of the affine building for $\underline{G}(K)$ is completely determined by the proper connected subdiagrams obtained from deleting one vertex at a time from the Coxeter diagram for $\underline{G}(K)$. Since each such diagram corresponds to a spherical building, the link of each vertex in an affine building in rank ≥ 2 is a spherical building.

The affine building of SL_2 over a nonarchimedean local field

Let $G = SL_2(\mathbb{F}_q((t^{-1})))$. Then G has a BN-pair of Tits, where

$$B = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q[[t^{-1}]]) \mid c \equiv 0 \; mod(t^{-1})\},$$

$$N = G \cap \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \cup \quad G \cap \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

Then B is the inverse image of the Borel subgroup of $SL_2(\mathbb{F}_q)$ in $SL_2(\mathbb{F}_q[[t^{-1}]])$.

It is straightforward to verify that $B \cup N$ generates G, $B \cap N$ is normal in N and that this data satisfies the other BN-pair axioms.

The Weyl group W is the infinite dihedral group

$$W = N/(B \cap N) = < w_1 > * < w_2 > \cong \mathbb{Z} \rtimes \{\pm I\}$$

where $w_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 & -t \\ 1/t & 0 \end{pmatrix}$.

Parabolic subgroups

The subgroup B is called a *minimal parabolic subgroup.*

The maximal standard parabolic subgroups of G are

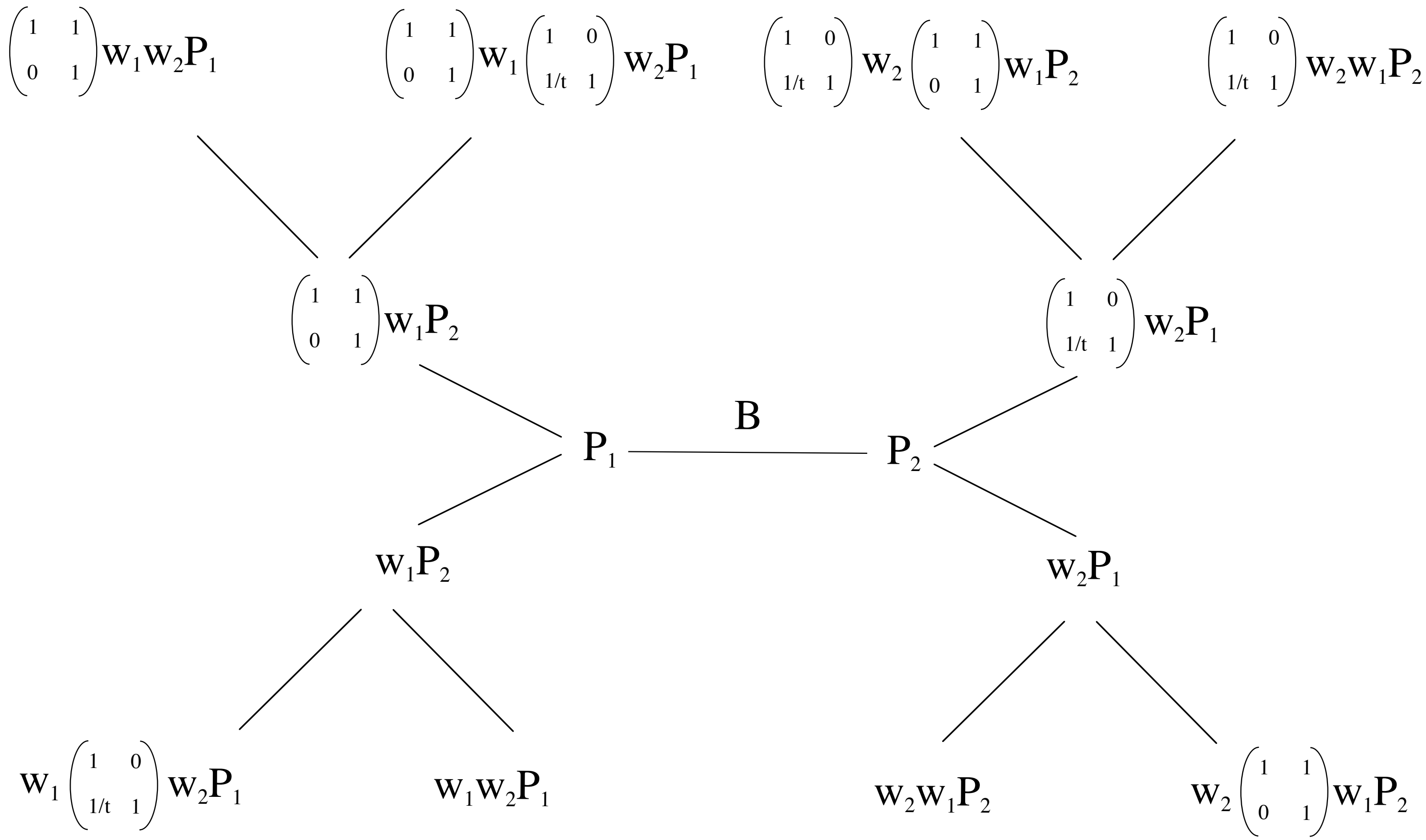
$$P_1 := B \sqcup Bw_1B, \; P_2 := B \sqcup Bw_2B.$$

Then $P_1 = SL_2(\mathbb{F}_q[[t^{-1}]])$, $P_2 \cong SL_2(\mathbb{F}_q[[t^{-1}]])$ and $P_1 \cap P_2 = B$.

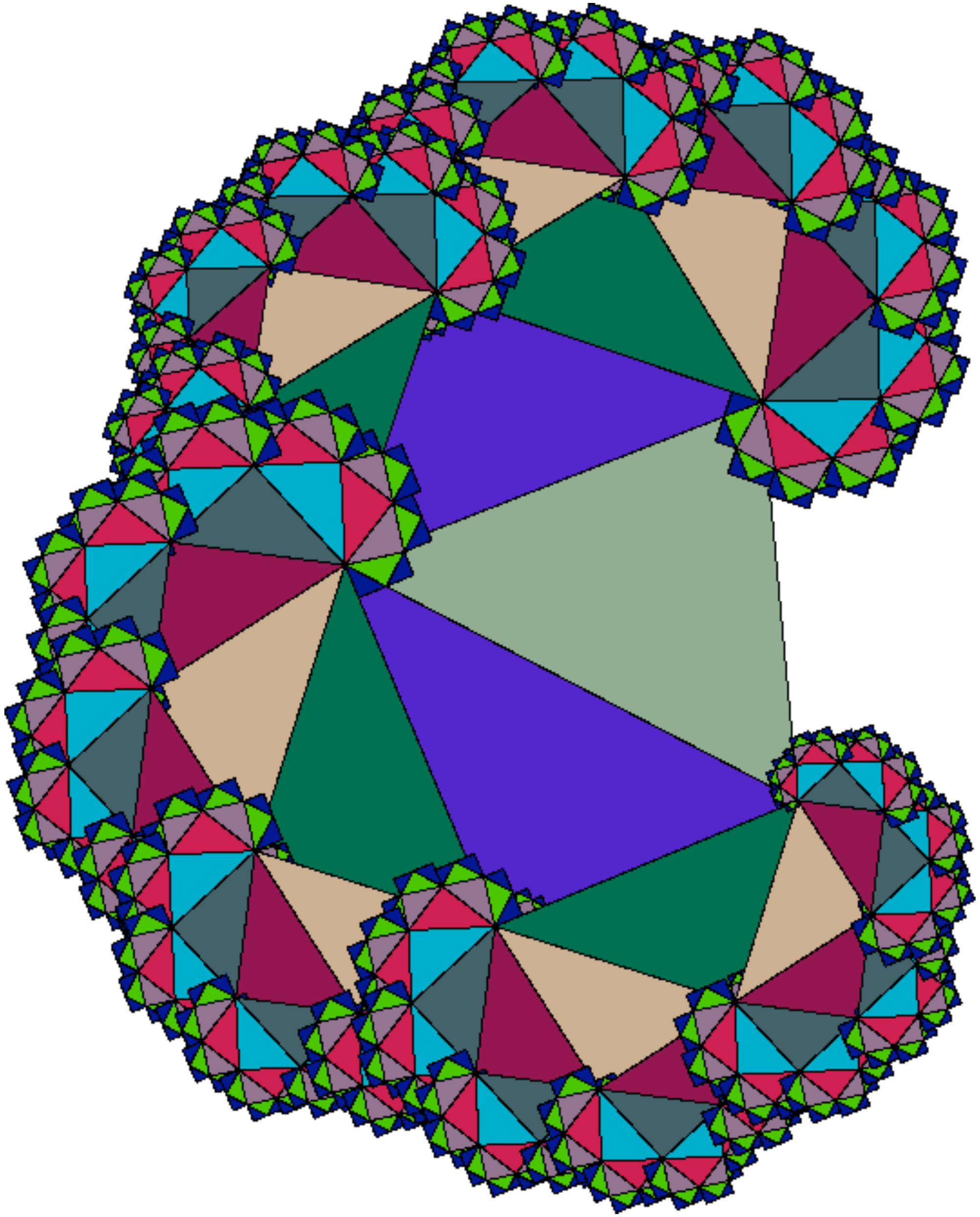
The Bruhat-Tits building of G is a simplicial complex of dimension $|S| - 1 = 1$, a tree X. The vertices of X are the conjugates of P_1 and P_2 in G.

If Q_1 and Q_2 are vertices, then there is an edge connecting Q_1 and Q_2 if and only if $Q_1 \cap Q_2$ contains a conjugate of B. We have an action of G on X by conjugation.

For $G = SL_2(\mathbb{F}_q((t^{-1})))$, the tree over the field of 2 elements



This picture by Paul Garrett shows a region of a 2 dimensional affine building.



Groups associated to Kac-Moody algebras

A Kac-Moody algebra is the most natural generalization to infinite dimensions of a finite dimensional simple Lie algebra.

Kac-Moody algebras were discovered by dropping the assumption that the matrix of Cartan integers is positive definite. The problem of associating groups to these algebras then arose, the difficulty being that there is no obvious definition of a general 'Kac-Moody group'.

Several appropriate definitions of a Kac-Moody group have been discovered, many of them using a variety of techniques as well as additional external data, such as a $\mathbb{Z}$*-form for the universal enveloping algebra (by Moody and Teo, Kac and Peterson, Goodman and Wallach,Tits, Slodowy, Mathieu, Garland, Segal, Carbone and Garland,...).*

Most constructions use some version of the *Tits functor*.

The Tits functor for Kac-Moody groups

Though there is no obvious infinite dimensional generalization of finite dimensional Lie groups, Tits' approach to the problem of defining the Kac-Moody groups was abstract in nature. Tits associated a group functor $\mathcal{G}_A$ on the category of commutative rings, such that for any symmetrizable generalized Cartan matrix A and any ring R there exists a group $\mathcal{G}_A(R)$.

Tits gave an axiomatic description of the functor $\mathcal{G}_A$ by five postulates, each of which are natural extensions of the properties of Chevalley group schemes. He showed that if K is a field, then $\mathcal{G}_A(K)$ is characterized uniquely up to isomorphism, apart from some degeneracy in the case of small fields.

Tits also showed that if K is a field containing a ring R, then there is an injective map

$$\mathcal{G}_A(R) \longrightarrow \mathcal{G}_A(K).$$

A natural generalization of the Steinberg presentation for finite dimensional simple algebraic groups gives generators and relations for $\mathcal{G}_A$ over fields. However, we do not yet have generators and relations for $\mathcal{G}_A$ over rings such as $\mathbb{Z}$.

A Tits building for Kac-Moody groups

The group of K-points of the Tits functor $\mathcal{G}_A$ over a field K is often called a *minimal Kac-Moody group.*

Tits showed that a minimal Kac-Moody group G over a field K admits a BN-pair. In fact G admits the more general structure of a *twin* BN-pair whose buildings are isomorphic as chamber complexes. In general, the building of a Kac-Moody group is neither affine nor spherical.

A complete Kac-Moody group G over a finite field is locally compact and totally disconnected and G admits an action on a locally finite building X.

G **affine type**, X *affine building*

G **hyperbolic type**, X *hyperbolic building*

Vertices correspond to cosets G/P_i, where P_i are the maximal parabolic subgroups of G. Since the Weyl group W is infinite, by the Solomon-Tits theorem X is contractible. The group G acts on X by left translation of cosets.

If G is of hyperbolic type, apartments in X are hyperbolic spaces tessellated by the action of the hyperbolic Weyl group W.

Example Let $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Then A is the generalized Cartan matrix of a Kac-Moody algebra of noncompact hyperbolic type. The Weyl group of A is the $(2,3,\infty)$-triangle group:

$$W \ = < w_1, w_2, w_3 \mid w_1^2 = w_2^2 = w_3^2 = 1, (w_1 w_2)^3 = (w_1 w_3)^2 = 1 > \cong \ PGL_2(\mathbb{Z}).$$

The standard apartment of the Tits building is a copy of the Poincaré upper half plane tessellated by the action of the extended modular group $W \cong PGL_2(\mathbb{Z})$.

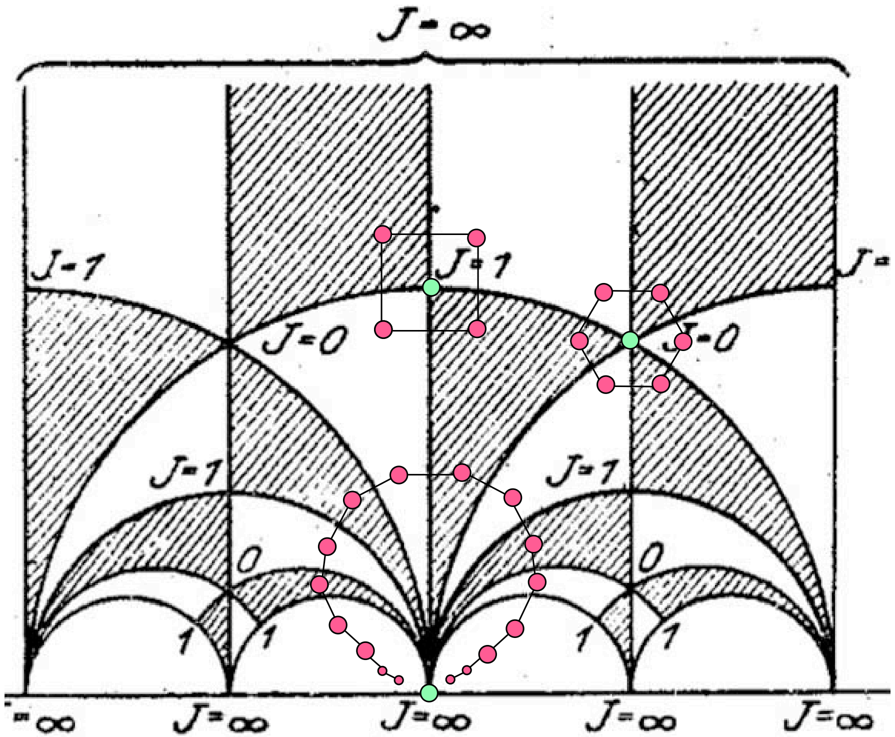


Fig. 7.

Types of Tits buildings, summary

Field	Group	Weyl group	Building type
$\mathbb{F}_q$	finite dim simple algebraic gp	finite	spherical, locally finite
Any infinite field	finite dim simple algebraic gp	finite	spherical, not locally finite
Nonarchimedean local field	finite dim simple algebraic gp	affine, infinite	affine, locally finite, spherical links
$\mathbb{F}_q$	Tits' Kac-Moody group functor in rank 2	finite	spherical, locally finite
Any field	Tits' Kac-Moody group functor in rank > 2		spherical building not known
$\mathbb{F}_q$	Tits' Kac-Moody group functor	affine, infinite	affine, locally finite, has spherical links
$\mathbb{F}_q$	Tits' Kac-Moody group functor	hyperbolic, infinite	hyperbolic, locally finite, may have spherical and affine links
$\mathbb{F}_q$	Tits' Kac-Moody group functor	Lorentzian, infinite	Lorentzian, locally finite may have spherical, affine and hyperbolic links
Any infinite field	Tits' Kac-Moody group functor	affine, infinite	affine, not locally finite, has spherical links
Any infinite field	Tits' Kac-Moody group functor	hyperbolic, infinite	hyperbolic, not locally finite, may have spherical and affine links
Any infinite field	Tits' Kac-Moody group functor	Lorentzian, infinite	Lorentzian, not locally finite, may have spherical, affine and hyperbolic links

Comparison of the Tits buildings of $PGL_3(\mathbb{F}_q)$ and $PGL_3(\mathbb{C})$

We recall that the Tits building of $PGL_3(\mathbb{F}_q)$ is the flag complex of a projective plane over $\mathbb{F}_q$. There is no convenient way to visualize the Tits building of $PGL_3(\mathbb{C})$.

If we let P_i denote the maximal standard parabolic subgroups of PGL_3 with respect to its spherical BN-pair over a field K, and B its Borel subgroup, then the set of edges P_i/B emanating from a vertex in the Tits building X can be indexed according to the following lemma:

Lemma. *We have*

$$Bw_iB/B = \{\chi_{\alpha_1}(t)w_1B/B \mid t \in K\},$$

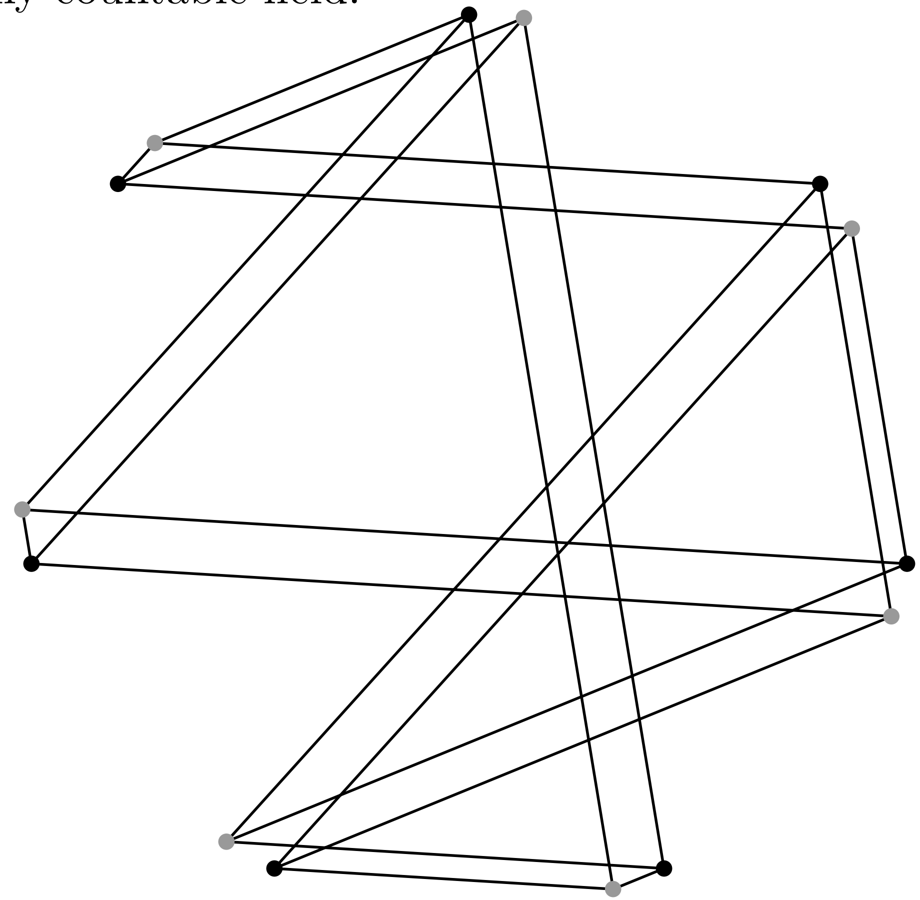
where $\chi_{\alpha_1}(t)$ are the Chevalley generators of PGL_3, that is, $\chi_{\alpha_1}(t) = exp(te_1)$, where e_1 is a generator of the Lie algebra.

When $K = \mathbb{F}_q$, the sets $|P_i/B|$ are *finite* and X is locally finite.

When $K = \mathbb{C}$, $|P_i/B| = \infty$ and X is *not* locally finite.

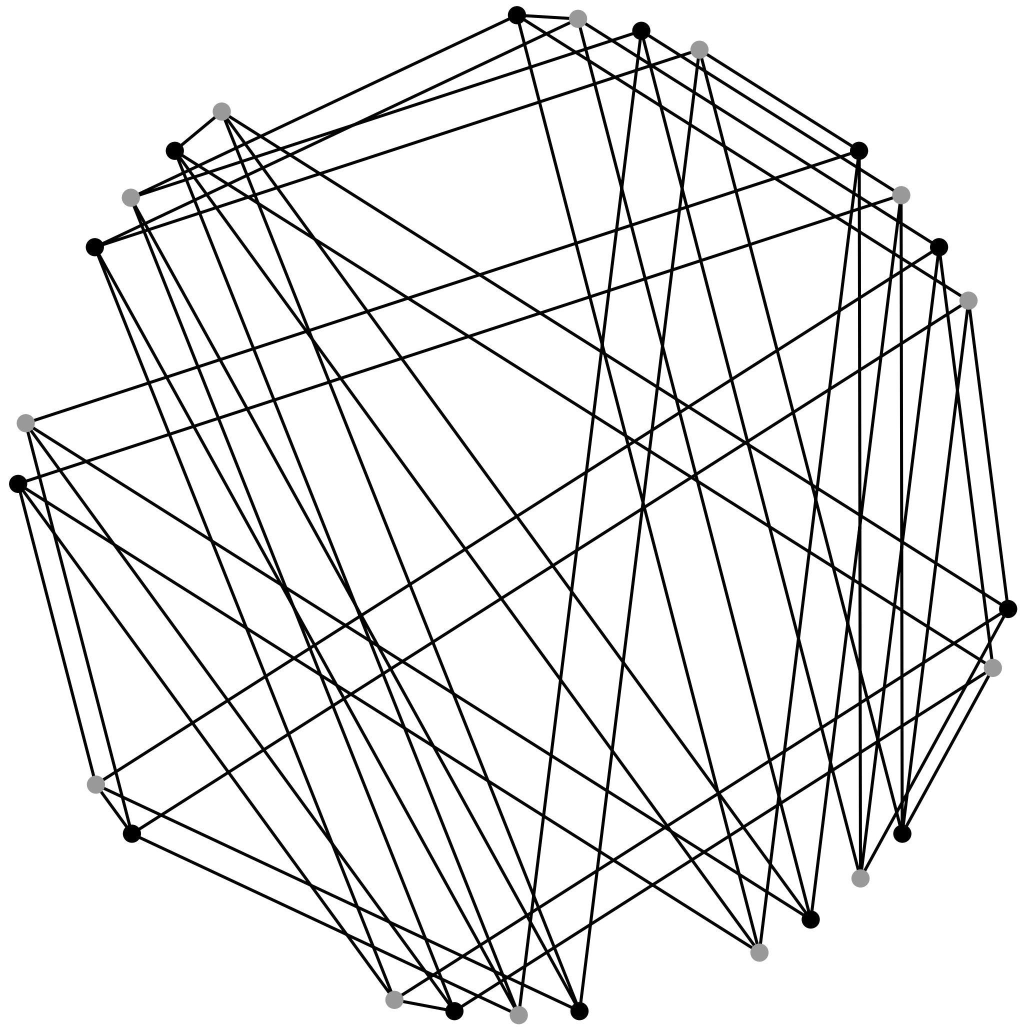
‘Thickening’ the Tits building of $PGL_3(\mathbb{F}_q)$

We can ‘thicken’ the Tits building of $G = PGL_3(\mathbb{F}_q)$ by increasing the number of edges emanating from a vertex. In this way we can obtain the Tits building of PGL_3 over any countable field.



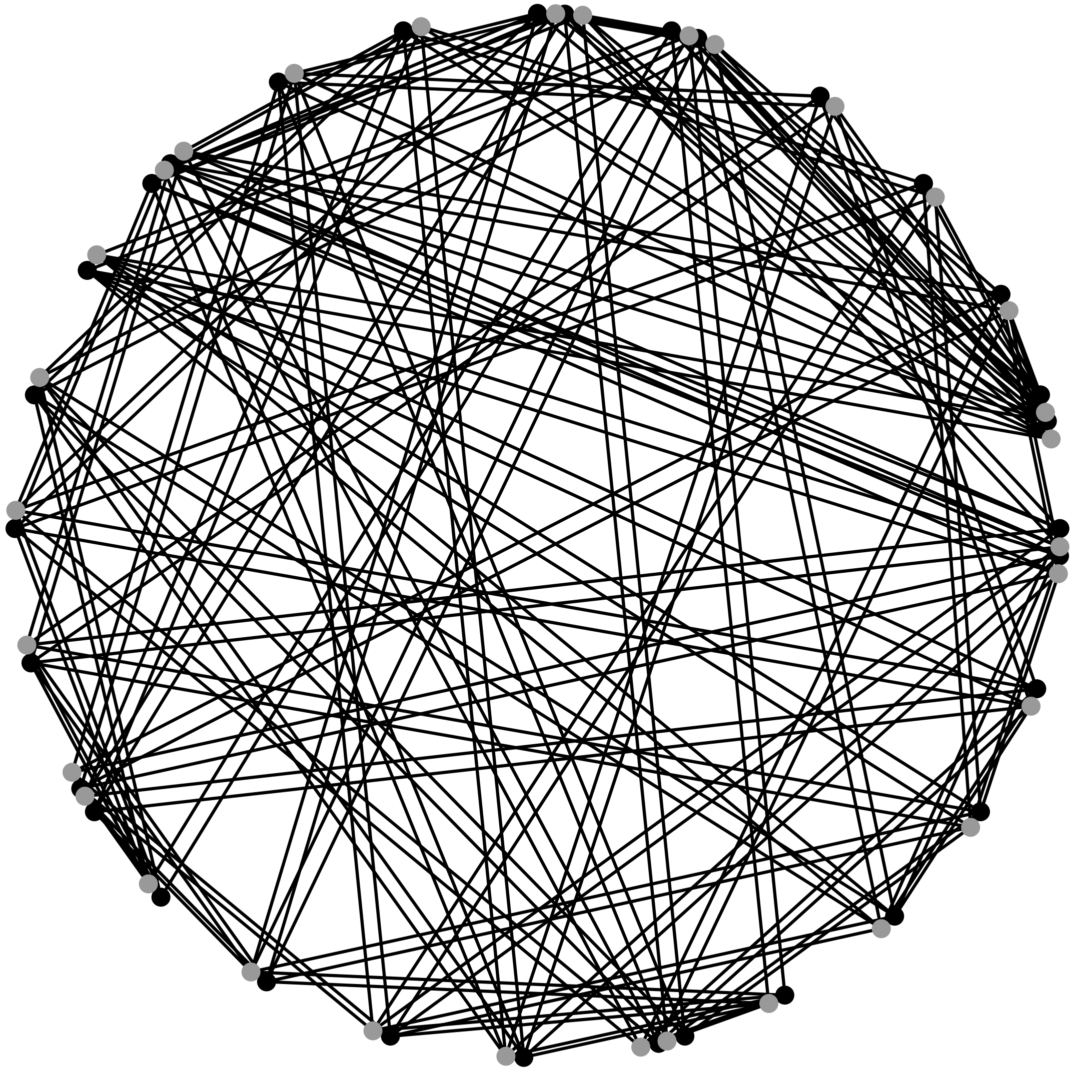
Another view of the spherical building for GL(3,Z/2)

Picture by Paul Garrett at http://www.math.umn.edu/ garrett/pix/



A view of the spherical building for GL(3,Z/3)

Picture by Paul Garrett at http://www.math.umn.edu/ garrett/pix/



A view of the spherical building for GL(3,Z/5)

Discrete to continuous

There is some sort of method for passing from $\{\mathbb{F}_p \mid p \text{ prime}\}$, with indexing over a countable set, to $\mathbb{C}$ using logic and model theory. That is, we can construct the 'ultraproduct' $\mathbb{U} = \Pi_{\mathcal{U}}(\mathbb{F}_p)_i$ of $\{\mathbb{F}_p \mid p \text{ prime}\}$ with respect to a nonprincipal ultrafilter $\mathcal{U}$.

Then $\mathbb{U}$ is a field, and there is a nonprincipal ultrafilter $\mathcal{U}$ such that $\mathbb{U}$ has characteristic zero and contains $\mathbb{C}$. However it is not straightforward to retrieve $\mathbb{R}$ from the embedding of $\mathbb{C}$ in $\mathbb{U}$.

Given finite fields K_i and Chevalley groups $G(K_i)$, F. Point proved that the ultraproduct

$$\mathbb{U} = \Pi_{\mathcal{U}} G(K_i)$$

of the $G(K_i)$ with respect to a nonprincipal ultrafilter $\mathcal{U}$ is isomorphic to $G(\Pi_{\mathcal{U}} K_i)$.

Then $G(\Pi_{\mathcal{U}} K_i)$ is the group of $\mathbb{U}$-points, a well-defined group functor over a field. Hence $G(\Pi_{\mathcal{U}} K_i)$ has a corresponding spherical building.

Relationship between $G(\mathbb{F}_q)$, $G(\mathbb{R})$ and $G(\mathbb{C})$ for G a Kac-Moody group

Let A be a generalized Cartan matrix, and let
$$G_A : \textit{Commutative rings} \longrightarrow \textit{Groups}$$
be the Tits functor for Kac-Moody groups.