Solutions for Homework 9, Math 477, Fall 2018

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December 14, 2018

From the Problems in Chapter 7:

11 A changeover occurs at step $j \geq 2$ if the result of the jth toss is different from the result of the $j-1$st toss. For $j = 1, \dots, n$, let $X_j = 1$ if the jth toss is heads, and $X_j = -1$ is the jth toss is tails. For $j = 2, \dots, n$, define $Z_j = \frac{1}{2}(1 - X_j X_{j-1})$. Then either $Z_j = 0$ or $Z_j = 1$, and a changeover occurs at step j. Hence the expected number of changeovers

$$\sum_{j=2}^{n} \mathrm{E}(Z_j) \ .$$

It is easy to compute that

$$P(Z_j = 1) = 2p(1-p)$$

and hence the expected number of changeovers is $(n-1)2p(1-p)$.

Note that we did not need the Bernoulli variables Z_j to be independent and in fact they are not.

38 Let $f(x,y) = \frac{2e^{-2x}}{x}$ for $0 \leq x \leq \infty$ and $0 \leq y \leq x$. We compute

$$\mathrm{E}(XY) = \int_0^\infty \frac{2e^{-2x}}{x} x \left(\int_0^x y \mathrm{d}y \right) \mathrm{d}x = \int_0^\infty x^2 e^{-2x} \mathrm{d}x = \frac{1}{4} \ .$$

$$\mathrm{E}(Y) = \int_0^\infty \frac{2e^{-2x}}{x} \left(\int_0^x y \mathrm{d}y \right) \mathrm{d}x = \int_0^\infty x e^{-2x} \mathrm{d}x = \frac{1}{4} \ .$$

$$\mathrm{E}(X) = \int_0^\infty \frac{2e^{-2x}}{x} x \left(\int_0^x \mathrm{d}y \right) \mathrm{d}x = \int_0^\infty 2x e^{-2x} \mathrm{d}x = \frac{1}{2} \ .$$

Hence

$$\mathrm{Cov}(X,Y) = \frac{1}{8} \ .$$

From the theoretical exercises in Chapter 7:

38 Let $U = \sum_{j=1}^n X_j$. By symmetry

$$\mathrm{E}(X_j | U = x)$$

is independent of j. Hence

$$\mathrm{E}(X_1 | U = x) = \frac{1}{n} \sum_{j=1}^{n} \mathrm{E}(X_j | U = x) = \frac{1}{n} \mathrm{E}(U | U = x) = \frac{x}{n} \ .$$

41 a No, X and Y are not independent: Suppose we know that $|X| \leq 1$. Then we know that $|Y| \leq 1$. **b** Yes, Y and I are independent: $P(Y > x|I = 1) = P(X > x)$ and $P(Y > x) = P(X > x, I = 1) + P(X < -x, I = 0) = \frac{1}{2}(P(X > x) + P)X < -x)) = P(X > x)$. So knowing that $I = 1$ does not change the distribution of Y. **c** We compute

$$P(Y > x) = \frac{1}{2}P(X > x) + \frac{1}{2}P(X < -x) = P(X > x) \ ,$$

so Y has the same distribution as X. **d** Given part **c** , it is clear that $\mathrm{Cov}(X, Y) = 0$.