Homework 2 Solutions, Math 477, Fall 2018

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October 2, 2018

Chapter 2 Problems:

43 The sample space consists S of all sequences of length N with distinct entries, two of which are A and B. The cardinality of S is $N!$, and each outcome is equally likely. Let E_1 be the event that A is the first entry. (A is seated at the left), let E_2 be the event that A is the last entry. (A is seated at the right), and E_3 be the event that A not at either end; i.e., that A is seated somewhere in between. These events are mutually disjoint and $E_1 \cup E_2 \cup E_3 = S$.

Now let F be the event that B is seated next to A. Then $F = (F \cap E_1) \cup (F \cap E_2) \cup (F \cap E_3)$ and

$$P(F) = P(F \cap E_1) + P(F \cap E_2) + P(F \cap E_3) \ .$$

Every outcome in $F \cap E_1$ has B in the second place, and the other $N-2$ people in any of the $(N-2)!$ possible orders. Hence

$$P(F \cap E_1) = \frac{(N-2)!}{N!} = \frac{1}{N(N-1)} \ ,$$

and by the same reasoning $P(F \cap E_2)$ has the same value. An outcome belongs to $F \cap E_3$ if and only if B is placed on either one side or the other of A, There are $N-2$ choices of where to seat A, and so there are 2 choices of where to seat B, and then the remaining people can be seated in $(N-2)!$ ways. Hence

$$P(F \cap E_3) = \frac{2(N-2)}{N(N-1)} \ .$$

Altogether then,

$$P(F) = \frac{2N-2}{N(N-1)} = \frac{2}{N} \ .$$

Now, if we have a circular table, we use the same spaces discussed earlier with the problem of seating couples at a round table that was discussed in connection with the inclusion-exclusion formula. Let A be seated first; where they sit does no matter because we regard "rotated" seating arrangements as the same. So the sample space has $(N-1)!$ outcomes. After A is seated, there are two places out of $N-1$ to seat B next to A. the remaining $N-2$ people can be seated arbitrarily. With F denoting the event that B is seated next to A,

$$P(F) = 2\frac{(N-2)!}{(N-1)!} = \frac{2}{N-1} \ .$$

47 The sample space consists of the 12^{12} ways to assign one of the 12 months to each of the 12 people. We regard each of these outcomes as equally likely since we are not given any information that would enable us to do otherwise. (It is not just the differing lengths of the months, there is a seasonal pattern in reproductivity rates, etc.)

If no two people have the same birth months, the outcome is one o the 12! ways of matching the 12 people to the 12 months. Hence the probability is

$$\frac{12!}{12^{12}} = \frac{1925}{35831808} = 0.0000537232170...$$

Chapter 2 Theoretical Exercises:

8 Let $S = \{A, B, C\}$. The partitions are

$$\begin{aligned} S_1 &= \{A, B, C\} \\ S_1 &= \{A, B\}\ , S_2 = \{C\} \\ S_1 &= \{A, C\}\ , S_2 = \{B\} \\ S_1 &= \{B, C\}\ , S_2 = \{A\} \\ S_1 &= \{A\}\ , S_2 = \{B\}\ , S_3 = \{C\} \end{aligned}$$

Hence $T_3 = 4$.

Now let $S = \{A, B, C, D\}$. The partitions are

$$\begin{aligned} S_1 &= \{A, B, C, D\} \\ S_1 &= \{A, B, C\}\ , S_2 = \{D\} \\ S_1 &= \{A, B, D\}\ , S_2 = \{C\} \\ S_1 &= \{A, C, D\}\ , S_2 = \{B\} \\ S_1 &= \{B, C, D\}\ , S_2 = \{A\} \\ S_1 &= \{A, B\}\ , S_2 = \{C, D\} \\ S_1 &= \{A, C\}\ , S_2 = \{B, D\} \\ S_1 &= \{A, D\}\ , S_2 = \{B, C\} \\ S_1 &= \{A, B\}\ , S_2 = \{C\}\ , S_3 = \{D\} \\ S_1 &= \{A, C\}\ , S_2 = \{D\}\ , S_3 = \{D\} \\ S_1 &= \{A, D\}\ , S_2 = \{B\}\ , S_3 = \{C\} \\ S_1 &= \{B, C\}\ , S_2 = \{Q\}\ , S_3 = \{D\} \\ S_1 &= \{B, D\}\ , S_2 = \{A\}\ , S_3 = \{C\} \\ S_1 &= \{C, D\}\ , S_2 = \{A\}\ , S_3 = \{B\} \\ S_1 &= \{A\}\ , S_2 = \{B\}\ , S_3 = \{C\}\ , S_4 = \{D\}. \end{aligned}$$

Hence $T_4 = 15$.

To deduce a recursive formula, consider a set of $n+1$ elements $\{x_1, \ldots, x_n, x_{n+1}\}$. The elements x_{n+1} must be assigned to some set in the partition, and this set could contain k other members of this set where j is any integer with $0 \le j \le n$. If $j = n$ we have the trivial partition, this is one choice. Now suppose $0 \le j < n$. there are $\binom{n}{j}$ ways to choose a set of j elements from $\{x_1, \dots, x_n\}$; these will be the "neighbors" of x_{n+1} in the partition. There remain $n - j$ elements, and they can be portioned in T_{n-j} ways.

Therefore, for $0 \le j < n$, there are

$$\binom{n}{j} T_{n-j}$$

ways to partition $\{x_1, \dots, x_n, x_{n+1}\}$ so that x_{n+1} has exactly j neighbors. We get the total number of partitions by summing on j, and including 1 for the trivial partition; i.e., $j = n$. The result is

$$T_{n+1} = 1 + \sum_{j=0}^{n-1} \binom{n}{j} T_{n-j}$$

There is an equivalent formula; Define k by $n - j = k$ so that $j = n - k$. The since j runs from 0 to $n - 1$, k runs from 0 to n, and since

$$\binom{n}{j} = \binom{n}{n-k} = \binom{n}{k} ,$$

we also have

$$T_{n+1} = 1 + \sum_{k=1}^{n} \binom{n}{k} T_k$$

19 First Solution: Here is what I think is the best solution, with the simplest formula as an answer. Let's take the sample space for drawing *all* of the balls out. This the the sets of sequences $n + m$ terms long, with each entry being either R or B, and there are exactly n entries that at R (and hence exactly m that are B). The sample space S has cardinality $\binom{n+m}{m}$, and each outcome is equally likely,

Fix $1 \le r \le n$, and then k with $k \ge r$, and let E be the event that exactly r red balls have been withdrawn at the kth trial, but fewer than r at any previous trial. Then the sequence must have an R in the kth place, and $r - 1$ R's among the first $k - 1$ places. There are $\binom{k-1}{r-1}$ ways to fill in the first k entries of the sequence and have it belong to E. There are $n - r$ red balls to assign to the remaining $n + m - k$ places in the sequence, so there are $\binom{n+m-k}{n-k}$ ways to complete the sequence. Altogether there are

$$\binom{k-1}{r-1}\binom{n+m-k}{n-k}$$

sequences belong to E. Hence the probability of E is

$$\frac{\binom{k-1}{r-1}\binom{n+m-k}{n-k}}{\binom{n+m}{m}} . \qquad (*)$$

Second solution: Most people who gave a correct solution gave one that gives a more complicated formula, but is correct. Here is how this goes: To have the rth red ball appear exactly at the kth drawing, when we have drawn the first $k - 1$ balls out of the urn, we must have already drawn *exactly* $r - 1$ red balls – only then can we drawn the rth red ball on the kth trial. Let E denote the event that that when we have drawn $k - 1$ balls from the urn, exactly $r - 1$ of them are red.

Let F denote then event that the kth ball drawn is red. Then $E \cap F$ is the event that the the rth ball shows exactly when the kth ball is drawn from the urn. Then

$$P(E \cap F) = P(E)P(F|E)$$

and we can answer the question by computing $P(E)$ and $P(F|E)$. It is easy to compute $P(F|E)$: Given E, what remains in the urn are $n+m-k+1$ balls, $n-r+1$ of which are red. Hence

$$P(F|E) = \frac{n-r+1}{n+m-k+1} \ .$$

It remains to compute $P(E)$. For this we may take the sample space to be the set of subsets of cardinality $k-1$ that may chosen from our set of $m+n$ balls. The cardinality of S is $\binom{n+m}{k-1}$, and all outcomes are equally likely.

To form a set of cardinality $k-1$ containing exactly of $r-1$ red balls, we must choose a set of $r-1$ red balls and a set of $k-r$ blue balls. There are $\binom{n}{r-1}$ ways to choose a subset of $r-1$ red balls from the set of n red balls. Then we must choose $k-r$ blue balls from the sets of m blue balls to complete the set of $k-1$ balls. there are $\binom{m}{k-r}$ ways to do this. Therefore, there are

$$\binom{n}{r-1}\binom{m}{k-r}$$

ways to choose a subset of $k-1$ balls, of which $r-1$ are red from our set of n red balls and m blue balls, and

$$P(E) = \frac{\binom{n}{r-1}\binom{m}{k-r}}{\binom{n+m}{k-1}} \ .$$

Altogether, the probability that the rth red ball show up at the kth draw is

$$\frac{\binom{n}{r-1}\binom{m}{k-r}}{\binom{n+m}{k-1}} \frac{n-r+1}{n+m-k+1} \ . \qquad (**)$$

This is not the same as the formula $(*)$ that we found in the first solution, but it is the same answer. Indeed, if we we write out the right side of $(**)$ explicitly, we find

$$\frac{n!}{(r-1)!(n-r+1)!} \frac{m!}{(k-r)!(m+r-k)!} \frac{(k-1)!(n+m-k+1)!}{(m+n)!} \frac{n-r+1}{n+m-k+1} = \frac{n!}{(r-1)!(n-r)!} \frac{m!}{(k-r)!(m+r-k)!} \frac{(k-1)!(n+m-k)!}{(m+n)!} \ .$$

If one expands $(*)$, one finds the exact same terms in the numerator and denominator, and hence the answers are the same – as they must be.

The second solution is more sophisticated; it uses conditional probabilities. The first solution is simpler and uses only elementary counting techniques, and it yields a simpler formula – as one might expect rom a simpler approach.