Solutions for Homework 8, Math 477, Fall 2018

Eric A. Carlen
Rutgers University

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From the Problems in Chapter 6:

40 (a) We frist compute $p_Y(1) = P(Y = 1) = p(1,1) + p(2,1) = \frac{1}{4}$ and hence $p_Y(2) = P(Y = 2) = \frac{3}{4}$. Then

$$P(X = 1|Y = 1) = \frac{p(1,1)}{p_Y(1)} = \frac{1}{2} \qquad P(X = 2|Y = 1) = \frac{p(2,1)}{p_Y(1)} = \frac{1}{2} \ .$$

and

$$P(X = 1|Y = 2) = \frac{p(1,2)}{p_Y(2)} = \frac{1}{3} \qquad P(X = 2|Y = 2) = \frac{p(2,2)}{p_Y(2)} = \frac{2}{3} \ .$$

(b) $P(X = 1) = p(1,1) + (1,2) = \frac{3}{8} \neq P(X = 1|Y = 2) = frac13$. Therefore, X and Y are not independent.

(c) $\{XY \leq 3\} = \{X \neq 2\ ,\ Y \neq 2\} = 1 - p(2,2) = \frac{1}{2}$. Next, $\{X + y > 2\} = \{X \neq 1\ ,Y \neq 1\} = 1 - p(1,1) = \frac{7}{8}$. Finally, $\{X/Y > 1\} = \{X = 2\ ,\ Y = 1\} = p(2,1) = \frac{1}{8}$.

44 Let $X = \min\{X_1, X_2, X_3\}$, $Z = \max\{X_1, X_2, X_3\}$, and let Y be the remaining random variable. The joint probability density function of (X, Y, Z) is $f(x, y, z)$ where

$$f(x,y,z) = \begin{cases} \frac{1}{6} & 0 < x < y < z < 1 \\ 0 & \text{otherwise} \end{cases} \ .$$

Suppose $0 < x < y < z < 1$ and $x + y < z$, Then $x + y < 1$ so $y < 1 - x$, Also $x + y < z$ and $x < y$ imply $2x < z$, so $0 < x < 1/2$. Therefore, the quantity we seek is

$$\begin{aligned} P(Z > X + Y) &= 6\int_0^1 \left(\int_x^{1-x} \left(\int_{x+y}^1 \mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x \\ &= 6\int_0^{1/2} \left(\int_x^{1-x} (1 - x - y)\,\mathrm{d}y \right) \mathrm{d}x \\ &= 6\int_0^{1/2} \left(\frac{1}{2}(2x-1)^2 \right) \mathrm{d}x = \frac{1}{2} \ . \end{aligned}$$

57 parts (b) and (c)

For **(b)**, we have $u(x, y) = x$ and $v(x, y) = x/y$ so $x(u, v) = u$ and $y(u, v) = u/v$ is follows that

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{u}{v^2} \ .$$

The region Ω where the joint density of (U, V) is not zero is the region

$$0 < x(u,v) < 1 \quad \text{and} \quad 0 < y(u,v) < 1 \ ,$$

and by the computations above, this is

$$0 < u < 1 \quad \text{and} \quad 0 < u < v \ .$$

Hence Ω is the region the u, v plane given by $0 < u < 1$ and $v > u$. We have

$$f(u,v) = \begin{cases} uv^{-2} & (u,v) \in \Omega \\ 0 & (u,v) \notin \Omega \end{cases} .$$

One readily checks that this is, indeed, a probability density.

For **(c)** we have $u(x,y) = x + y$ and $v(x,y) = \frac{x}{x+y}$ so $x(u,v) = uv$ and $y(u,v) = u(1-v)$ is follows that

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = u \ .$$

The region Ω where the joint density of (U, V) is not zero is the region

$$0 < x(u,v) < 1 \quad \text{and} \quad 0 < y(u,v) < 1 \ ,$$

and by the computations above, this is

$$0 < uv < 1 \quad \text{and} \quad 0 < u(1-v) < 1 \ .$$

Hence Ω is the region the positive quadrant of the u, v plane that lies above the curve $v = 1 - 1/u$ and below the curve $v = 1/u$ and below the line $v = 1$. We have

$$f(u,v) = \begin{cases} u & (u,v) \in \Omega \\ 0 & (u,v) \notin \Omega \end{cases} .$$

To check, we compute

$$\int_\Omega f(u,v) \mathrm{d}u \mathrm{d}v = \int_0^{1/2} \frac{1}{2}(1-v)^{-2} \mathrm{d}v + \int_{1/2}^1 \frac{1}{2} v^2 \mathrm{d}v = 1 \ .$$

From the theoretical exercises in Chapter 6:

19 Let X, Y, and Z be independent and have the same probability density functions f.

For **(a)** and **(b)**, we define $U = X - Y$, $V = X - Z$ and $W = X$. We need to compute $P(U > 0 | V > 0)$ and $P(U > 0 | V < 0)$. We note that

$$(U, V, W) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} .$$

Since

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} ,$$

and since the determinant of this matrix is 1, the joint density function of (U, V, W) is

$$f(w)f(w-u)f(w-v) \ .$$

and then the joint density of (U, V) is

$$\int_{\mathbb{R}} f(w)f(w-u)f(w-v)\mathrm{d}w \ .$$

It then follows that

$$P(U > 0 \ , \ V > 0) = \int_0^\infty \left(\int_0^\infty \left(\int_{\mathbb{R}} f(w)f(w-u)f(w-v)\mathrm{d}w \right) \mathrm{d}u \right) \mathrm{d}v \ ,$$

and

$$P(V > 0) = \int_0^\infty \left(\int_{\mathbb{R}} f(w)f(w-v)\mathrm{d}w \right) \mathrm{d}v \ ,$$

Therefore,

$$P(U > 0 | V > 0) = \frac{\int_0^\infty \left(\int_0^\infty \left(\int_{\mathbb{R}} f(w)f(w-u)f(w-v)\mathrm{d}w \right) \mathrm{d}u \right) \mathrm{d}v}{\int_0^\infty \left(\int_{\mathbb{R}} f(w)f(w-v)\mathrm{d}w \right) \mathrm{d}v} \ ,$$

and

$$P(U > 0 | V >< 0) = \frac{\int_{-\infty}^0 \left(\int_0^\infty \left(\int_{\mathbb{R}} f(w)f(w-u)f(w-v)\mathrm{d}w \right) \mathrm{d}u \right) \mathrm{d}v}{\int_{-\infty}^0 \left(\int_{\mathbb{R}} f(w)f(w-v)\mathrm{d}w \right) \mathrm{d}v} \ .$$

The other two cases are very similar; you get full credit for this much.