Solutions for Homework 10, Math 477, Fall 2018

Eric A. Carlen
Rutgers University

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Instructions Do all of these problems – at least – but neatly write up and turn in *only* those that are marked with a $*$, including the first one below:

1* Let $\{X_j\}_{j\in\mathbb{N}}$ be an independent, identically distributed sequence of random variables that are Poisson with parameter 1.

(a) Compute $p(\lambda) = \ln(\mathrm{E}(e^{\lambda X_1}))$, and then compute

$$s(x) := \sup_{\lambda>0}\{\lambda x - p(\lambda)\} \ .$$

(b) Let $\overline{X}_N$ denote the sample mean. Use Cramér's Theorem to estimate $P(\overline{X}_N \geq 1.02)$ for $N = 10^4$, $N = 10^5$ and $N = 10^6$.

SOLUTION (a) We compute

$$\mathrm{E}e^{\lambda X_1} = \sum_{n=0}^{\infty} e^{\lambda n}\frac{1}{n!}e^{-1} = e^{e^\lambda - 1} \ .$$

Hence

$$p(\lambda) = e^\lambda - 1 \ .$$

We then compute $s(x) = \sup\lambda > 0\{\lambda x - p(\lambda)\}$ by fixing x and defining $\varphi(\lambda) = \lambda x - p(\lambda)$. Then $0 = \varphi'(\lambda)$ is the equation

$$0 = x - c^\lambda$$

so $\lambda = \ln(x)$ is the solution, and this is positive for $x > 1$. Hence for all $x > 1$, we plug $\lambda = \ln(x)$ into $\varphi(\lambda)$ to find

$$s(x) = x\ln x - x + 1 \ .$$

If $x <!$, the best we can do is take $\lambda = 0$ and then $s(x) = 0$.

(b) By Carmer's Theorem, for $x > \mathrm{E}(X_1) = 1$,

$$P(\overline{X}_N > x) \leq e^{-Ns(x)} \ .$$

Then

$$s(1.02) = 0.00019867985...$$

It then follows that $P(\overline{X}_{10^4} \geq 1.02) \leq .1371337571...$, $P(\overline{X}_{10^5} \geq 1.02) \leq 2.352034826... \times 10^{-9}$, $P(\overline{X}_{10^6} \geq 1.02) \leq 5.181284977... \times 10^{-87}$.

From the Problems in Chapter 8:

6 Let X_1 be the number on one toss of a fair die. Then simple computations give

$$\mu := \mathrm{E}(X_1) = \frac{21}{6}\ , \quad \mathrm{E}(X_1^2) = \frac{91}{6} \quad \sigma^2 := \mathrm{Var}(X_1) = \frac{35}{12}\ .$$

By the Central Limit Theorem, the distribution for the sum of the results for 80 tosses is approximately that of

$$80\mu + \sqrt{80}\sigma Z$$

where Z is standard normal. This simplifies to

$$280 + (15.275)Z\ .$$

So we need to compute

$$P(280 + (15.275)Z < 300) = P(Z < 20/(15.275)) = P(Z < 1.309) \approx 0.9049\ .$$

11 The total change in the price of the stock is

$$W = \sum_{j=1}^{10} X_j\ ,$$

and this is normal with zero mean and variance 10. Hence $W = \sqrt{10}Z$ where Z is standard normal. Then

$$P(W > 5) = P(Z > 5/\sqrt{10}) \approx P(Z > 1.581) = 1 - P(Z \leq 1.581) = 1 - 0.9429\ .$$

From the Problems in Chapter 9: **4** If there are zero while balls in the first urn, we will surely select a black ball from the first urn and a while ball from the second, After swapping, there will be one white ball in the first urn. So

$$P(X_{n+1} = 1 | X_n = 0) = 1$$

and all of other probabilities are zero. Hence the first row of the transition matrix is $(0, 1, 0, 0,)$.

If there is one while ball in the left urn, the probability that we select WW or BB is $\frac{4}{9}$. This is the probability of no change. If we select BW, the number increases by 1. The probability of this is $\frac{4}{9}$. Otherwise, with probability $\frac{1}{9}$, the number decreases by 1. Hence the second row is

$$\frac{1}{9}(1, 4, 4, 0)\ .$$

Likewise, we find the third row is

$$\frac{1}{9}(0, 4, 4, 1)\ ,$$

and the fourth row is $(0, 0, 1, 0)$. Hence the transition matrix is

$$P := \frac{1}{9}\begin{bmatrix} 0, 9, 0, 0 \\ 1, 4, 4, 0 \\ 0, 4, 4, 1 \\ 0, 0, 9, 0 \end{bmatrix}\ .$$

10 See the notes on Markov chains where this is worked out. The final answer is $1/6$.