Homework 6 Solutions, Math 477, Fall 2018

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From the Problems in Chapter 5:

10: **(a)** Since the probability that the passenger arrives *exactly* at $7:00$ is zero, we are concerned with arrivals after $7:00$. Since a train (for A) arrives exactly at $8:00$, this is the last train that concerns us. If the passenger arrives in any of the intervals $(7:00, 7:05)$, $(7:15, 7:20)$, $(7:30, 7:355)$ 0r $(7:45, 7:50)$, they will take the train for B, and otherwise they take a train for A. The total length of the intervals of arrival in which the passenger will travel to B is 20 minutes, 1/3 of the total. Hence the probability that the passenger travels fo B is 1/3, and the probability that the passenger travels fo A is 2/3.

(b) Once again, there are exactly 4 intervals of length 5 minutes such that if the passengers arrives in one of these intervals, they travel to B. Hence the answer is the same as in part **a**.

23: For the first part, we define independent Bernoulli variables as follows: $T_j = 1$ if the result of the jth toss is 6, and $T_j = 0$ otherwise. Evidently $P(T_j = 1) = p = \frac{1}{6}$, and $P(T_j = 0) = q = \frac{5}{6}$. Let $S_n = \sum_{j=1}^{n} T_j$. We want to estimate

$$P(150 \leq S_{1000} \leq 200) \ .$$

For $n = 1000$, $np = 166\frac{2}{3} \approx 166.7$, and $\sqrt{npq} = \frac{25}{3}\sqrt{2} \approx 11.78$. Then

$$\begin{aligned} P(150 \leq S_{1000} \leq 200) &= P(-16.7 \leq S_{1000} - 166.7 \leq 33.3) \\ &= P\left(-\frac{16.7}{11.78} \leq \frac{S_{1000} - 166.7}{11.78} \leq \frac{11.78}{3}3.3\right) \\ &\approx P\left(-1.418 \leq \frac{S_{1000} - 166.7}{11.78} \leq 2.827\right) \end{aligned}$$

By the DeMoivre-Laplace Theorem, the random variable $\dfrac{S_{1000} - 166.7}{11.78}$ is approximately standard norm – to an excellent degree of approximation since there are 1000 trials. Hence

$$P\left(-1.418 \leq \frac{S_{1000} - 166.7}{11.78} \leq 2.28\right) \approx \frac{1}{\sqrt{2\pi}} \int_{-1.418}^{2.827} e^{-x^2/2} \mathrm{dx} = \Phi(2.817) - \Phi(-1.418) \ .$$

Using the table, or otherwise numerically evaluating the integral, one finds

$$P(150 \leq S_{1000} \leq 200) \approx 0.9198 \ .$$

For the second part, the conditional probability of tossing a 5 given that one does not toss a 6 is $\frac{1}{5}$. So now we have $n = 800$ independent tosses, each with a probability $p = \frac{1}{5}$ of resulting in a 5. As above we compute

$$np = \frac{800}{5} = 160 \qquad \text{and} \qquad \sqrt{npq} = 8\sqrt{2} \approx 11.31\ .$$

By the DeMoivre-Laplace Theorem, $X := \dfrac{S_{800} - 160}{11.31}$ is approximately standard normal, so that

$$P(S_{800} < 150) \approx P\left(X \leq -\frac{10}{11.31}\right) \approx P(X < -0.8841) = \Phi(-0.8841) = 0.8117\ .$$

From the theoretical exercises in Chapter 5:

5: Using the fact that

$$\mathrm{E}(X^n) = \int_0^\infty P(X^n > t)\mathrm{d}t\ ,$$

which is proved in exercise 5.2, but see below, we make the change of variables $t = x^n$, so that $\mathrm{d}t = nx^{n-1}\mathrm{d}x$. Then

$$\int_0^\infty P(X^n > t)\mathrm{d}t = n\int_0^\infty P(X^n > x^n)x^{n-1}\mathrm{d}x = n\int_0^\infty P(X > x)x^{n-1}\mathrm{d}x$$

since the event $\{X^n > x^n\}$ is the same as the event $\{X > x\}$.

This was all you needed to do. But to show that for all non-negative random variables Y, $\mathrm{E}(Y) = \int_0^\infty P(Y > t)\mathrm{d}t$, suppose that Y has a density f_Y so that

$$P(Y > t) = \int_t^\infty f_Y(s)\mathrm{d}s \quad \text{and hence} \quad \frac{\mathrm{d}}{\mathrm{d}t}P(Y > t) = -f_Y(y)\ .$$

Then integrate by parts:

$$\begin{aligned}
\int_0^\infty P(Y > t)\mathrm{d}t &= \int_0^\infty P(Y > t)\left(\frac{\mathrm{d}}{\mathrm{d}t}t\right)\mathrm{d}t \\
&= tP(Y > t)\Big|_0^\infty - \int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}t}P(Y > t)\right)t\mathrm{d}t \\
&= \int_0^\infty tf_Y(t)\mathrm{d}t = \mathrm{E}(Y)\ .
\end{aligned}$$

There is no loss of generality in assuming Y has a density since we may approximate Y by $Y + \epsilon Z$ where Z is an independent standard normal variable. Then $Y + \epsilon Z$ does have a density, and the above argument is valid. Taking ϵ to zero yields the general case, though a bit of thought is required to dot the i's and cross the t's.

12: If X is exponential with parameter λ, then $P(X > t) = e^{-\lambda t}$ and hence, integrating by parts,

$$\mathrm{E}(X^2) = 2\int_0^\infty te^{-\lambda t}\mathrm{d}t = 2\int_0^\infty t\left(-\frac{1}{\lambda}e^{-\lambda t}\right)'\mathrm{d}t = \frac{2}{\lambda}\int_0^\infty e^{-\lambda t}\mathrm{d}t = \frac{2}{\lambda^2}\ .$$