Fundamental domains for nonuniform lattices in Kac–Moody groups

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Overview

Let $G = G_A(\mathbb{F}_q)$ be a locally compact complete Kac-Moody group over a finite field $\mathbb{F}_q$, corresponding to a generalized Cartan matrix A. Let $\Gamma \leq G$ be a nonuniform lattice subgroup, that is, a discrete subgroup $\Gamma \leq G$ whose quotient $\Gamma\backslash G$ is not compact but carries a finite invariant measure. The group G is known to contain nonuniform lattices by work of Carbone and Garland ([CG]) and by Rémy ([Re]). Let X denote the Tits building of the (positive or negative) BN-pair for G. If $rank(G) = 2$ then X is the homogeneous tree X_{q+1}. The following conjecture was posed by Howard Garland.

Conjecture: Let G be a locally compact rank 2 Kac-Moody group over a finite field $\mathbb{F}_q$ with Tits building $X = X_{q+1}$. Let Γ be a nonuniform lattice subgroup of G. The quotient $\Gamma\backslash X$ of X by Γ consists of a finite core graph together with finitely many cusps which are semi-infinite rays.

This conjecture holds for all known lattice subgroups of G. Let B^- be the minimal parabolic subgroup of the negative BN-pair for G and let $P_1^- = B^- \sqcup B^- w_1 B^-$ be the standard parabolic subgroup corresponding to a simple root reflection denoted w_1. The full Weyl group W is generated by 2 simple root reflections, and is the infinite dihedral group. Let B denote the minimal parabolic subgroup of the positive BN-pair for G. Then B is compact, in fact a profinite neighborhood of the identity in G, and B^- is discrete. It turns out that B^- and P_1^- are nonuniform lattices in G ([CG]).

In [CG], the authors established that there are bijective correspondences

$$B^-\backslash G/B \quad \cong \quad W$$
$$P_1^-\backslash G/B \quad \cong \quad W^+,$$

where W^+ denotes an index 2 subgroup of W. As follows naturally, the action of B^- on X yields a quotient graph with 2 cusps (semi-infinite rays) and a core graph consisting of a single edge. The lattice P_1^- has quotient graph with 1 cusp and a core graph consisting of a single vertex.

If G is of affine type and G is completed using the Rémy-Ronan method then we can identify G with $PSL_2(\mathbb{F}_q((t)))$. The conjecture holds here by the work of Lubotzky ([L1] and [L2]) and also by Ragunathan ([R]). Work of Morgenstern ([M]) shows that for congruence subgroups of $PGL_2(\mathbb{F}_q[t]) \leq PGL_2(\mathbb{F}_q((t^{-1})))$, the core graph is a bihomogeneous bipartite graph.

An analog of the conjecture also holds for $\mathbb{R}$-rank one, $\mathbb{R}$-simple Lie groups, where Garland and Raghunathan showed that fundamental domains for nonuniform lattices are finite unions of Siegel sets ([GR]).

A positive answer to the conjecture for rank 2 hyperbolic Kac-Moody groups would have a number of important consequences. For example, it would imply the 'Kazhdan-Margulis' property for Kac-Moody groups, namely that covolumes of lattices are bounded away from zero.

It would also have consequences for the study of automorphic forms on arithmetic quotients of Kac-Moody groups, a subject that is not yet well understood. It is known that the automorphic spectrum of the combinatorial Laplacian with respect to a nonuniform lattice Γ acting on a homogeneous tree X is determined by the 'edge-indexed' quotient graph of Γ on X. If G is the projective affine Kac-Moody group $PGL_2(\mathbb{F}_q((t)))$, Efrat ([E]) used this to study Eisenstein series on quotients of X by the lattice $\Gamma = PGL_2(\mathbb{F}_q[t])$ and its congruence subgroups.

To develop an analogous theory for hyperbolic Kac-Moody groups, it is desirable then to obtain a structure theorem for quotients of the Tits building by nonuniform lattices.

Background, setting and main conjecture

Let G be a locally compact complete rank 2 Kac-Moody group over a finite field $\mathbb{F}_q$. Let X be the Tits building of G, a homogeneous tree X_{q+1}. Let Γ be a nonuniform lattice subgroup of G. The *Dirichlet fundamental domain* for Γ around a vertex $x_0 \in VX$ is defined as ([L1] and [CG]):

$$D = D(\Gamma, x_0) := \{x \in VX \mid d(x, x_0) \leq d(\gamma x, x_0) \text{ for every } \gamma \in \Gamma\}.$$

It follows that D is a connected subtree of X and that $\Gamma D = X$ ([L1]). Since Γ is a lattice, the orders of the Γ-stabilizers of vertices along some (in fact any) semi-infinite ray in $D(\Gamma, x_0)$ are unbounded. To state the main conjecture precisely, we make the following definitions, following [B].

A *geometric end* of X is an equivalence class of infinite paths in X, where two paths are equivalent if and only if their intersection is infinite. A *geometric end* of $\Gamma\backslash X$ is a lifting to X of a semi-infinite ray in $\Gamma\backslash X$.

We may identify the set of geometric ends of X with the spherical building of G. In [CG], the authors showed how to associate a spherical building to a rank 2 locally compact complete rank 2 Kac-Moody group G. It follows that every geometric end ϵ of X corresponds to a proper parabolic subgroup $\mathcal{P}_\epsilon$ in the spherical BN-pair for G.

A geometric end ϵ of X is called Γ-*cuspidal* if there is a non-trivial 'good unipotent element' in Γ fixing ϵ. We have not yet defined 'unipotent elements' or specified what it means for them to be 'good'. This will be crucial for our work (see below).

To each Γ-cuspidal end ϵ, there is an infinite ascending chain of finite *cusp subgroups* Γ_n, $n = 1, 2, \ldots$ with $\Gamma_n \subset \Gamma_{n+1}$ which correspond to the Γ-stabilizers of vertices $x_1, x_2, \ldots$ along an infinite path towards the end ϵ. For each $n = 1, 2, \ldots$, we have $\Gamma_n \subset \mathcal{P}_\epsilon$.

Main conjecture *Every semi-infinite ray in the Dirichlet fundamental domain for* Γ *represents a* Γ*-cuspidal end.*

If we can establish the main conjecture, we should be able to obtain the following corollaries.

Corollary 1 There are only finitely many semi-infinite rays in the Dirichlet fundamental domain for Γ.

Corollary 2 The quotient graph $\Gamma\backslash X$ consists of a finite core graph together with finitely many cusps which are semi-infinite rays.

This is essentially the line of argument for the analogous cases in [L1] and [GR]. In further analogy with [L2], we propose:

Conjecture The subgroup $P_1^- \leq G$ is the lattice of minimal covolume in the Kac-Moody group G.

This should follow from the main conjecture and the methods of [L1].

Characterization and existence of good unipotents

The crucial work in the conjecture is to find the correct definition of 'good unipotent elements' in a rank 2 locally compact complete Kac-Moody group G and to prove that they exist. We should try to work in analogy with [L1], [L2] and [B]. The following discussion is in the language of the Carbone-Garland completion in [CG]. The groups B and B^- are generated by a subgroup $H \cong (\mathbb{F}_q^\times)^2$ and all positive (respectively negative) root groups U_α, where for each real root α, U_α is isomorphic to the additive group of $\mathbb{F}_q$. Let

$$\mathcal{U} \quad = \quad \text{closure of } \mathcal{U}_0, \text{ where}$$

$$\mathcal{U}_0 \quad = \quad \text{group generated by all root groups } U_\alpha, \text{ with } \alpha \in \Phi_1,$$

where

$$\Phi_1 \ := \ \{-\alpha_2, -w_2 \cdot \alpha_1, -w_2 w_1 \cdot \alpha_2 \dots\} \cup \{\alpha_1, w_1 \cdot \alpha_2, w_1 w_2 \cdot \alpha_1 \dots\}$$

corresponds to 'half' the real roots. The group $\mathcal{U}$ is topologically isomorphic to the additive group of $\mathbb{F}_q((t))$, the field of formal Laurent series in t over $\mathbb{F}_q$. The group $\mathcal{U}$ plays the role of the (additive) subgroup of $SL_2(\mathbb{F}_q((t^{-1})))$ of upper triangular unipotent matrices in Lubotzky's paper [L1].

We view $\mathcal{U}$ as the 'unipotent radical' of the parabolic subgroup in the spherical BN-pair for G corresponding to the end associated to the standard apartment of X. We may hence propose the following definition: An element $u \in G$ is called a *good unipotent* if it lies in a conjugate of the subgroup $\mathcal{U}$.

Any conjugate of $\mathcal{U}$ corresponds to a unique parabolic subgroup in the spherical BN-pair for G, hence to a unique end stabilizer. It follows that any good unipotent fixes a unique end.

Let P_i, $i = 1, 2$ be the maximal parabolic subgroups for the positive BN-pair for G. In analogy with [L2] may define

$$N_i = Ker\{P_i \longrightarrow \langle H, U_{\alpha_i}, U_{-\alpha_i} \rangle\},$$

that is, the kernel of the surjective homomorphism from P_i onto its Levi factor, with α_i denoting the simple roots. In [RR], in analogy with SL_2, the authors showed that for each $i = 1, 2$, N is the maximal normal pro-p subgroup of P_i. In [L2] Lubotzky called a unipotent element 'good' or dominant' at a vertex x if it lies in the kernel

$$N = Ker\{G_x \longrightarrow SL_2(\mathbb{F}_q)\},$$

for $G = SL_2(\mathbb{F}_q((t^{-1})))$, where here the stabilizer G_x is conjugate to P_i, $i = 1$ or 2. We ask if this is the case for rank 2 Kac-Moody groups.

Question 1 Let $u \in G$ be a good unipotent, that is, u lies in a conjugate of the subgroup $\mathcal{U}$. Does u belong to a kernel N_i for $i = 1$ or 2 (or corresponding to a conjugate of P_i)?

If G is a simply connected simple algebraic group over a nonarchimedean local field, then every unipotent element of G is a good unipotent ([BT]). We ask if this is the case for our rank 2 Kac-Moody group.

Question 2 Is every unipotent element of G a good unipotent?

In analogy with $SL_2(K)$, K a nonarchimedean local field of characteristic p, we ask:

Question 3 Is it true that every good unipotent in G has p-power order, where p is the characteristic of $\mathbb{F}_q$? Does every element of G of p-power order lie in a conjugate of $\mathcal{U}$?

Question 4 Is it true that every cusp subgroup Γ_n, $n = 1, 2, \dots$ in an infinite ascending chain corresponding to a Γ-cuspidal end lies in a conjugate of $\mathcal{U}$?

As in [B], Corollary 3.3, we should be able to establish that if a lattice $\Gamma \leq G$ contains nontrivial good unipotent elements, then it it must be a nonuniform lattice.

Question 5 Is it true that cocompact lattices in G cannot contain nontrivial good unipotent elements?

Finally we stress that we are considering only 'split' forms of Kac-Moody groups, thus our work proceeds in analogy with SL_2 rather than a general simple algebraic group over a nonarchimedean local field. We may thus try to proceed in analogy with the simpler case of $G = SL_2$ in [L2].

References

[B] Baumgartner, U. *Cusps of Lattices in Rank 1 Lie Groups over Local Fields*, Geometriae Dedicata, Volume 99, Number 1, June 2003 , pp. 17–46(30)

[BT] Borel, A and Tits, J *Complment l' article "Groupes rductifs* Publ. Math. I.H.E.S. , 41 (1972) pp. 253–276

[E] Efrat, I. *Automorphic Spectra on the Tree of PGL_2*, Enseign. Math. (2) 37, No. 1–2, 1991, 31–43.

[GR] Garland, H. and Raghunathan, M. S. *Fundamental Domains for Lattices in Rank One Semisimple Lie Groups* PNAS, February 15, 1969, vol. 62, no. 2 —, 309–313.

[L1] Lubotzky, A. *Lattices in rank one Lie groups over local fields*, Geom. Funct. Anal. 1(1991), 405–431.

[L2] Lubotzky, A. *Lattices of minimal covolume in SL_2 : a nonarchimedean analogue of Siegel's theorem $\mu \geq \pi/21$*, J. Amer. Math. Soc. 3 (1990) 961–975

[M] Morgenstern, M *Natural Bounded Concentrators*, Combinatorica 15(1): 111- 122 (1995)

[R] Raghunathan, M. S. *Discrete subgroups of algebraic groups over local fields of positive characteristics*, Proc. Indian Acad. Sci. Math. Sci. 99(1989), 127–146.

[Re] Rémy, B *Formes presque dployes des groupes de Kac-Moody sur un corps. quelconque*, Thse Universit Henri Poincar Nancy 1 1999

[RR] Rémy, B and Ronan, M *Topological groups of Kac-Moody type, right-angled twinnings and their lattices*, Commentarii Mathematici Helvetici 81 (2006) 191–219.