Homework 4 SOLUTIONS, Math 477, Fall 2018

Eric A. Carlen
Rutgers University

October 13, 2018

Problems in Chapter 3:

17 The probability space (S, P) consists of S, the set of all families ω in the community. (The families are the"outcomes"; think of randomly selecting a family.) For each family ω, $P(\omega)$ is the inverse of the number of families in the community.

Define events as follows: D is the event consisting of families owning a dog. C is the event consisting of families owning a cat. B is the event consisting of families owning both a dog and a cat, i.e,, $B = D \cap C$. We are given that $P(D) = 0.36$, $P(C|D) = 0.22$, and $P(C) = 0.30$.

(a) $P(C \cap D) = P(D)P(C|D) = (0.36) * (0.22) = 0.0792$, so lets say 8% of families own both a cat and a dog.

(b) $P(D|C) = \frac{P(C\cap D)}{P(C)} = \frac{0.792}{0.30} = 0.264$, so 26.4% of families that own a cat also own a dog.

41 First Solution: After the ace is inserted into the second half of the deck, and this is then reshuffled, there are 27 cards from which one is selected randomly. Let E be the event the the selected card is the inserted acs. Let F be the event that the selected card is some ace.

$$P(F) = P(E)P(F|E) + P(E^c)P(F|E^c)$$

It is clear that $P(E) = \frac{1}{27}$ and hence $P(E^c) = \frac{26}{27}$. It is also clear that $P(F|E) = 1$, and $P(F|E^c) = \frac{3}{51}$, because when we select any other card out of the cut deck, we are selecting a random card from the whole deck, minus the ace that was selected from the other part. The second card is equally likely to be any of the 51 remaining cards, and 3 of those are aces. Hence

$$P(F) = \frac{1}{27} + \frac{26}{27}\frac{3}{51} = \frac{43}{459} \ .$$

Second Solution The second solution is longer, but also instructive. When the deck is divided in half, the half deck on the left will contain k aces, and the half deck on the right will contain $4 - k$ aces, where $k \in \{0, 1, 2, 3, 4\}$. Let E_k, $k = 0, 1, 2, 3, 4$, be the event that the half of the deck on the left contains k aces.

We are given that a randomly selected card from the left half deck is an ace. Let F denote this event. It is evident that

$$P(F|E_k) = \frac{k}{26} \ .$$

We would rather know $(E_k|F)$ for $k = 0, 1, 2, 3, 4$. By Bayes' formula,

$$P(E_k|F) = \frac{P(E_k)}{P(F)}P(F|E_k) \ .$$

When we shuffle the deck, cut the deck, and then select one card form the left half, we are selecting a random card from the whole deck. Hence $P(F) = \frac{4}{52}$. We need to compute $P(E_k), = 0, 1, 2, 3, 4$.

There are $\binom{4}{k}$ ways to choose the k aces that will go into this half of the deck, and then there are $\binom{48}{26-k}$ ways to choose a set of $26-k$ cards that are not aces. There are of course $\binom{52}{26}$ ways to choose a set of half of the cards. Thus

$$P(E_k) = \frac{\binom{4}{k}\binom{48}{26-k}}{\binom{52}{26}} \ .$$

Altogether,

$$P(E_k|F) = \frac{\binom{4}{k}\binom{48}{26-k}}{\binom{52}{26}}\frac{k}{2}.$$

Explicitly,

$$P(E_1|F) = \frac{46}{833} \ , \quad = P(E_2|F) = \frac{104}{833} \ , \quad P(E_3|F) = \frac{325}{833} \quad \text{and} \quad P(E_4|F) = \frac{92}{833} \ .$$

You can check that these probabilities sum to 1 as they must.

Now, after placing the ace in the right half of the deck and shuffling, let F_k denote the event that this half of the deck contains exactly k aces. Evidently, $k \in \{1, 2, 3, 4\}$. If E_j occurs, then there were j aces in the left half and $4-j$ in the right half before moving one ace over. Afterwards, there are $4-j+1$ aces on the right. That is, the event F_k occurs exactly when E_{4-k+1} occurs. Given F_k, i.e., that there are k aces in the 27 cards now on the right, the probability that the next card drawn is an ace is $\frac{k}{27}$. Let G be the event that the next card drawn is an ace:

$$P(G) = P(E_1|F)\frac{4}{27} + P(E_2|F)\frac{3}{27} + P(E_3|F)\frac{2}{27} + +P(E_4|F)\frac{1}{27} = \frac{43}{459} \ .$$

as we found before.

59 The probability of $HHHH$ is p^4. The probability of $THHH$ is $(1-p)p^3$. If the pattern $HHHH$ occurs first, it must occur as the result of the first four tosses because once the initial run of heads is broken, it is followed by a run of tails, after which a run of heads begins again. When this happens, the initial T will already be in place, and before a run of four heads can occur, a run of three heads will occur, following an initial T. Hence the probability that $HHHH$ occurs first is p^4, and hence the probability that $THHH$ occurs first is $1-p^4$.

82 Consider first the games in which the players seek 2 or 3 heads in a row, let E_1 be the event that A wins on their first turn. Let E_2 be the event that B wins on their first turn, and let E_3 be the event that neither A nor B win on their first turns. Let F be the event that A eventually wins. Since the events $\{E_1, E_2, E_3\}$ are mutually exclusive an exhaustive,

$$P(F) = P(F \cap E_1) + P(F \cap E_2) + P(F \cap E_3) \ .$$

Since $F \cap E_2 = \emptyset$, and since $F \cap E_1 = E_1$ this can be simplified to

$$P(F) = P(E_1) + P(F \cap E_3) = P(E_1) = P(E_3)P(F|E_3) \ .$$

However, given that neither A or B have won their first turns, the game is no back to its starting point, and so $P(F|E_3) = P(F)$. Altogether, $P(F) = P(E_1) + P(E_3)P(F)$, or

$$P(F) = \frac{P(E_1)}{1 - P(E_3)} \ .$$

Now suppose the condition for winning is two heads in a row. Then $P(E_1) = P_1^2$, and $P(E_3) = (1 - P_1^2)(1 - P_2^2)$. Hence

$$P(F) = \frac{P_1^2}{1 - (1 - P_1^2)(1 - P_2^2)} \ .$$

By the same reasoning, if the condition for winning is k heads in a row, for any $k \geq 1$, including 3, the answer is

$$P(F) = \frac{P_1^k}{1 - (1 - P_1^k)(1 - P_2^k)} \ .$$

Before moving on, let's consider the case $P_1 = P_2 = \frac{1}{2}$. Then, by the computation just above, if k heads in a row is the winning combination, the probability that A wins is

$$\frac{2^{-k}}{2^{1-k} - 2^{-2k}} = \frac{1}{2} \frac{1}{1 - 2^{-k-1}} \ .$$

Notice that as k increases, the advantage A has in going first quickly goes to zero, which is natural because the advantage that A has lies in the fact that they can win before B even gets a chance at all, but the probability of this goes to zero exponentially fast in k.

Now consider the games in which the condition for winning it 2 or 3 heads total. This is different because now it matters how close to winning the players came in previous turns, while before it did not. We use recursion, and solve a more general problem: Let $W_{m,n}$ denote the probability that A wins if they need m heads to win, while B needs m heads to win. Evidently $W_{0,1} = 1$ and $W_{!,0} = 0$, while from the above we have

$$W_{1,1} = \frac{P_1}{1 - (1 - P_1)(1 - P_2)} \ .$$

We will condition on A winning in their first turn (the event E_1) , B winning in their first turn (the event E_2), and neither player winning in their first turn, as before, but we must break up this last event to keep track of how much progress was made by each player. Let $G_{k,\ell}$ be the event that neither player has won, but player A got k heads in their first turn, and player B got ℓ heads in their first turn. Of course, we only consider $0 \leq k < m$ and $0 \leq \ell < n$. Then, conditioning as before, with F denoting the event that player A wins,

$$W_{m,n} = P(F) = P(E_1) + \sum_{k=0}^{m-1} \sum_{\ell=0}^{n-1} P(G_{k,\ell}) W_{m-k,n-\ell} \ .$$

Evidently, the event $G_{k,\ell}$ occurs when the tossing sequence begins with k heads for A, followed by a tail, and then ℓ heads for B, followed by a tail. Thus, $P(G_{k,\ell}) = P_1^k(1-P_1)P_2^\ell(1-P_2)$. Rearranging terms, we have

$$W_{m,n} = \frac{(1 - P_1)(1 - P_2)}{1 - (1 - P_1)(1 - P_2)} \sum_{k<m, \ell<n, k+\ell>0} P_i^k P_2^\ell W_{m-k,n-\ell} \ .$$

For example,

$$W_{2,1} = \frac{(1-P_1)(1-P_2)}{1-(1-P_1)(1-P_2)} P_1 W_{1,1} = \frac{(1-P_1)(1-P_2)}{(1-(1-P_1)(1-P_2))^2} P_1^2$$

and then

$$W_{12} = \frac{(1-P_1)(1-P_2)}{1-(1-P_1)(1-P_2)} P_2 W_{1,2} = \frac{(1-P_1)(1-P_2)}{(1-(1-P_1)(1-P_2))^2} P_1 P_2 \ .$$

We can now compute

$$\begin{aligned} W_{2,2} &= \frac{(1-P_1)(1-P_2)}{1-(1-P_1)(1-P_2)} (P_1 W_{1,2} + P_2 W_{2,1} + P_1 P_2 W_{1,1}) \\ &= \frac{(1-P_1)^2(1-P_2)^2}{(1-(1-P_1)(1-P_2))^3} (P_1^3 + P_1^2 P_2) \\ &+ \frac{(1-P_1)(1-P_2)}{(1-(1-P_1)(1-P_2))^2} P_1^2 P_2 \ . \end{aligned}$$

This answers the question concerning two heads total. To do 3 heads total, we first apply the recursion to compute

$$W_{3,1} = \frac{(1-P_1)(1-P_2)}{1-(1-P_1)(1-P_2)} (P_1 W_{2,1} + P_1^2 W_{1,1}$$

and similarly for $W_{1,3}$. Then compute

$$W_{3,2} = \frac{(1-P_1)(1-P_2)}{1-(1-P_1)(1-P_2)} (P_1 W_{2,2} + P_1^2 W_{1,2} + P_2 W_{3,1} + P_2 P_1 W_{2,1} + P_2 P_1^2 W_{1,1}) \ ,$$

and likewise compute $W_{2,3}$. Now one has everything on the right side of

$$W_{3,3} = \frac{(1-P_1)(1-P_2)}{1-(1-P_1)(1-P_2)} \sum_{k<3,\ell<3,k+\ell>0} P_i^k P_2^\ell W_{3-k,3-\ell} \ ,$$

computed explicitly.

From the theoretical exercises in Chapter 3:

30 Let C_i, $i = 0, \dots, k$ be the event that coin i is selcted. Let E be the event that the first n flips result in exactly r heads. Let F be the event that the $n+1$st flip results in heads. We are to compute $P(F|E)$

We write

$$P(F \cap E) = \sum_{i=0}^{k} P(E \cap F \cap C_i) = \sum_{i=0}^{k} P(C_i) P(E \cap F | C_i) \ .$$

Then we have $P(C_i) = \frac{1}{k+1}$ and $P(E \cap F | C_i) = \binom{n}{r} \left(\frac{i}{k}\right)^{r+1} \left(1 - \frac{i}{k}\right)^{n-r}$. Finally,

$$P(E) = \frac{1}{k+1} \sum_{i=0}^{k} \binom{n}{r} \left(\frac{i}{k}\right)^r \left(1 - \frac{i}{k}\right)^{n-r} \ .$$

Then from

$$P(F|E) = \frac{P(F \cap E)}{P(E)} \ ,$$

$$P(F|E) = \frac{\frac{1}{k+1}\sum_{i=0}^{k}\binom{n}{r}\left(\frac{i}{k}\right)^{r+1}\left(1-\frac{i}{k}\right)^{n-r}}{\frac{1}{k+1}\sum_{i=0}^{k}\binom{n}{r}\left(\frac{i}{k}\right)^{r}\left(1-\frac{i}{k}\right)^{n-r}} \ .$$

Then we can replace the factors of $\frac{1}{k+1}$ top and bottom by $\frac{1}{k}$ since they cancel, and then the sums are Reimann sums for the ratio of integrals

$$\frac{\int_0^1 x^{r+1}(1-x)^{n-r}\mathrm{d}x}{\int_0^1 x^{r+1}(1-x)^{n-r}\mathrm{d}x} \ .$$

By the identity, which follws from recursive integration by parts,

$$\frac{\int_0^1 x^{r+1}(1-x)^{n-r}\mathrm{d}x}{\int_0^1 x^{r+1}(1-x)^{n-r}\mathrm{d}x} = \frac{(r+1)!(n-r)!}{(n+2)!}\frac{(n+1)!}{r!(n-r)!} = \frac{r+1}{n+2} \ .$$