Solutions for Homework 7, Math 477, Fall 2018

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From the Problems in Chapter 6:

14 Let X denote the position of the ambulance at the time of the accident; X is uniformly distributed on $[0, L]$. Let Y denote the position of the accident; it is also uniformly distributed on $[0, L]$. We are asked for the distribution of the distance between the accident and the ambulance; i.e., $|X - Y|$.

Define $U = Y - X$ and $V = X$. We seek the distribution of $|U|$, but let's first find the distribution of U. We consider the change of variables

$$\mathbf{U}(x, y) = (y - x, x)$$

which has the inverse

$$\mathbf{X}(u, v) = (v, u + v) \ .$$

The joint density of (U, V) is 0 unless $0 \leq x(u, v), y(u, v) \leq L$ which is the same as

$$0 \leq v \leq L \qquad \text{and} \qquad -v \leq u \leq L - v \ ,$$

and this region is the parallelepiped Ω bounded by $v = 0$, $v = L$, $u = -v$ and $u = L - v$.

In the interior of this region, the joint density of $(U.V)$ is given by

$$\frac{1}{L^2}\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \frac{1}{L^2} \ ,$$

The probability density function of U is then the u marginal of the joint probability density function

$$g(u, v) = \begin{cases} L^{-2} & (u, v) \in \Omega \\ 0 & (u, v) \notin \Omega \end{cases} \ .$$

The probability density of U is $g_U(u)$, the marginal $g(u, v)$, which is $\int_{\mathbb{R}} g(u, v)\mathrm{d}v$. This is simply L^{-2} times the length of the segment that is the intersection of the vertical line though $(0, u)$ and Ω. That is,

$$g_U(u) = \begin{cases} L^{-2}(L + u) & u \in [-L, 0] \\ L^{-2}(L - u) & u \in [0, L] \\ 0 & u \notin [-L, L] \end{cases} \ .$$

Finally, $|U| \le a$ if and only if $-a \le U \le a$, and so, for $a \in [0, L]$

$$P(|U| \le a) = \int_{-a}^{a} g_U(u)\mathrm{d}u = 2\int_0^a (1-u)\mathrm{d}u = L^{-2}(2La - a^2) \ .$$

Evidently, for $a \ge 2$, $P(U| \le a) = 1$.

Alternatively, one could use convolutions: $X - Y = X + (-Y)$, and since X and $-Y$ are independent, and $-Y$ is uniformly distributed on $[-L, 0]$, the convolution formula gives another way to arrive at the formula for g_U.

20 Since

$$f(x,y) = \begin{cases} xe^{-(x+y)} & x, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$f_X(x) = \begin{cases} xe^{-x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} e^{-y} & y \ge 0 \\ 0 & \text{otherwise} \end{cases} ,$$

X and Y are independent.

In the second case,

$$f(x,y) = \begin{cases} 2 & 0 \le x \le y \ , 0 \le y \ , 0 \le 1 \ge 0 \\ 0 & \text{otherwise} \end{cases} ,$$

X and Y are not independent since the joint density function os not a product function: Indeed, since $f(x,y) = 0$ unless $0 < x < y$, once you know, say $Y \le 1/2$, you know $X \le 1/2$., while without being given this information about Y, one knows only that $X \le 1$.

32 Let X_j denote the sales in month j. We are given that X_j has mean $\mu = 100$ and variance $\sigma^2 = 25$, and that these normal variables are independent. For each j, $P(X_j > 100) = \frac{1}{2}$. Hence the probability that in exactly 3 of the next 6 months sales exceed 100 is the same as the probability that in tossing a fair coin 6 times, we get exactly three heads, and this probability is

$$\binom{6}{3} 2^{-6} = \frac{5}{16} \ .$$

For the second part, let $Z = X_1 + X_2 + X_3 + X_4$. Then X is normal with mean 100 and variance 100. Hence $(Z - 400)/10$ is standard normal, and

$$P(Z > 420) = P\left(\frac{Z - 400}{10} > 2\right) = \Phi(2) \ .$$

From the theoretical exercises in Chapter 6:

8 Recall that the hazard rate function of a continuous random variable X is given by

$$\lambda(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \ln(P(X > t)) \ .$$

Let X and Y be independent and let $W := \min\{X, Y\}$. Then

$$P(W > t) = P(\{X < t\} \cap \{Y > t\}) = P(X > t)P(Y > t) \ .$$

Thus

$$\overline{F}_W(t) = \overline{F}_X(t)\overline{F}_Y(t)\ .$$

Next, taking the logarithmic derivative,

$$\lambda_W(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln(P(W > t)) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln P(X > t) + \ln P(Y > y)\right) = \lambda_X(t) + \lambda_Y(t)\ .$$