Homework 3 Solutions, Math 477, Fall 2018

Eric A. Carlen
Rutgers University

October 13, 2018

From the Self Test in Chapter 2:

11. We work with the usual probability space consisting of the 52! possible shuffles of the deck, each with equal probability. Number the suits 1 to 4. Let E_j be the event that the hand contains no card from suit j. The event that at least one suit is missing is $\cup_{j=1}^4 E_j$. The event that no suit is missing is therefore $(\cup_{j=1}^4 E_j)^c$. Then answer than is

$$1 - P(\cup_{j=1}^4 E_j) \ .$$

We now proceed to compute $P(\cup_{j=1}^4 E_j)$ using the inclusion-exclusion formula.

First, we compute $P(Ej)$. The number of 5 card hands that can be dealt avoiding suit j is $\binom{39}{5}$, and the total number of 5 card hands that can be dealt is $\binom{52}{5}$. Therefore,

$$P(E_j) = \binom{39}{5}\binom{52}{5}^{-1} \ ,$$

independent of j.

Now consider any set $X = \{i, j\}$ of two of the suits. Let $E_X = E_1 \cap E_j$, as usual. The number of 5 card hand that can be dealt avoiding both suits i and j is $\binom{26}{5}$, and the total number of 5 card hands that can be dealt is $\binom{52}{5}$. Therefore, for such X

$$P(E_X) = \binom{26}{5}\binom{52}{5}^{-1} \ .$$

Now consider any set $X = \{i, j, k\}$ of three of the suits. Let $E_X = E_1 \cap E_j \cap E_k$, as usual. The number of 5 card hand that can be dealt avoiding all three of suits i, j and k is $\binom{13}{5}$, and the total number of 5 card hands that can be dealt is $\binom{52}{5}$. Therefore, for such X

$$P(E_X) = \binom{13}{5}\binom{52}{5}^{-1} \ ,$$

Finally, at least one suit must be present, so $E_1 \cap E_2 \cap E_3 \cap E_4 = \emptyset$, and this event has zero probability.

Since there are $\binom{4}{1} = 4$ way of choosing one suit or three of the suits, and $\binom{4}{2} = 6$ ways of choosing two of suits, the inclusion-exclusion formula gives us

$$\begin{aligned} P(\cup_{j=1}^4 E_j) &= \left[4\binom{39}{5} - 6\binom{26}{5} + 4\binom{13}{5}\right]\binom{52}{5}^{-1} \\ &= 4\frac{2109}{9520} - 6\frac{253}{9996} + 4\frac{33}{66640} = \frac{6133}{8330} \approx 0.736 \ . \end{aligned}$$

Finally, the probability we seek is

$$1 - \frac{6133}{8330} = \frac{2197}{8330} \approx 0.264 \ .$$

From the Problems in Chapter 3:

10. We work with the usual probability space consisting of the 52! possible shuffles of the deck, each with equal probability.

Let E be the event that the first card is a spade. Let F be the event that the second and third cards are spades.

To determine the cardinality of $E \cap F$, we must choose 3 of the 13 spades to put in the first 3 places, and there are 3! ways to arrange these 3 spades in the first three places. Then the remaining 49 cards can be arranged in any of the 49! possible ways. Hence

$$P(E \cap F) = 3!\binom{13}{3}\frac{49!}{52!} \ .$$

To determine the cardinality of F, we must choose 2 of the 13 spades to put in the places 2 and 3, and there are $2! = 2$ ways to order them in the second and third places. Then the remaining 50 cards can be arranged in the remaining 50 places in any of the 50! possible ways. Hence

$$P(F) = 2\binom{13}{2}\frac{50!}{52!} \ .$$

Therefore,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{33}{150} \ .$$

22. The sample space consists of all triples (x_1, x_2, x_3) where each $x_j \in \{1, \ldots, 6\}$. Here, x_1 is the result for the red die, x_2 is the result for the blue die, and x_3 is the result for the yellow die. Each of the $6^3 = 216$ outcomes has equal probability. Let E be the event that that outcome (x_1, x_2, x_3) satisfies $x_2 < x_3 < x_1$. Let F be the event that x_1, x_2 and x_3 are all different.

The number of outcomes in which no two dice show the same number is $6 \times 5 \times 4 = 120$. Hence the probability that no two dice show the same number is $\frac{120}{216} = \frac{5}{9}$. Hence $P(F) = \frac{5}{9}$.

Given that no two dice show the same number, the $3! = 6$ orderings of the 3 distinct numbers are equally likely, so $P(E|F) = \frac{1}{6}$.

Finally, $P(E) = P(E|F)P(F) = \frac{5}{9}\frac{1}{6} = \frac{5}{54}$.

43. Let E_j be the event that the jth coin is selected, $j = 1, 2, 3$. Let H be the event that the toss is heads. We are asked to compute $P(E_1|H)$. By Bayes' formula,

$$P(E_1|H) = \frac{P(E_1)}{P(H)}P(H|E_1) \ .$$

It is evident that $P(E_1) = \frac{1}{3}$ and that $P(H|E_1) = 1$. It remains to compute $P(H)$. But

$$P(H) = \sum_{j=1}^{3} P(E_j)P(H|E_j) = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{3}{4}\right) = \frac{3}{4} \ .$$

Therefore,

$$P(E_1|H) = \frac{4}{9} .$$

From the theoretical exercises in Chapter 3: 7*, 14, 19*

7. The sample space S consists of all vectors $(x_1, \ldots, x_{m+n})$ where each x_j is 0 or 1, and $x_j = 1$ for exactly n of the entries. Here, $x_j = 1$ indicates the jth ball drawn is white. Each outcome is equally likely. Notice we set up the probability space (S, P) to describe the extraction of *all* of the balls from the urn.

Let E denote the event that when only balls of one color remain, those balls are white. Evidently, $\omega = (x_1, \ldots, x_{m+n})$ belongs to E if and only if for some $k < m+n$, $x_j = 1$ for all $j > k$, and this is the case if and only if $x_{m+n} = 1$. The probability that the last ball is white is the same as the probability that the first ball is white since $\omega = (x_1, \ldots, x_{m+n})$ and the reversed outcome $\omega' = (x_{m+n}, \ldots, x_1)$ are equally likely. And since any of the $m+n$ balls are equally likely to be chosen first, and n of them are white,

$$P(E) = P(\{x_1 = 1\}) = \frac{n}{m+n} .$$

Alternatively, the cardinality of E is the number of ways we can arrange the m black balls in the first $m+n-1$ places, while the cardinality of S is the number of ways we can arrange the m black balls in the full $m+n$ places. Hence

$$P(E) = \binom{m+n-1}{m}\binom{m+n}{m}^{-1} = \frac{n}{m+n} .$$

19. In the notation of the notes, if the total fortune is N, and the game is played until one player wins or else n trials have happened, whichever comes first, $P_{i,N,n}$ is the probability that player A wins if their initial fortune is i. (The order of the subscripts is a bit different in the way the problem is posed in the text.)

Let p be the probability that A wins in each trial. As we saw in class, and as explained in the notes, for $0 < i < N$,

$$P_{i,N,n} = pP_{i+1,N,n-1} + (1-p)pP_{i-1,N,n-1} ,$$

and $P_{0,N,n} = 0$ and $P_{N,N,n} = 1$. Also, Player A cannot win in fewer than $N - i$ trials, so

$$P_{i,N,n} = 0 \quad \text{for} \qquad n < N - i . \tag{$*$}$$

also, if $n = N - i$, A wins the game if and only if A wins the remaining n trials, and the probability of this is p^n. Hence

$$P_{n,N,n} = p^n . \tag{$**$}$$

Hence, repeatedly applying the recursion relation, and $P_{N,N,n} = 1$,

$$\begin{aligned} P_{3,5,7} &= pP_{4,5,6} + (1-p)P_{2,5,6} \\ &= p[p + (1-p)P_{3,5,5}] + (1-p)[pP_{3,5,5} + (1-p)P_{1,5,5}] \\ &= p^2 + 2p(1-p)P_{3,5,5} + (1-p)^2 P_{1,5,5} \end{aligned}$$

Now, by the recursion relation and $P_{0,N,n} = 0$ and $(*)$

$$P_{1,5,5} = pP_{2,5,4} = p^2 P_{3,5,3} = p^3 P_{4,5,2} = p^4 \ .$$

Indeed, if A looses any of the first four trials, A needs 5 trial to come out with a net gain of 4, but one of the 5 trials has been used for B's win, and this is impossible.

Next, $P_{3,5,5}$:

$$P_{3,5,5} = pP_{4,5,4} + (1-p)P_{2,5,4} = p^2 + 2p(1-p)P_{3,5,3} = p^2 + 2p^3(1-p)$$

where we used $(*)$ and $(**)$.

Altogether,

$$P_{3,5,7} = p^2 + 2(1-p)p^2 + 4(1-p)^2 p^4 \ .$$

This is readily checked: With H denoting a win by A and T denoting a win by B, there is exactly one outcome in which A wins in 2 trials, namely

$$HH \ ,$$

which has probability p^2.

There are 2 outcomes in which A wins in exactly 4 trials, namely

$$THHH \quad \text{and} \quad HTHH \ .$$

If B does not win in one of the first two trials, then A wins in 3 trials. But in this case, A must win trials 3 and 4 to win in 4 trials. Each of these outcomes has probability $(1-p)p^3$.

Finally, there 4 outcomes in which A wins in exactly 6 trials, namely

$$TTHHHH \ , \qquad THTHHH \ , \qquad HTHTHH \quad \text{and} \quad HTTHHH \ ,$$

and each of these has probability $(1-p)^2 p^4$. There are no outcomes in which A wins in an odd number of trials.