Hyperbolic Kac-Moody Weyl groups, lattices and actions on trees

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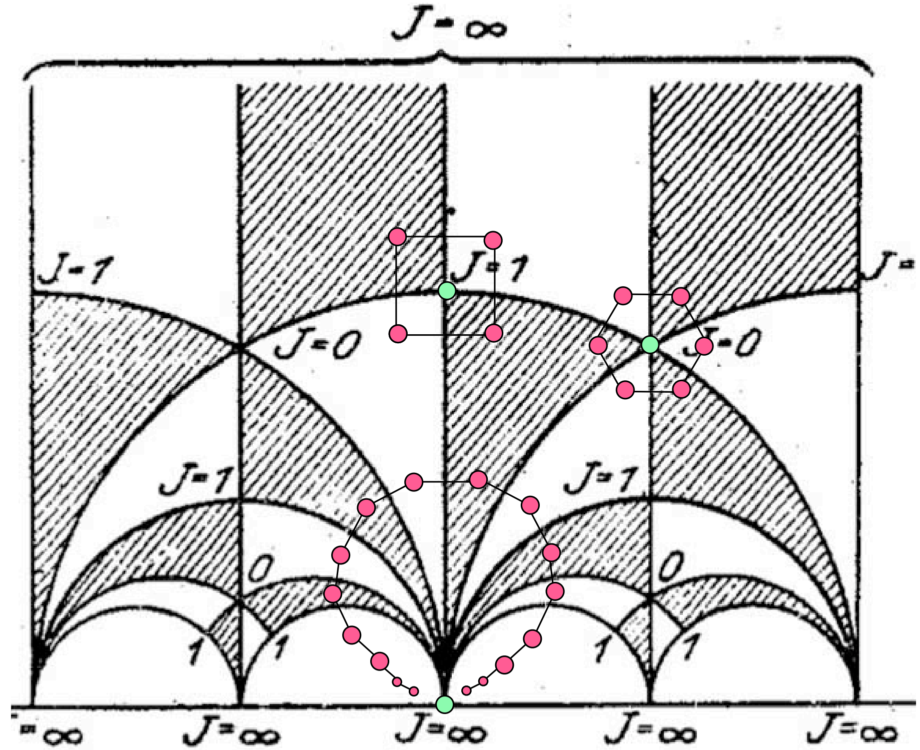
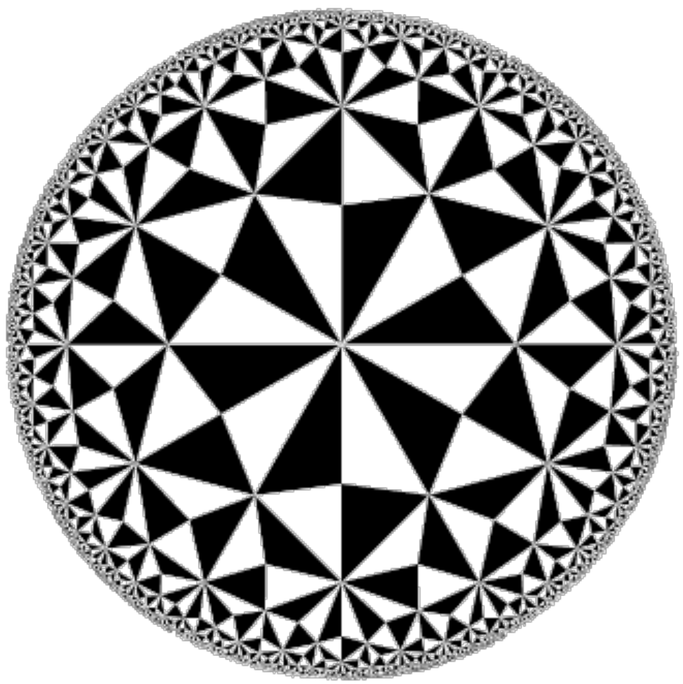


Fig. 7.

The underlying diagram is Figure 7 in Felix Klein's paper 'Uber die Transformation der elliptischen Funktionen und die Auflosung der Gleichungen funften Grades' which appeared in May 1878 in Mathematische Annalen. This shows the standard apartment of the Tits building of a hyperbolic Kac-Moody group whose Weyl group is $PGL_2(\mathbb{Z})$.

HYPERBOLIC TRIANGLE GROUPS

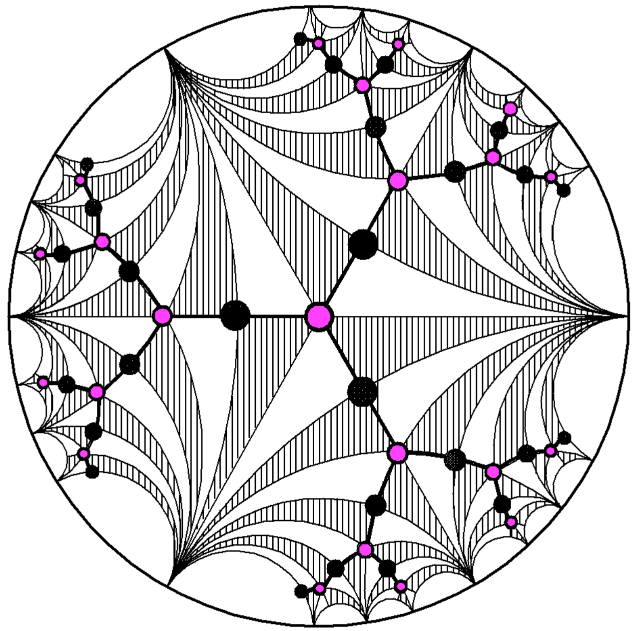
Triangle groups of compact hyperbolic type are groups $W = W(r, s, t)$ generated by reflections in the sides of a triangle in the hyperbolic plane (Poincaré disk) with angles π/r, π/s, π/t, with $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} < 1$ and all angles greater than 0.



While such groups have been widely studied, there is only sparse literature on triangle groups of noncompact hyperbolic type, which are generated by reflections in the sides of a hyperbolic triangle with one or more angles equal to zero, that is, one or more of r, s, t equal ∞. Such a hyperbolic triangle has a point on the boundary of the Poincaré disk for every zero angle.

NONCOMPACT HYPERBOLIC TESSELLATIONS

Some noncompact hyperbolic tessellations were described in the book of Magnus, *Noneuclidean tessellations and their groups*, Academic Press, 1974, but only those arising from the modular group $W(\infty, 3, 2) \cong PGL_2(\mathbb{Z})$ and its finite index subgroups.



We will describe all the noncompact hyperbolic reflection groups that arise as Weyl groups of rank 3 symmetrizable Kac-Moody algebras of noncompact hyperbolic type and describe their noncompact tessellations of the Poincaré disk.

We will also construct actions of these groups on simplicial trees that are naturally related to their tessellations.

We first discuss the representation theoretic properties of locally compact groups associated to Kac-Moody algebras.

KAC-MOODY ALGEBRAS

A Kac-Moody algebra is the most natural generalization to infinite dimensions of a finite dimensional simple Lie algebra.

The data for constructing a Kac-Moody algebra includes a 'generalized Cartan matrix' A which is a generalization of the notion of a Cartan matrix of a finite dimensional Lie algebra, and which encodes the same information as a Dynkin diagram.

The *rank* of a Kac-Moody algebra equals the number of rows and columns in the generalized Cartan matrix. 'Symmetrizability' is an important property of a generalized Cartan matrix, necessary for the existence of a well-defined symmetric invariant bilinear form $(\cdot \mid \cdot)$ on the Kac-Moody algebra.

TYPES OF GENERALIZED CARTAN MATRICES

Finite type: A is positive definite, $det(A) > 0$.

Affine type: A is positive semi-definite but not positive definite, $det(A) = 0$.

Hyperbolic type: A is neither of finite nor affine type, but every proper indecomposable submatrix is either of finite or affine type, $det(A) < 0$.

Compact hyperbolic type: Every proper indecomposable submatrix of A is of finite type.

Noncompact hyperbolic type: A has a proper indecomposable submatrix of affine type.

WEYL GROUP OF A KAC-MOODY ALGEBRA

Let $\mathfrak{g}$ be a Kac-Moody algebra and let $\mathfrak{h}$ denote its Cartan subalgebra. Then $\mathfrak{g}$ has an associated root system Φ.

Let a_{ij} denote the entries of the generalized Cartan matrix A. For each simple root $\alpha_i \in \Phi$ we define the simple root reflection

$$w_i(\alpha_j) := \alpha_j - a_{ij}\alpha_i.$$

The w_i generate a subgroup $W \subseteq Aut(\mathfrak{h}^*)$, called the *Weyl group*.

For $i, j \in I$, and for $i \neq j$, we set

$$c_{ii} := 1, \ c_{ij} := 2, 3, 4, 6 \text{ or } \infty$$

if

$$a_{ij}a_{ji} = 0, 1, 2, 3, \text{ or } \geq 4 \text{ respectively .}$$

Then W is the Coxeter group with presentation:

$$W = \langle m_i \mid i \in I, (m_i m_j)^{c_{ij}} = 1, \ if \ c_{ij} \neq \infty \rangle,$$

and W acts on the set of all roots, preserving a symmetric bilinear form.

Then W is infinite $\iff$ Φ is infinite $\iff$ $\mathfrak{g}$ is infinite dimensional.

Example: Rank 2 affine or hyperbolic
Let

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$$

If $ab = 4$, A is affine. If $ab > 4$, A is hyperbolic. For $ab > 4$, A has Dynkin diagram

We have

$$W = < w_1, w_2 \mid w_1^2 = 1, w_2^2 = 1 > .$$

Then W is the infinite dihedral group

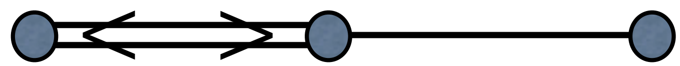
$$W \quad = \quad \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \quad \cong \quad \mathbb{Z} \rtimes \{\pm 1\}.$$

Example: Rank 3 noncompact hyperbolic type

Let

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

A has Dynkin diagram



The Weyl group of A is the $(\infty, 3, 2)$-triangle group:

$$W \ = < w_1, w_2, w_3 \mid w_1^2 = w_2^2 = w_3^2 = 1, (w_2 w_3)^3 = (w_1 w_3)^2 = 1 > \cong \ PGL_2(\mathbb{Z})$$

A has affine submatrix

$$A_1^{(1)} \quad = \quad \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

GROUPS ASSOCIATED TO KAC-MOODY ALGEBRAS

In order to associate an analog of a Lie group to a Kac-Moody algebra, Tits associated a group functor G_A on the category of commutative rings, such that for any symmetrizable generalized Cartan matrix A and any ring R there exists a group $G_A(R)$. He also showed that over a field K the group $G_A(K)$ is unique up to isomorphism.

Locally compact forms of $G = G_A(K)$ can be constructed over finite fields, that is, for $K = \mathbb{F}_q$. This was first observed by Tits (1980s) and distinct constructions were given by Carbone and Garland (1999), Rémy and Ronan (2006). When constructed over a finite field $\mathbb{F}_q$, G is locally compact and totally disconnected.

ACTIONS ON HYPERBOLIC BUILDINGS

Moreover, $G = G_A(\mathbb{F}_q)$ admits an action on a locally finite simplicial complex known as the Tits building X. The Tits building has a geometric realization which can be expressed as a union of subcomplexes (apartments) which are isomorphic Coxeter complexes satisfying certain axioms.

In our case of interest, when G is of rank 3 noncompact hyperbolic type, apartments in X are hyperbolic planes tessellated by the action of a hyperbolic Weyl group W of noncompact type.

LATTICES IN LOCALLY COMPACT KAC-MOODY GROUPS

We recall that a discrete subgroup Γ of a locally compact group G is a *lattice* if the quotient $\Gamma\backslash G$ carries a finite invariant measure. If further, $\Gamma\backslash G$ is *compact*, we say that Γ is cocompact. Otherwise we say that Γ is *nonuniform*.

A classical example of a nonunifom lattice is $\Gamma = SL_2(\mathbb{F}_q[t]) \leq G = SL_2(\mathbb{F}_q((t^{-1})))$.

A locally compact Kac-Moody group $G = G_A(\mathbb{F}_q)$ is a source of lattice subgroups with interesting representation theoretic, K-theoretic and 'arithmetic' properties.

The group $G = G_A(\mathbb{F}_q)$ has a twin BN-pair corresponding to positive and negative roots. The minimal parabolic subgroup B^- of the negative BN-pair is known to be a nonuniform lattice subgroup of G (Carbone and Garland (1999), Rémy (1999)).

We study $G = G_A(\mathbb{F}_q)$ and B^- in analogy with lattices in Lie groups.

SUMMARY OF INGREDIENTS SO FAR

A, a symmetrizable generalized Cartan matrix of affine or hyperbolic type

$\mathfrak{g} = \mathfrak{g}_A(K)$, a Kac-Moody algebra over a field K

$\mathfrak{g}_{\mathbb{F}_q} = \mathfrak{g}_A(\mathbb{F}_q)$, a form of $\mathfrak{g}$ over $\mathbb{F}_q$

$\mathfrak{h}$, the Cartan subalgebra of $\mathfrak{g}$

$W \subseteq Aut(\mathfrak{h}^*)$, the Weyl group of $\mathfrak{g}$

$G = G_A(\mathbb{F}_q)$, a locally compact totally disconnected group associated to $\mathfrak{g}_{\mathbb{F}_q}$

X, the locally finite Tits building of a BN-pair for G

B^-, the minimal parabolic subgroup of the negative BN-pair for G

$\Gamma = B^- \leq G$, a nonuniform lattice

We will now discuss the representation theoretic and K-theoretic properties of the groups $G = G_A(\mathbb{F}_q)$ in the setting described above.

Summary of representation theoretic and K-theoretic properties of groups

A locally compact group G satisfies the *Haagerup property* if it admits a continuous, isometric, proper action on an affine Hilbert space.

The Haagerup property is a strong negation of *Kazhdan's Property (T)* which states that every continuous action of G by isometries on a Hilbert space has a fixed point.

The *Baum-Connes conjecture* in non-commutative geometry conjectures an isomorphism between K-homology and K-theory, relating the analytic and topological properties of a group.

The strongest form of the Baum-Connes conjecture: the conjecture with coefficients in any C^*-algebra.

Representation theoretic properties and results on the Baum-Connes conjecture for Kac-Moody groups can be summarized in the following table, where G is a symmetrizable locally compact affine or hyperbolic Kac-Moody group over a finite field $\mathbb{F}_q$ (Carbone (2009)).

Rank r of G	**Assumptions on G**	**Properties of G**
$r = 2$	affine or hyperbolic type	Haagerup property Baum-Connes conjecture with coefficients
$r = 3$	noncompact hyperbolic type q sufficiently large	Haagerup property Baum-Connes conjecture with coefficients
$r = 3$	compact hyperbolic type q sufficiently large	Property (T) Baum-Connes assembly maap is injective
$r \geq 3$	affine type q sufficiently large	Property (T) Baum-Connes assembly map is injective
$4 \leq r \leq 10$	hyperbolic type q sufficiently large	Property (T) Baum-Connes assembly map is injective

Special case - G has rank 3 noncompact hyperbolic type

Let G be a locally compact symmetrizable rank 3 Kac-Moody group of noncompact hyperbolic type. Then the minimal parabolic subgroup B^- of the negative BN-pair for G is a nonuniform lattice subgroup of G (Carbone and Garland (1999), also Rémy (1999)) and B^- has the Haagerup property (Carbone (2009)).

It is also known that the Haagerup property predicts a continuous proper action by isometries on a 'median space' (Chatterji, Drutu and Haglund (2007)). In joint work with Yusra Naqvi (2009), we explicitly construct a proper action of the lattice $B^- \leq G$ on a bihomogeneous tree $\mathcal{X}$, which is an example of a median space.

Construction of a lattice acting properly on a tree cannot be made in a locally compact Kac-Moody group G of affine or hyperbolic type (compact or noncompact) if $rank(G) \geq 4$ since such groups have Kazhdan's Property (T). By a theorem of de la Harpe and Valette, every action of a Property (T) group on a tree fixes a vertex. In particular this applies to lattice subgroups of G.

On the other hand, if G is a rank 2 locally compact Kac-Moody group, then G is symmetrizable and is of affine or hyperbolic type, The Tits building of G is then itself a homogeneous tree X. It is known that G has the Haagerup property and that the subgroup B^- acts properly on X (Carbone and Garland). Our results therefore give an action on a simplicial tree for any locally compact Kac-Moody group with the Haagerup property.

PROPER ACTION OF $B^- \leq G$ ON A TREE

In joint work with Yusra Naqvi (2009), we explicitly construct a proper action of the lattice $B^- \leq G$ on a bihomogeneous tree $\mathcal{X}$ for a symmetrizable locally compact rank 3 Kac-Moody group G of noncompact hyperbolic type.

The Weyl group W of G is a hyperbolic triangle group $W = W(r, s, t)$ with at least one of r, s, t equal to ∞.

The tree $\mathcal{X}$ is constructed from a tree $\mathcal{Y}$ which we associate to the Weyl group W. The tree $\mathcal{X}$ is naturally inscribed in the Tits building X of G, a rank 3 locally finite hyperbolic building.

Tree $\mathcal{Y}$ associated to W $\hookrightarrow$ Poincaré disk $\mathbb{D}$ tessellated by W

subtree $\downarrow$ $\qquad$ $\downarrow$ fundamental apartment

Tree $\mathcal{X}$ $\hookrightarrow$ Tits building X of G

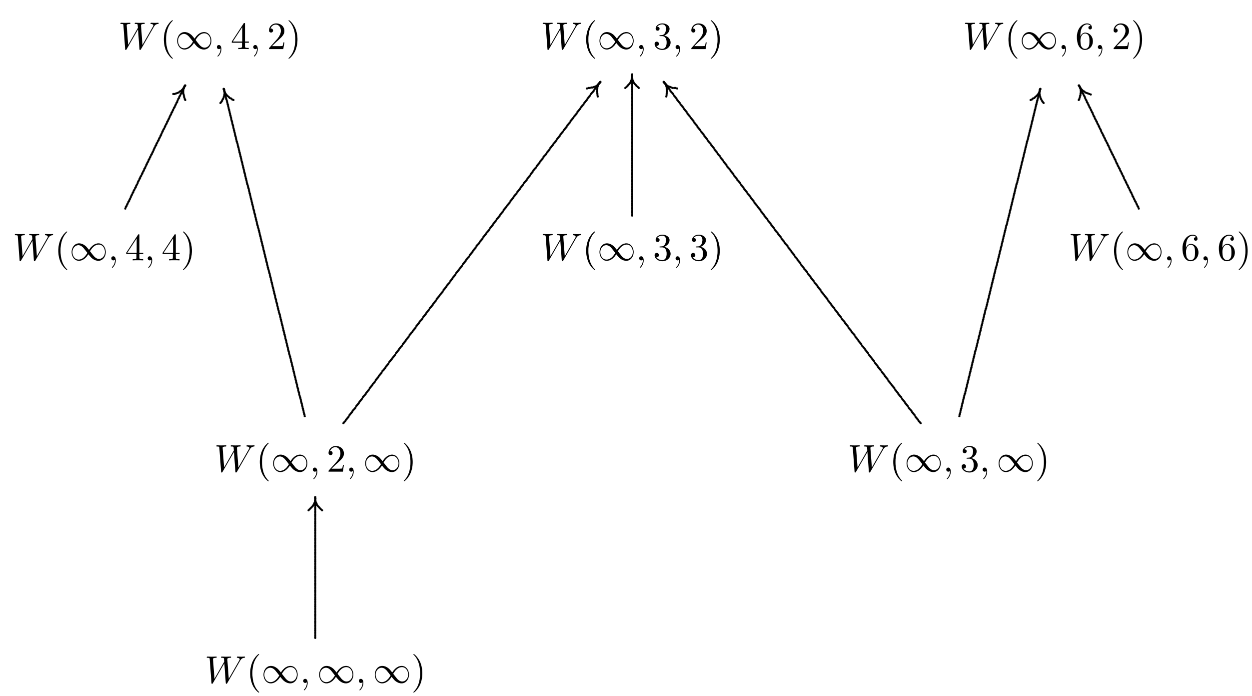
The natural retraction of X onto $\mathbb{D}$ induces a retraction of $\mathcal{X}$ onto $\mathcal{Y}$.

WEYL GROUPS OF RANK 3 NONCOMPACT HYPERBOLIC TYPE

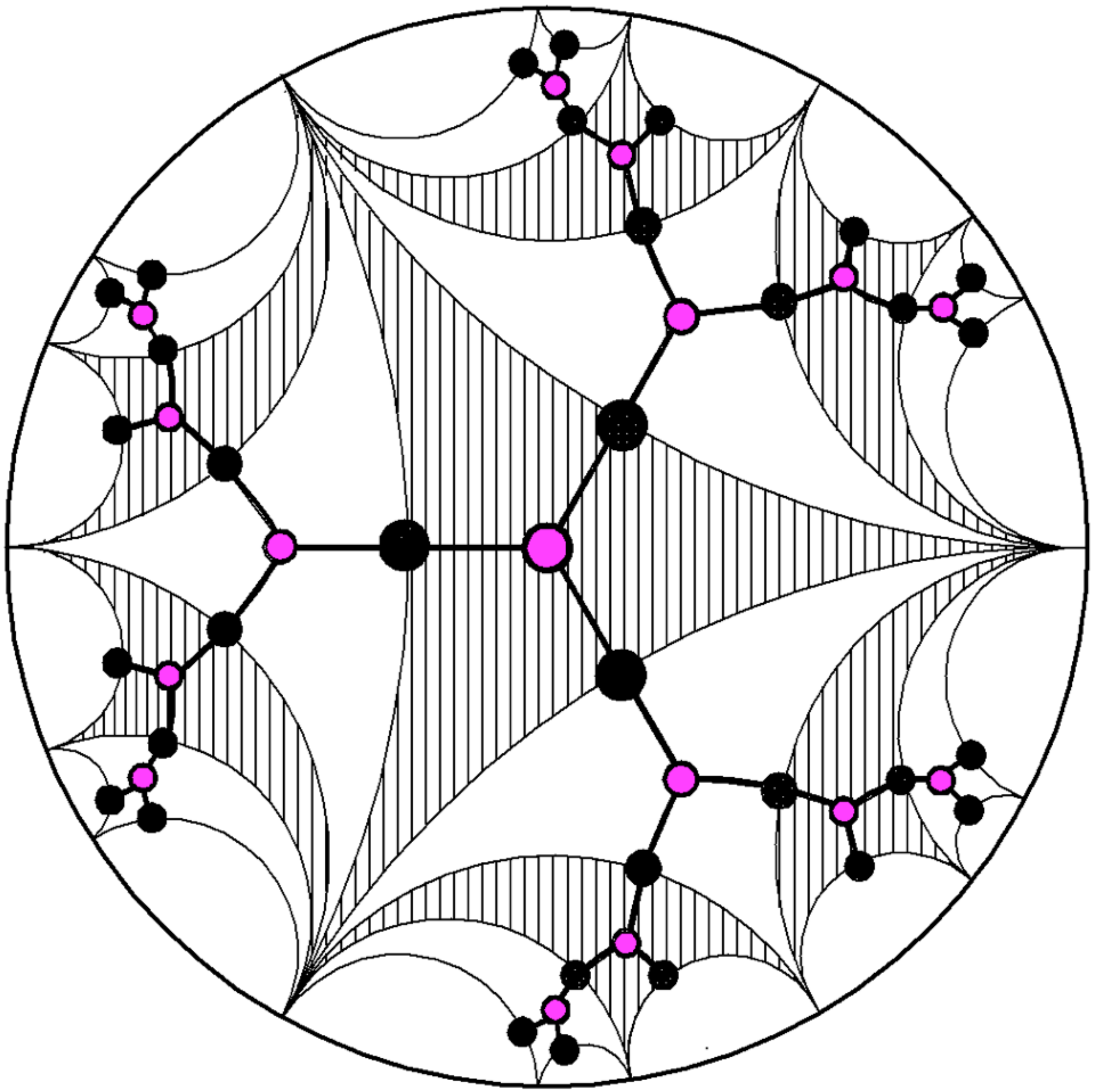
The classification of hyperbolic Dynkin diagrams shows that there are 33 possible symmetrizable generalized Cartan matrices of rank 3 noncompact hyperbolic type. However, there are only 9 isomorphism classes of Weyl groups:

$$W(\infty,\infty,\infty),\ W(\infty,2,\infty),\ W(\infty,3,\infty),$$
$$W(\infty,3,2),\ W(\infty,4,2),\ W(\infty,6,2),$$
$$W(\infty,3,3),\ (\infty,4,4),\ (\infty,6,6).$$

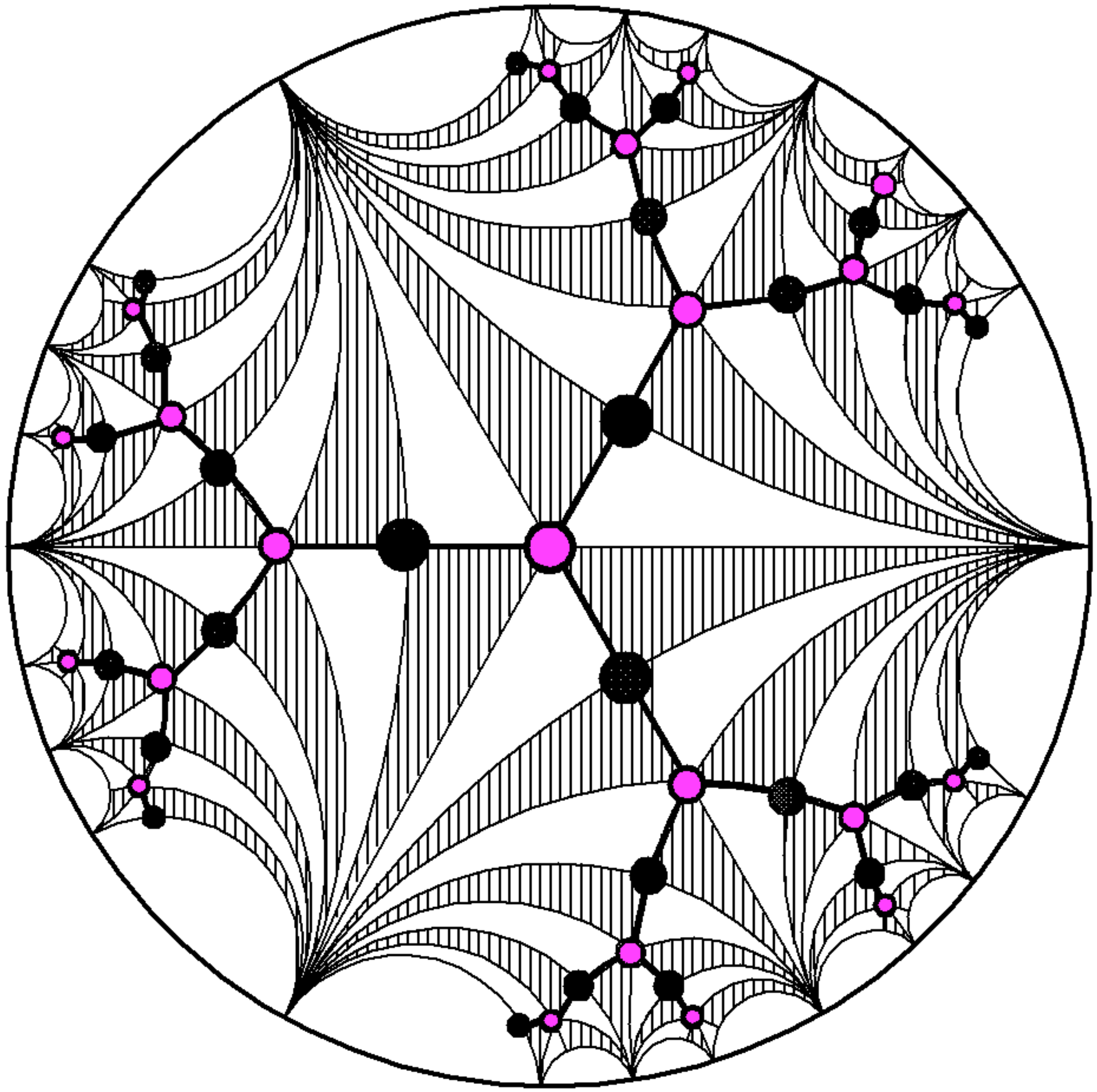
There are many subgroup relations among these:



When $W = W(\infty, \infty, \infty)$, the tree associated to W is $\mathcal{Y} = \mathcal{Y}_{3,2}$.

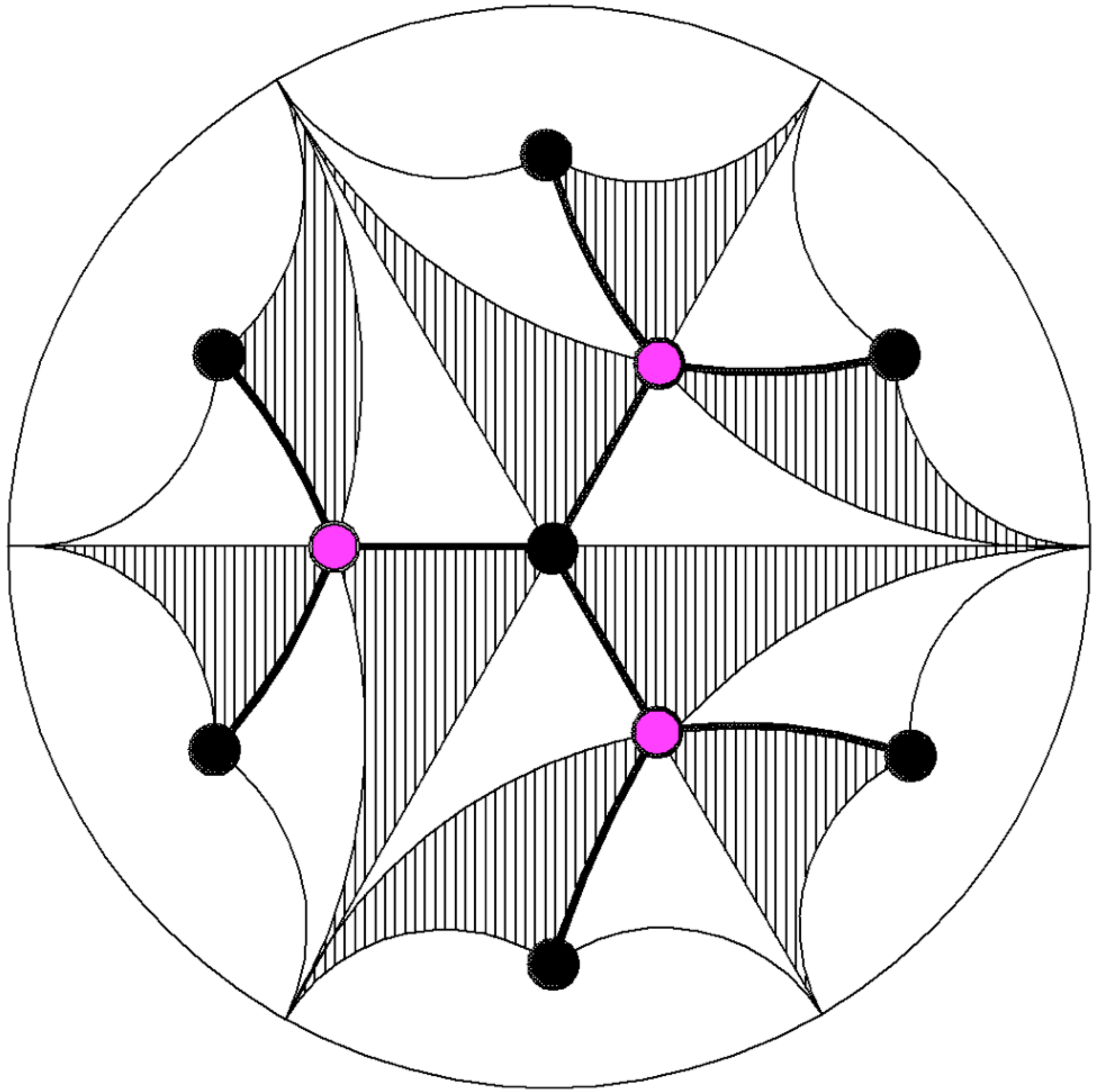


When $W = W(\infty, 3, 2) \cong PGL_2(\mathbb{Z})$, the tree associated to W is $\mathcal{Y} = \mathcal{Y}_{3,2}$.

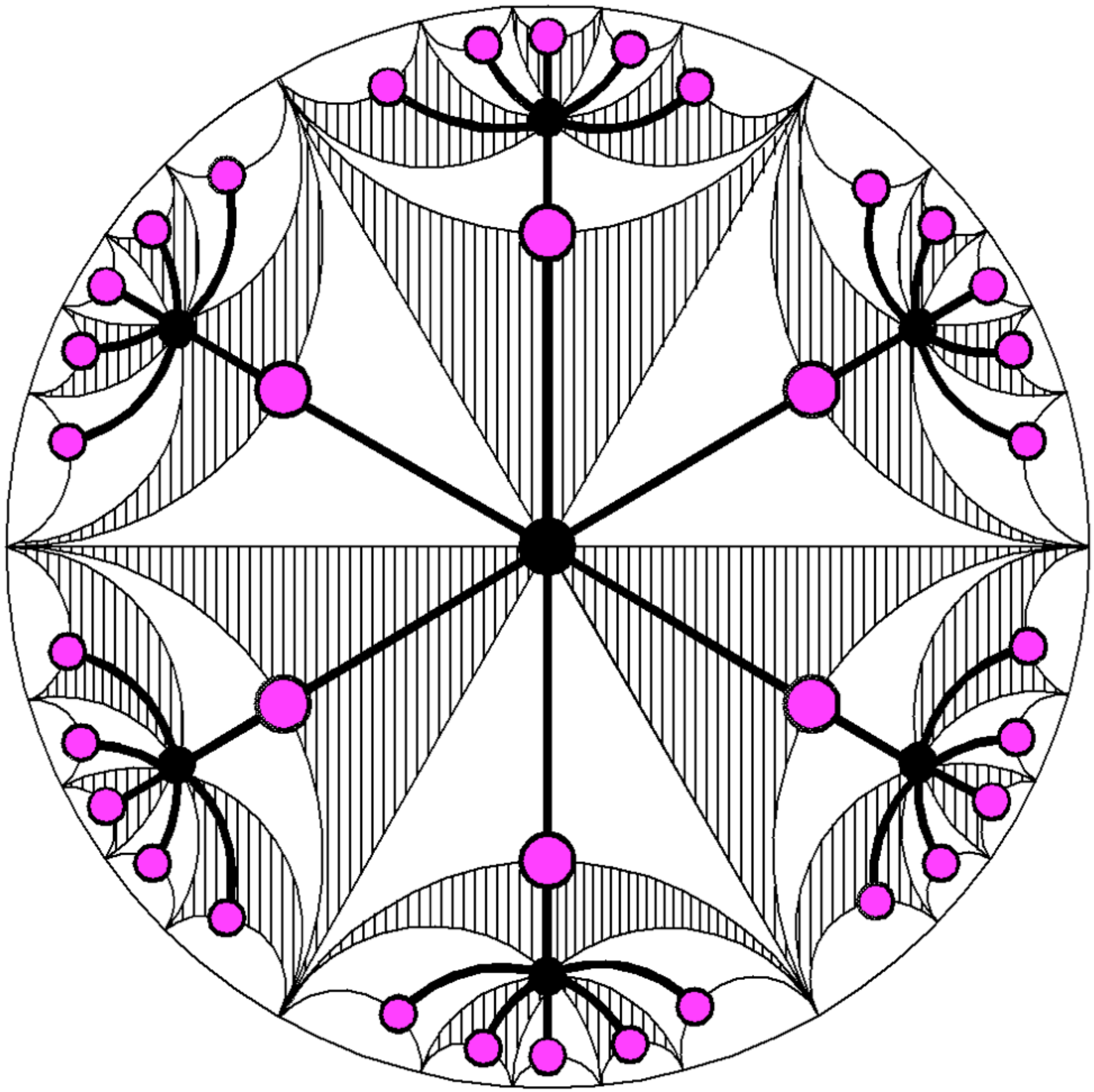


Note that $W(\infty, 3, 2)$ has the same tree as its subgroup $W(\infty, \infty, \infty)$ of index 6.

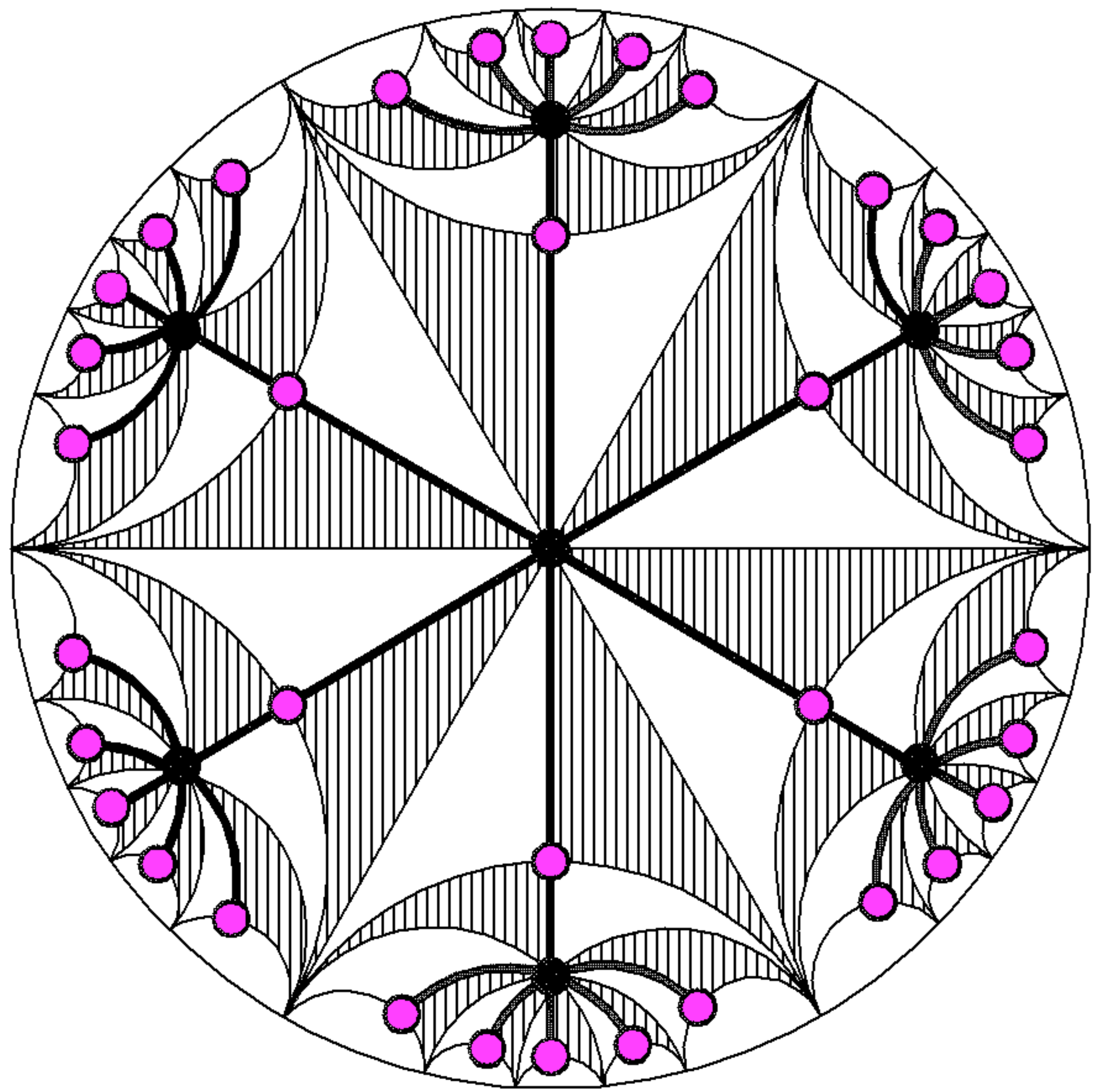
When $W = W(\infty, 3, 3)$, the tree associated to W is $\mathcal{Y} = \mathcal{Y}_{3,3}$.



When $W = W(\infty, \infty, 3)$, the tree associated to W is $\mathcal{Y} = \mathcal{Y}_{6,2}$.



When $W = W(\infty, 6, 2)$, the tree associated to W is $\mathcal{Y} = \mathcal{Y}_{6,2}$.



Note that $W(\infty, 6, 2)$ has the same tree as its subgroup $W(\infty, \infty, 3)$ of index 2.

AMALGAM DECOMPOSITIONS OF HYPERBOLIC WEYL GROUPS

A group G is said to have Property (FA) if every action of G on a tree has a global fixed point. Serre showed that if a group has Property (FA), then it cannot split as a free product with amalgamation or HNN extension.

Theorem (Carbone and Naqvi (2009)) Let A be a symmetrizable affine or hyperbolic generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a locally compact Kac-Moody group associated to A and the finite field $\mathbb{F}_q$, with q sufficiently large.

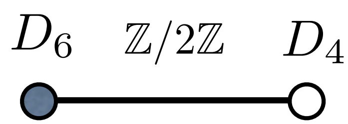
(i) If $r = rank(G) = 2$ or if $rank(G) = 3$ and G has noncompact hyperbolic type, then G has the Haagerup property and $W = W(A)$ has a nontrivial amalgamated product decomposition. Thus W does not have Property (FA).

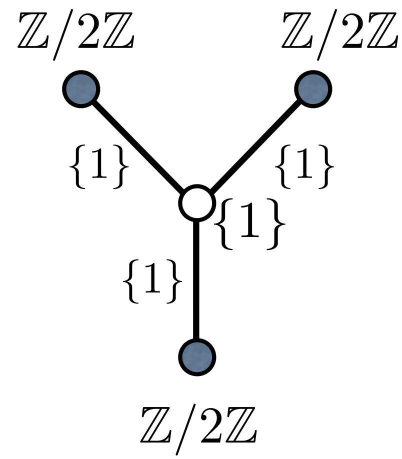
(ii) If $r = rank(G) = 3$ and G has compact hyperbolic type, or if $rank(G) \geq 3$ and G has affine type, or if $4 \leq r \leq 10$ and G has hyperbolic type, then G has Property (T) and $W = W(A)$ has Property (FA).

GRAPH OF GROUPS PRESENTATION FOR W ON $\mathcal{Y}$

The action of W the Poincaré disk induces an action of W on $\mathcal{Y}$. If $\mathcal{F}$ is a fundamental triangle for the action of W on the Poincaré disk, then the intersection of the closure of $\mathcal{F}$ with $\mathcal{Y}$ will be a fundamental domain for the action of W on $\mathcal{Y}$.

The quotient graphs of groups for $W(\infty, 3, 2)$ and $W(\infty, \infty, \infty)$ on $\mathcal{Y}$ are:



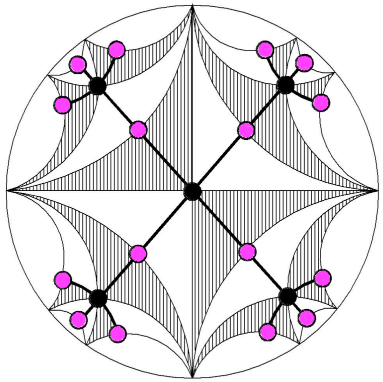


Using the Bass-Serre correspondence between group actions on trees and quotient graphs of groups we obtain:

$$W(\infty, 3, 2) \cong PGL_2(\mathbb{Z}) \cong D_6 *_{\mathbb{Z}/2\mathbb{Z}} D_4$$

$$W(\infty, \infty, \infty) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$$

These amalgam decompositions coincide with the JSJ-decompositions for Weyl groups given by Mihalik and Ratcliffe-Tschantz.



OTHER DIRECTIONS, OPEN QUESTIONS

○ Other noncompact tessellations for A of rank 3 noncompact hyperbolic type but not symmetrizable ?

○ Strengthen the Baum-Connes conjecture for Kac-Moody groups and other groups with Property (T) ?

○ Let G be a rank 2 locally compact Kac-Moody group over a finite field $\mathbb{F}_q$. If $q = 2^s$ then G contains a cocompact lattice $\Gamma \cong M_q *_{M_q \cap \widetilde{M_q}} \widetilde{M_q}$ with quotient a simplex. When $q = 2$, G also contains an infinite descending chain of cocompact lattices ... $\Gamma_3 \leq \Gamma_2 \leq \Gamma_1 \leq \Gamma$ (Carbone and Cobbs (2009)).

○ If our Kac-Moody group G has the Haagerup property, we claim that the graph of groups presentation for $B^- \leq G$ acting on a simplicial tree satisfies the Kac-Peterson conjecture on the structure of B^-. We have a proof of this when G has rank 2. What is the quotient graph of groups $B^- \backslash \mathcal{X}$ when G has rank 3 noncompact hyperbolic type?

○ Development of a theory of congruence subgroups for nonuniform lattices in locally compact hyperbolic Kac-Moody groups is in progress.

○ Construction of Eisenstein series on quotients of Tits buildings by nonuniform lattices in locally compact hyperbolic Kac-Moody groups is in progress.

○ $\mathbb{R}$-forms of hyperbolic Kac-Moody groups play a role in supergravity theories.