ON THE CLASSIFICATION OF RANK 1 GROUPS OVER NON–ARCHIMEDEAN LOCAL FIELDS

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Abstract. We outline the classification of K–rank 1 groups over non–archimedean local fields K up to strict isogeny, as obtained by Tits. We describe the possible Bruhat–Tits buildings that can arise from such groups.

1. Introduction and preliminaries

The classification of absolutely simple algebraic groups over non–archimedean local fields up to strict isogeny is classical. Accounts of it have been written by Tits ([Ti1], [Ti2]) and Satake ([Sa]). Tits compiled tables of 'admissible indices' from which groups can be constructed.

Here we explain Tits' tables for groups of relative rank 1 over non–archimedean fields, we describe the groups that occur, as well as their Bruhat–Tits buildings which are trees.

The non–archimedean local fields K have been classified. If $Char(K) = 0$, then K is the field $\mathbb{Q}_p$ of p–adic numbers, or a finite extension of $\mathbb{Q}_p$. If $Char(K) = p > 0$, then K is isomorphic to $\mathbb{F}_q((t^{-1}))$, the field of formal Laurent series in one variable over $\mathbb{F}_q$, where q is a power of a prime p. In each case, K has finite residue class field $\mathbb{F}_q$.

Let K be a non–archimedean local field, let $\overline{K}$ be its algebraic closure, and let

$$\mathbb{G}_m/K = GL_1(K) \cong K^\times$$

be the multiplicative group of K. Let G be a linear algebraic group defined over K. We recall that a subgroup $T \subseteq G$ is called a *torus* if T is isomorphic to $(\mathbb{G}_m)^l$ over $\overline{K}$. Since G is finite dimensional, there is a torus of maximal dimension, called a *maximal torus*. Over $\overline{K}$, all maximal tori are conjugate and hence have the same dimension, called the *absolute rank* of G. If the isomorphism $T \cong (\mathbb{G}_m)^l$ is defined over K, T is called *split* over K, or *K–split*. Since G is finite dimensional, there is a K–split torus of maximal dimension, called a *maximal K–split torus*. Over $\overline{K}$, all maximal K–split tori are conjugate and hence have the same dimension, called the *relative rank* of G.

We say that G *is split over* K, if G contains a split maximal torus, and *non–split* otherwise. If G is split over K, then the absolute rank of G equals the relative rank of G.

Let G and G' be linear algebraic groups defined over K. Then G' is called a K–form of G if $G \cong G'$ over $\overline{K}$ or a finite extension of K.

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We recall that G is *simple* over K if G contains no infinite normal algebraic K–subgroup, and *absolutely simple* over K if G is simple over $\overline{K}$.

An *isogeny* of algebraic groups is a surjective homomorphism with finite kernel. An isogeny is *strict* (or *central*) if the kernel is central.

We now assume that G is absolutely simple over K. Let Γ denote the Galois group $Gal(\overline{K}/K)$. Let S be a maximal K–split torus of G, and let T be a maximal torus containing S and defined over K. Let N be the normalizer of T, and let $W = N/T$ be the Weyl group. Let $X^*(T) = Hom_{alg}(T, \overline{K}^{\times})$ be the character group of T, and let $\Sigma \subseteq X^*(T)$ be the set of all roots of G relative to T. Let Δ be the simple roots of G relative to T, and let δ be the corresponding–Dynkin diagram (δ has one vertex for each $\alpha \in \Delta$). Let $\Delta_0 \subseteq \Delta$ be the simple roots that vanish on S, and let δ_0 be the corresponding Dynkin diagram, a subdiagram of δ.

Let $Z(S)$ be the centralizer of S and let $\mathcal{D}Z(S)$ be its derived group, called the *semisimple anisotropic kernel* of G. If $\mathcal{D}Z(S)$ is trivial, G is called *quasi–split*. The Galois group $\Gamma = Gal(\overline{K}/K)$ acts on Δ and on the Dynkin diagram δ.

A *Tits index* consists of:

(1) The simple roots Δ and corresponding Dynkin diagram δ.

(2) The action of Γ on Δ and corresponding action on δ (called the **–action*).

We call a vertex of the Dynkin diagram δ *distinguished* and circle it, if the corresponding simple root does not belong to Δ_0. Vertices of the Dynkin diagram in the same orbit of Γ are drawn 'close together', and if they are both distinguished, a common circle is drawn around them.

All orbits of $\Gamma = Gal(\overline{K}/K)$ on δ are distinguished (all vertices of δ are circled) if and only if G is quasi–split.

There is a unique involutory permuation of Δ, called the *opposition involution*, such that for $\alpha \in \Delta$, the mapping $\alpha \mapsto -i(\alpha)$ extends to an action of the Weyl group: i leaves each connected component of the Dynkin diagram δ invariant, and induces a non–trivial automorphism on a connected component if and only if this component is of type A_n, D_{2n+1}, or E_6, in which case there is only one possible non–trivial automorphism.

2. Classification theorems

Theorem 2.1. *We have the following.*

(1) (Thm 1, [Ti2]) Over $\overline{K}$, G is characterized up to strict isogeny by its Dynkin diagram.

(2) (2.7.2(b), [Ti2]) G is determined up to strict isogeny by its strict $\overline{K}$–isogeny class, its Tits index, and the K–isogeny class of its semisimple anisotropic kernel.

(3) ([Ti2]) If G is quasi–split, G is determined up to strict isogeny by its Tits index.

Tits' strategy for classification is the following ([Ti2]):

(Step 1) Find all admissible Tits indices over K, subject to the constraints 'self–opposition' and 'induction' described below.

(Step 2) For a given Tits index, determine the 'anisotropic index', by erasing all circled vertices and incident edges.

(Step 3) From the anisotropic index, determine all possible semisimple anisotropic kernels.

(Step 4) Find a group with all of the above data.

The following are necessary conditions for admissibility of Tits indices ([Ti2]):

(1) *Self–opposition*: The Tits index of a group G should be invariant under the opposition involution.

(2) *Induction*: If one removes a distinguished orbit from the Tits index, together with all edges that have at least one endpoint in the orbit, the result should again be admissible.

Tits introduces the following notation for a Tits index: ${}^gX^t_{n,r}$ ([Ti2]), where

n = absolute rank = dim$T$,

r = relative rank = dim$S$,

g = order of quotient of Galois group Γ which acts faithfully on δ,

t = degree(=$\sqrt{dim}$) of division algebra which occurs in the definition of the considered form.

X = type of group over $\overline{K}$.

Let D be a division algebra of degree $d \geq 1$ over K, and let σ be an involution on D. We assume that D is central, that is, the center of D is equal to the fixed field with respect to σ. The absolutely simple groups of relative rank 1 over non–archimedean local fields K are, up to strict isogeny, either $SL_2(D)$, or $SU_n(D)$ for some n. The possible Tits indices are listed in Figure 1.

We let E denote a quadratic extension of K, and h a non–degenerate form on $V = D^m$, hermitian or skew hermitian relative to σ. We recall that the involution σ on an algebra is of the *first kind* if its restriction to the center is trivial (such as transpose), and of the *second kind* otherwise. Moreover, σ is of the :

$$\textit{first type}\text{, if the dimension of the field fixed by } \sigma \text{ is } \frac{d}{2}(d+1),$$

$$\textit{second type}\text{, if the dimension of the field fixed by } \sigma \text{ is } \frac{d}{2}(d-1).$$

In considering unitary groups over division algebras, varying the form h as hermitian or skew–hermitian, and varying the kind and type of the involution σ, allows us to determine the type of unitary group over the algebraic closure, as is summarized by the following proposition:

Proposition 2.2. *([PR], p86) Let $G = SU_m(D, h)$, where D is a division algebra of degree n, with involution σ, and h is a non–degenerate hermitian or skew–hermitian form on D^m. Then over $\overline{K}$ we have:*

(1) $G \cong Sp_{mn}$ (type $C_{(mn)/2}$) if σ is of the first kind and first type, and h is skew–hermitian, or if σ is of the first kind and second type, and h is hermitian.

(2) $G \cong SO_{mn}$ (type $B_{(mn-1)/2}$ or $D_{(mn)/2}$) if σ is of the first kind and first type, and h is hermitian, or if σ is of the first kind and second type, and h is skew–hermitian (here, type B occurs only when $n = 1$ in which case $D = K$).

(3) $G \cong SL_{mn}$ (type A_{mn-1}) if σ is of the second kind.

3. Groups over non–archimedean local fields

Here we describe the possible groups over non–archimedean local fields as in [Ti1].

(1) Tits index ${}^1A^1_{1,1}$, corresponding group $G = SL_2(D)$, D is a central division algebra of degree $d = 1$ over K, that is, $D = K$. The group G is a split form of SL_2, and G is simply connected.

(2) Tits index ${}^1A^d_{2d-1,1}$, $d \geq 2$, corresponding group $G = SL_2(D)$, D is a central division–algebra of degree $d \geq 2$ over K. We have

$$G(\overline{K}) = SL_2(D \otimes_K \overline{K}) \cong SL_2(M_d(\overline{K})) \cong SL_{2d}(\overline{K}).$$

The group G is a non–split form of SL_{2d}, and over $\overline{K}$, G is simply connected.

(3) Tits index ${}^2A^1_{2,1}$, corresponding group $G = SU^E_3(h)$, D is a central division algebra of degree $d = 1$ (that is, $D = K$) over a quadratic extension E of K, with an involution σ of the second kind, h is a non–degenerate hermitian form relative to σ. Over $\overline{K}$, $SU^E_3 \cong SL_3$. The group G is a non–split form of SL_3, and over $\overline{K}$, G is simply connected.

(4) Tits index ${}^2A^1_{3,1}$, corresponding group $G = SU^E_4(h)$, D is a central division algebra of degree $d = 1$ (that is, $D = K$) over a quadratic extension E of K, with an involution σ of the second kind, h is a non–degenerate hermitian form relative to σ. Over $\overline{K}$, $SU^E_4 \cong SL_4$. The group G is a non–split form of SL_4, and over $\overline{K}$, G is simply connected.

(5) Tits index $C^2_{2,1}$, corresponding group $G = SU_2(D, h)$, D is a quaternion division algebra of degree $d = 2$ over K, with an involution σ of the first kind, first type, h is a non–degenerate skew–hermitian form relative to σ. The group G is a non–split form of Sp_4, and over $\overline{K}$, G is simply connected.

(6) Tits index $C^2_{3,1}$, corresponding group $G = SU_3(D, h)$, D is a quaternion division algebra of degree $d = 2$ over K, with an involution σ of the first kind, first type, h is a non–degenerate skew–hermitian form relative to σ. The group G is a non–split form of Sp_6, and over $\overline{K}$, G is simply connected.

(7) Tits index ${}^2D^2_{3,1}$, corresponding group $G = SU_3(D, h)$, D is a central division algebra of degree $d = 2$ over K, with an involution σ of the first kind, first type, h is a non–degenerate hermitian form relative to σ. The group G is a non–split form of SO_6. Over $\overline{K}$, G is not simply connected.

(8) Tits index ${}^2D^2_{4,1}$, corresponding group $G = SU_4(D, h)$, D is a central division algebra of degree $d = 2$ over K, with an involution σ of the first kind, first type, h is a non–degenerate

hermitian form relative to σ. The group G is a non–split form of SO_8. Over $\overline{K}$, G is not simply connected.

(9) Tits index ${}^1D^2_{5,1}$, corresponding group $G = SU_5(D,h)$, D is a central division algebra of degree $d = 2$ over K, with an involution σ of the first kind, first type, h is a non–degenerate hermitian form relative to σ. The group G is a non–split form of SO_{10}. Over $\overline{K}$, G is not simply connected.

INDEX	GROUP	OVER $\overline{K}$
${}^1A^1_{1,1}$	$SL_2(D)$	SL_2
${}^1A^d_{2d-1,1}$ d>1	$SL_2(D)$	SL_{2d}
${}^2A^1_{2,1}$	$SU^E_3(h)$	SL_3
${}^2A^1_{3,1}$	$SU^E_4(h)$	SL_4
$C^2_{2,1}$	$SU_2(D,h)$	Sp_4
$C^2_{3,1}$	$SU_3(D,h)$	Sp_6
${}^2D^2_{3,1}$	$SU_3(D,h)$	SO_6
${}^2D^2_{4,1}$	$SU_4(D,h)$	SO_8
${}^1D^2_{5,1}$	$SU_5(D,h)$	SO_{10}

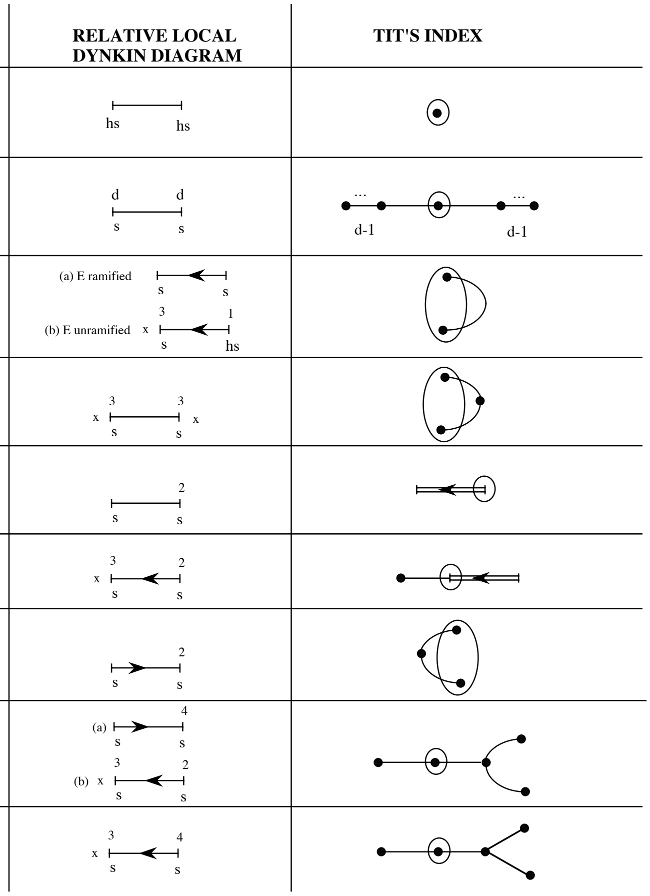

FIGURE 1. The rank 1 groups

Notes.

(i) $C_2 = B_2$ reflects the isogeny of Sp_4 with SO_5.

(ii) The Tits indices for ${}^2D^2_{3,1}$ and ${}^2A^1_{3,1}$ are the same, but Tits remarks ([Ti2]) that if a cyclic group of order 2 acts on the Dynkin diagram, then there is a quadratic extension of K fixed by the Galois group. In the case that this fixed field is ramified, the group is more naturally described as type D_3. For ${}^2A^1_{3,1}$, the fixed field of the Galois action is unramified.

(iii) ${}^2D^2_{4,1}$ has 2 different affine root systems ((a) and (b) in Table 2.3) which give different relative local Dynkin diagrams, and different isogeny classes of the group.

(iv) In every strict isogeny class, there is a unique simply connected group (see Proposition 2, section 2.6, [Ti2]).

4. The Bruhat–Tits tree of a K–rank 1 group

The Bruhat-Tits tree X, of a K–rank 1 simple algebraic group G over a non-archimedean local field K, is the building associated to G and is the non–archimedean analogue (in rank 1) of the symmetric space of a real Lie group.

Here we describe how the Bruhat–Tits tree of $G = SL_2(K)$, for $K = \mathbb{F}_q((t^{-1}))$, is constructed from a *Tits system*, or *BN–pair* for G. We have

$$K = \mathbb{F}_q((t^{-1})) = \{\sum_{n \geq n_0} a_n t^{-n} \mid a_n \in \mathbb{F}_q\},$$

which has infinitely many negative powers of t and finitely many positive powers. The field K has ring of integers

$$\mathcal{O} = \mathbb{F}_q[[t^{-1}]] = \{\sum_{n \geq 0} a_n t^{-n} \mid a_n \in \mathbb{F}_q\},$$

with infinitely many negative powers of t and no positive powers. The ring of integers $\mathcal{O}$ contains the 'prime element' t^{-1}. Also, $\mathcal{O} = \mathbb{F}_q[[t^{-1}]]$ is open and compact in $K = \mathbb{F}_q((t^{-1}))$.

For $G = SL_2(K)$, we have the *affine* Tits system (G, B, N), (see [S] pp 89–91) where

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) \mid c \equiv 0 \ mod \frac{1}{t} \right\},$$

is the *Iwahori subgroup*, or the *minimal parabolic subgroup*. We let T denote the diagonal subgroup of G.

We have

$$N = \left[G \cap \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right] \cup \left[G \cap \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right] = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in K^\times \} \cup \{ \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \mid b \in K^\times \right\}.$$

The conjugation action of N on T permutes the diagonal entries, so N normalizes T. The group

$$B \cap N = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathcal{O}^\times \right\},$$

is the *integral torus*, and is normal in N. Set

$$S = \left\{ w_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ w_2 = \begin{pmatrix} 0 & -t \\ 1/t & 0 \end{pmatrix} \right\}.$$

Then the *Weyl group*

$$W = N/(B \cap N) \cong \mathbb{Z} \rtimes \{\pm Id\} \cong D_\infty$$

is generated by S.

If we consider the Bruhat cells Bw_1B and Bw_2B, we obtain the *standard parabolic subgroups*

$$P_1 := B \sqcup Bw_1B,$$

and

$$P_2 := B \sqcup Bw_2B.$$

For $G = SL_2(K)$, we have (see [S], p 91)

$$P_1 = SL_2(\mathcal{O}),$$

$$P_2 = \{\begin{pmatrix} a & tb \\ c/t & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O})\},$$

and $P_1 \cap P_2 = B$.

The Bruhat-Tits building of G is a simplicial complex of dimension 1, a tree X. The vertices of X are the conjugates of P_1 and P_2 in G. If Q_1 and Q_2 are vertices, then there is an edge connecting Q_1 and Q_2 if and only if $Q_1 \cap Q_2$ contains a conjugate of B. We have an action of G on X by conjugation.

We will make use of the following basic facts:

(1) P_1 and P_2 are conjugate in $GL_2(K)$, by

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

but not conjugate in $SL_2(K)$.

(2) Each P_i is its own normalizer in G, $i = 1, 2$, and B is its own normalizer in G.

Let $g \in G$. Then (1) and (2) imply that the maps:

$$f_0 : gB \longrightarrow gBg^{-1},$$

$$f_1 : gP_1 \longrightarrow gP_1g^{-1},$$

$$f_2 : gP_2 \longrightarrow gP_2g^{-1}$$

are bijections. Thus

$$G/P_1 \cong \text{all conjugates of } P_1 \text{ in } G,$$

$$G/P_2 \cong \text{all conjugates of } P_2 \text{ in } G,$$

$$G/B \cong \text{all conjugates of } B \text{ in } G.$$

We have

$$G = P_1 *_B P_2.$$

We obtain the following description of the Bruhat-Tits tree X:

$$VX = G/P_1 \sqcup G/P_2,$$

$$EX = G/B \sqcup \overline{G/B},$$

where $\overline{G/B}$ is a copy of the set G/B, giving an orientation to EX, so that positively oriented edges come from G/B, and negatively oriented edges come from $\overline{G/B}$.

The group G acts by left multiplication on cosets. There is a natural projection on cosets induced by the inclusion of B in P_1 andP_2:

$$\pi : G/B \longrightarrow G/P_i, \ i = 1, 2.$$

If v_i is a vertex, and $St^X(v_i) = \pi^{-1}(v_i)$ is the set of edges with origin v_i, then $St^X(v_i)$ is indexed by $P_i/B \subseteq G/B$, $i = 1, 2$.

For $G = SL_2(K)$, X is a *homogeneous*, bipartite tree of degree

$$[P_1 : B] = [P_2 : B] = q + 1.$$

The following lemma describes how the cosets Bw_1B and Bw_2B are indexed modulo B:

Lemma 4.1. *[IM] For $G = SL_2(K)$, $K = \mathbb{F}_q((t^{-1}))$, we have:*

$$Bw_1B/B = \{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} w_1B/B \mid s \in \mathbb{F}_q\},$$

$$Bw_2B/B = \{\begin{pmatrix} 1 & 0 \\ s/t & 1 \end{pmatrix} w_1B/B \mid s \in \mathbb{F}_q\}.\square$$

It follows that the edges emanating from P_1 and P_2 can be indexed as follows:

$$St^X(P_1) = \{B, \ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} w_1B/B \mid s \in \mathbb{F}_q\},$$

$$St^X(P_2) = \{B, \ \begin{pmatrix} 1 & 0 \\ s/t & 1 \end{pmatrix} w_2B/B \mid s \in \mathbb{F}_q\},$$

and the stars of other vertices are obtained by translating (conjugating) these ones.

Apartments in X are infinite lines in one–to–one correspondence with the maximal K-split tori of G. The *standard apartment* $\mathcal{A}_0$ in X, corresponding to the diagonal subgroup T, consists of all Weyl group translates of the standard simplex.

Moreover, $G = SL_2$ acts transitively on edges of X, and has 2 orbits for the vertices, corresponding to vertices that come from P_1 or P_2

The group B of the Tits system is the stabilizer of the standard simplex $\mathcal{C}_0$, and N is the stabilizer of the standard apartment $\mathcal{A}_0$.

For $G = SL_2(\mathbb{F}_q((t^{-1})))$, Figure 2 shows the tree of the field with 2 elements.

We may obtain additional information about G and in particular its Bruhat–Tits tree from the *relative local Dynkin diagram* arising from an 'affine root system' for G (see [Ti1]).

The relative local Dynkin diagrams of K–rank 1 groups are given in [Ti1], and are included in Figure 1. The labels s and hs on the vertices of the relative local Dynkin diagram for G denote 'special' and 'hyperspecial' vertices. We refer the reader to ([Ti1], pp35) for definitions.

In general, the degrees of homogeneity of the Bruhat–Tits tree of a simple rank 1 group over a non-archimedean local field can be determined as follows:

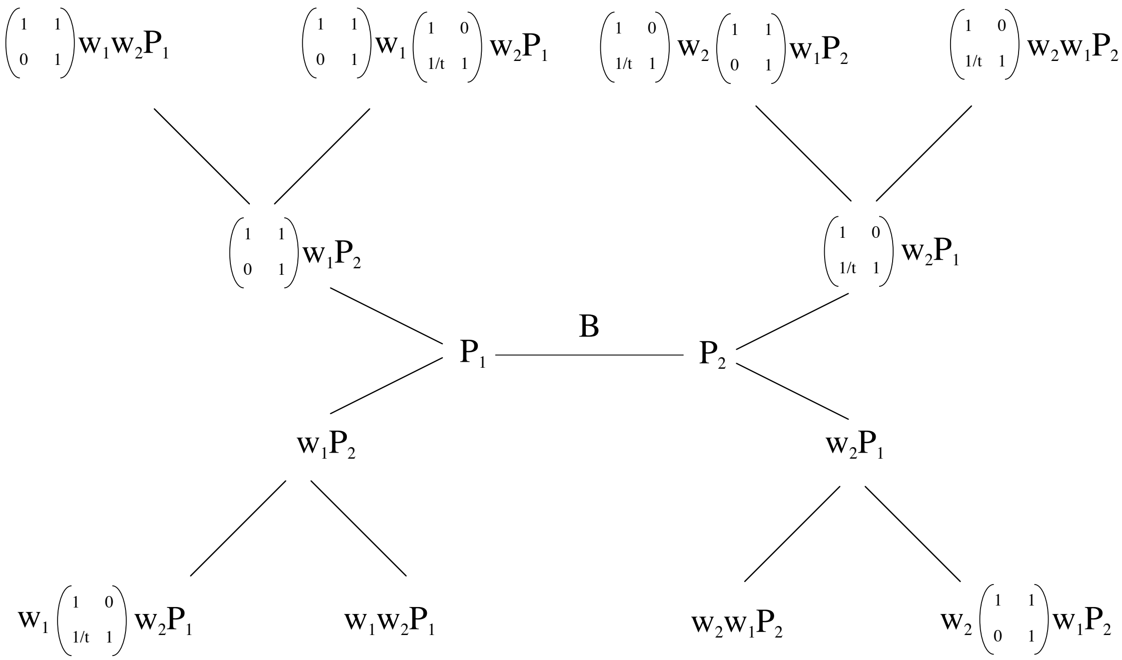

FIGURE 2. The tree of $SL_2(\mathbb{F}_2((t^{-1})))$

(1) Let G be a K–rank 1 simple algebraic group over a non–archimedean local field K. If K has residue class field $\mathbb{F}_q$, and if d and d' are the integers attached to the two vertices of the relative local Dynkin diagram for G, then G has Bruhat–Tits tree $X = X_{q^d+1,q^{d'}+1}$ ([Ti1]). Moreover, if B is the minimal parabolic subgroup of G, and P_1 and P_2 are the two maximal proper parabolic subgroups of G, then $q^d + 1 = [P_1 : B]$ and $q^{d'} + 1 = [P_2 : B]$. For a general rank 1 group, the indices $[P_1 : B]$ and $[P_2 : B]$ can be different, but for $G = SL_2$ they are the same.

(2) Let K' be a ramified extension of K whose ramification index equals the degree of the extension. If K has residue class field k, then K' also has residue class field k. By (1) above, if G is a simple algebraic group of–relative rank 1, defined over both K and K', then G/K and G/K' have the same Bruhat–Tits tree.

For a general rank 1 group G, if G is simply connected, then G acts without inversions on the Bruhat–Tits tree X, but if G is not simply connected, G may act with inversions. In this case, we may replace G by a subgroup of index 2 which acts without inversions. We recall, however, that in each strict isogeny class, there is a unique simply connected group.

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