TENSORS, WEDGES AND REPRESENTATIONS

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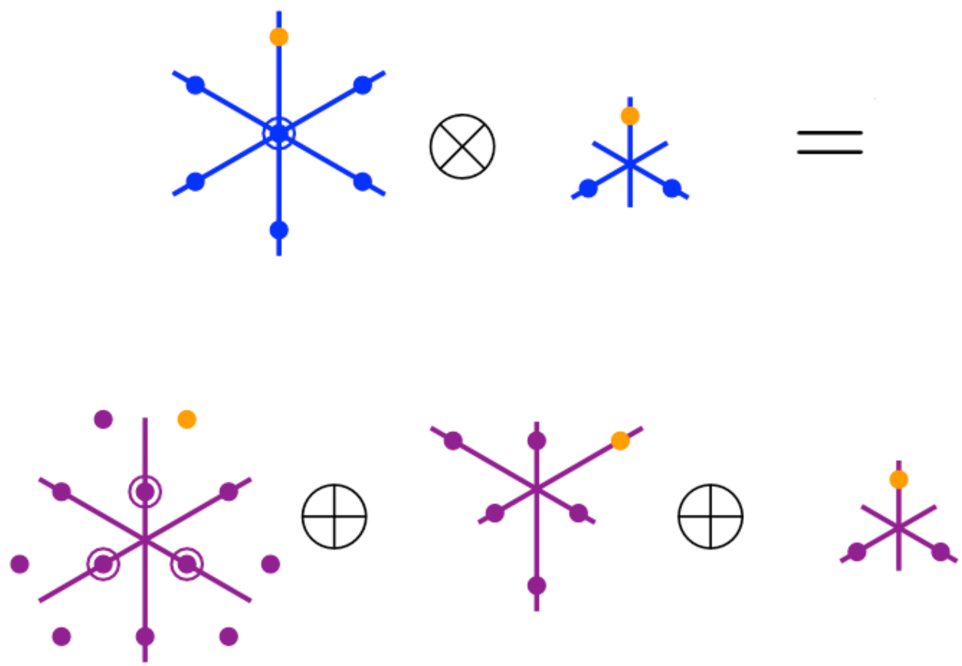

We introduce the tensor product '$\otimes$' and the exterior (or wedge) product '$\wedge$' and briefly discuss their roles in representation theory.

THE SETTING

Let $\mathbb{F}$ be a field such as $\mathbb{R}$ or $\mathbb{C}$.

Let U and V be vector spaces over $\mathbb{F}$ with dual spaces U^* and V^*.

We assume for now that U and V are finite dimensional, though much of what we discuss will be true for infinite dimensional vector spaces.

More generally, we may take U and V to be modules over a ring such as $\mathbb{Z}$.

We recall that a *module over a ring* is a generalization of the notion of a vector space, where the scalars lie in a ring rather than a field.

A left R-module V over a ring R with identity element consists of an abelian group $(V, +)$ and an operation $R \times V \to V$ such that for all $r, s \in R$ and $x, y \in V$, we have

$$r(x + y) = rx + ry, \quad (r + s)x = rx + sx, \quad (rs)x = r(sx), \quad 1_R x = x$$

Today we will introduce two types of products of modules: the tensor product '$\otimes$ 'and the exterior (or wedge) product '$\wedge$' and give examples of their roles in representation theory.

Roughly speaking, we will view $x \otimes y$ as a formal noncommutative product of vectors x, $y \in V$ and $x \wedge y$ as a formal antisymmetric product of vectors x, $y \in V$.

BILINEAR AND MULTILINEAR MAPS

Let U, V and W be vector spaces (or modules) over the same base field $\mathbb{F}$. A *bilinear map* is a function

$$B : U \times V \to W$$

such that for any $v \in V$ the map

$$u \mapsto B(u, v)$$

is a linear map from U to W, and for any $u \in U$, the map

$$v \mapsto B(u, v)$$

is a linear map from V to W.

In other words, if we hold the first entry of the bilinear map fixed, while letting the second entry vary, the result is linear and similarly if we hold the second entry fixed. We say that B *is linear in each variable.*

More generally, a multilinear map is a function of several variables that is linear in each variable. More precisely, a multilinear map is a function

$$f : U_1 \times \cdots \times U_n \to V$$

where U_i, V are vector spaces (or modules) and for each i, if each variable except for $u_i \in U_i$ is held constant, then f is linear in u_i.

TENSOR PRODUCTS

Let R be a commutative ring and U and V be R-modules. You are familiar with the direct sum $U \oplus V$ which is an addition operation on modules.

Here we describe a product operation, called the *tensor product* $U \otimes V$, which is a formal bilinear multiplication of two modules or vector spaces.

This notion first arose in differential geometry and physics in order to accurately define the stress and curvature tensors.

Let R be a commutative ring, and let U, V be R-modules. There is an R-module $U \otimes V$, called the *tensor product* of U and V over R, together with a canonical bilinear homomorphism

$$\otimes : U \times V \to U \otimes V,$$

which is characterized up to isomorphism by the following universal property. For every other R-module W, the bilinear R-module homomorphism π

$$\pi : U \times V \to W,$$

lifts to a unique R-module homomorphism

$$\tilde{\pi} : U \otimes V \to W,$$

such that

$$\pi(u, v) = \tilde{\pi}(u \otimes v)$$

for all $u \in U,\ v \in V$.

CONSTRUCTION OF TENSOR PRODUCTS

Let R be a commutative ring and U and V be R-modules. The tensor product $U \otimes V$ can be constructed by taking the free R -module generated by all formal symbols

$$u \otimes v, \quad u \in U,\ v \in V$$

modulo the bilinear relations:

$$(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v, \qquad u_1, u_2 \in U,\ v \in V$$

$$u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2, \qquad u \in U,\ v_1, v_2 \in V$$

$$r(u \otimes v) = (ru) \otimes v = u \otimes (rb), \qquad r \in R,\ u \in U,\ v \in V$$

In particular, if U and V are finite dimensional vector spaces with bases $\{u_i\}$, $\{v_j\}$ respectively, then $u_i \otimes v_j$ is a basis for $U \otimes V$ and an element x of $U \otimes V$ looks like

$$x = \sum_{i,j} a_{ij}\ u_i \otimes v_j,$$

where $a_{ij} \in R$.

In order to stress the choice of base ring R, the tensor product $U \otimes V$ is often written as $U \otimes_R V$.

BASIC PROPERTIES OF TENSOR PRODUCTS

Let R be a commutative ring and U, V, W be R-modules. Since the tensor product $U \otimes V$ is a bilinear homomorphism $\otimes : U \times V \to U \otimes V$ that uses the direct product (ordered pairs), in general

$$u \otimes v \neq v \otimes u$$

for $u \in U$, $v \in V$. However, we have the following isomorphisms of modules:

(1) $R \otimes V \cong V$, invariance under tensoring by scalars

(2) $U \otimes V \cong V \otimes U$, commutative up to isomorphism

(3) $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$, associative

(4) $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$, distributive over addition

If U and V are finite dimensional, then $dim(U \otimes V) = dim(U) \times dim(V)$.

WARNING: COLLAPSING OF TENSOR PRODUCTS

Lemma. *If $gcd(m, n) = 1$ then $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} = 0$.*

Proof: Since $gcd(m, n) = 1$, we have

$$n(x \otimes y) = (nx) \otimes y = 0$$

$$m(x \otimes y) = x \otimes (my) = 0.$$

Hence $x \otimes y = 0$ for all $x \in \mathbb{Z}/n\mathbb{Z}$, $y \in \mathbb{Z}/m\mathbb{Z}$. But $x \otimes y$ generate $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$. Hence the tensor product is zero. $\square$

EXAMPLE: EXTENSION OF SCALARS

Let R be a commutative ring with identity and let V be an R-module.

If $f : R \rightarrow R^{'}$ be a ring homomorphism then $R^{'}$ is an R-module.

The tensor product $V \otimes R^{'}$ is an $R^{'}$-module, called the *extension of* V *over* $R^{'}$.

Given any field extension $K < K^{'}$, one can extend scalars from K to $K^{'}$.

For example, if $V_{\mathbb{R}}$ is a vector space over $\mathbb{R}$, then $V_{\mathbb{R}} \otimes \mathbb{C}$ is a vector space over $\mathbb{C}$.

If $dim(V_{\mathbb{R}}) = n$ and $V_{\mathbb{R}}$ has basis $v_1, \ldots, v_n$ then an element $x \otimes c \in V_{\mathbb{R}} \otimes \mathbb{C}$ has a unique expression

$$x \otimes c = \sum_{i=1}^{n} a_i v_i \otimes c_i$$

where $a_i \in \mathbb{R}$, $c_i \in \mathbb{C}$.

GROUP REPRESENTATIONS

○ $G = SL_n(\mathbb{F})$ the group of $n \times n$ matrices of determinant 1

○ V a finite dimensional vector space over $\mathbb{F}$

○ $\phi : SL_n(\mathbb{F}) \to GL(V)$, a group homomorphism, gives a representation of $SL_n(\mathbb{F})$ on V

○ $GL(V)$, the general linear group of V, is the group of all bijective linear transformations $V \to V$ where the group operation is composition of functions

We note that $G = SL_n(\mathbb{F})$ has a natural representation on $\mathbb{F}^n$, but representations on more general finite dimensional vector spaces V also arise.

LIE ALGEBRAS

A *Lie algebra* is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ together with a binary operation

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

called the Lie bracket, which satisfies the following axioms:

Bilinearity:

$$[ax + by, z] = a[x, z] + b[y, z], \quad [z, ax + by] = a[z, x] + b[z, y]$$

for all scalars a, b in $\mathbb{F}$ and all x, y, z in $\mathfrak{g}$

Alternating on $\mathfrak{g}$: $[x, x] = 0$ for all x in $\mathfrak{g}$

The Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all x, y, z in $\mathfrak{g}$.

REPRESENTATIONS OF LIE ALGEBRAS

Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$ be the Lie algebra of $n \times n$ matrices of trace 0. The Lie bracket can be interpreted here as the matrix operation

$$[x, y] = xy \; - \; yx.$$

Let V be a finite dimensional vector space over $\mathbb{F}$.

Then $\rho : \mathfrak{sl}_n(\mathbb{F}) \to End(V)$, a Lie algebra homomorphism, gives a representation of $\mathfrak{sl}_n(\mathbb{F})$ on V.

The algebra $End(V)$ is the commutator Lie algebra of V. That is, $\rho \in End(V)$ is a linear map satisfying the commutator relation

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$$

for $x, y \in \mathfrak{g}$.

ACTION OF SYMMETRIC GROUP ON ITERATED TENSORS

Let R be a commutative ring with identity and let V be an R-module.

Let $T^n(V) = V \otimes \cdots \otimes V = V^{\otimes n}$ be the iterated tensor product.

Let S_n be the symmetric group acting as permutations on a set of n letters $\{1, 2, \ldots, n\}$.

Then S_n acts on $V^{\otimes n}$ by permuting the indices.

If $\sigma \in S_n$, then σ acts on $v_1 \otimes \cdots \otimes v_n$ as $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$.

REPRESENTATIONS USING TENSOR PRODUCTS

Let $\mathfrak{g}$ be a Lie algebra and let

$$\rho_1 : \mathfrak{g} \to End(V_1)$$

$$\rho_2 : \mathfrak{g} \to End(V_2)$$

be representations of $\mathfrak{g}$ on vector spaces V_1 and V_2 respectively.

Then there is a representation $\rho_3 = \rho_1 \otimes \rho_2 : \mathfrak{g} \to End(V_1 \otimes V_2)$ given by

$$\rho_3(x)(v_1 \otimes v_2) = \rho_1(x)(v_1) \otimes v_2 + v_1 \otimes \rho_2(x)(v_2),$$

$x \in \mathfrak{g}$, $v_i \in V_i$, called the tensor product of the representations.

An example is given by the vector-spinor representation of $\mathfrak{so}(2k)$ which is described as follows.

Take the standard representation of $\mathfrak{so}(2k)$ on $V_1 = \mathbb{R}^{2k}$ and the spin representation on $V_2 = \mathbb{R}^{2^k}$. Then the tensor product is a representation on the $2k \times 2^k$ dimensional vector space $V_1 \otimes V_2$.

EXTERIOR POWERS

Let R be a commutative ring with identity and let V be an R-module of dimension n. Let $e_1, \ldots, e_n$ be a basis for V.

A multilinear map $f : V^{\otimes n} \to W$ is called *alternating* if for $x_i \in V$

$$f(x_1, \ldots, x_n) = 0$$

whenever $x_i = x_j$ for some $i \neq j$.

Let $\mathfrak{U}_n$ be the submodule of $T^n(V) = V^{\otimes n}$ generated by all elements of the form

$$x_1 \otimes \cdots \otimes x_n$$

where $x_i \in V$, $x_i = x_j$ for some $i \neq j$.

Define

$$\Lambda^n V = V^{\otimes n} / \mathfrak{U}_n.$$

For $k \leq n$, a basis of $\Lambda^k V$ consists of the vectors

$$e_{i_1} \wedge \ldots \wedge e_{i_k} = \sum_{\sigma \in \mathcal{S}_k} \varepsilon(\sigma)\, e_{i_{\sigma(1)}} \otimes \ldots \otimes e_{i_{\sigma(k)}}\,,$$

with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, where $\mathcal{S}_k$ is the permutation group on k letters and $\varepsilon(\sigma)$ is the signature of σ.

EXTERIOR (WEDGE) PRODUCT

Let R be a commutative ring with identity and let V be an R-module. Given $u, v \in V$, the *exterior product* $u \wedge v \in \Lambda^2 V$ has the defining properties

$$u \wedge u = 0$$

which is equivalent to

$$u \wedge v = -v \wedge u$$

and for $a_1, a_2 \in R$, $u_1, u_2 \in V$

$$(a_1 u_1 + a_2 u_2) \wedge v = a_1 u_1 \wedge v + a_2 u_2 \wedge v.$$

If $u \in V$ is non zero, then $u \wedge v = 0$ if and only if $v = \lambda u$ for some $\lambda \in R$.

In general, for $u_1, \ldots, u_n \in V$, an element $u_1 \wedge \cdots \wedge u_n \in \Lambda^n V$ is linear in each variable u_i and interchanging two variables changes the sign of the product.

REPRESENTATIONS USING EXTERIOR PRODUCTS

Let $\mathfrak{g}$ be a Lie algebra. A fundamental representation is an irreducible finite-dimensional representation whose highest weight is a fundamental weight.

Let V be a fundamental representation for $\mathfrak{sl}_n(\mathbb{R})$. Then other fundamental representations can be constructed from V by using the exterior product. For instance, if $\{v_1, \ldots, v_n\}$ is a basis for V, then

$$\{v_1 \wedge v_2, v_1 \wedge v_3, \ldots, v_1 \wedge v_n\}$$

is the basis of another fundamental representation.

For finite dimensional simple Lie algebras, all fundamental representations can be obtained from a subset of given ones, called basic modules.

Tensor products and exterior products are built into nature!.