Notes on the Poisson Process

December 6, 2018

Abstract

These are some notes on the Poisson process.

0.1 The Poisson Process

Let $\{T_j\}_{j\in\mathbb{N}}$ be an independent identically distributed sequence such that for some $\lambda > 0$, and all $t \geq 0$.

$$P(T_1 > t) = e^{-\lambda t} \ .$$

That is, each T_j is exponentially distributed with parameter λ.

Define random variables U_k, $k \in \mathbb{N}$, by

$$U_k = \sum_{j=1}^{k} T_j \ .$$

Think of the T_j's as the times when a random alarm clock goes off: After each ring, the clock is reset, so U_k is the time at which the kth random ring occurs. We now define a family of random variables N_t parameterized by $t \geq 0$ by defining N_t be the number of rings that have occurred by time t. That is

$$\{N_t = k\} = \{U_k \leq t\} \cap \{U_{k+1} > t\} \ . \tag{0.1}$$

0.1 LEMMA. *for each $k \in \mathbb{N}$, the probability density function of U_k is*

$$g_k(u) = \frac{(\lambda u)^{k-1}}{(k-1)!}\lambda e^{-\lambda u} \ .$$

Proof. The joint probability density function of $(T_1, \ldots T_k)$ is

$$f(t_1, \ldots, t_k) = \lambda^k \prod_{j=1}^{k} e^{-\lambda t t_j} = \lambda^k e^{-\lambda(t_1+\cdots+t_k)} \ .$$

Consider the change of variables

$$u_k(t_1, \ldots, t_k) = \sum_{j=1}^{k} t_j \ .$$

The the Jacobian matrix of this linear transformation is the $k \times k$ matrix with every entry on or below the diagonal being 1, and every entry above the diagonal being 0.

It follows that

$$\left|\frac{\partial(t_1, \ldots, t_k)}{\partial(u_1, \ldots, u_k)}\right| = \left|\frac{\partial(u_1, \ldots, u_k)}{\partial(y_1, \ldots, y_k)}\right| = 1 \ .$$

Then by the change of variables formula, the joint probability density function of $(U_1, \ldots, U_k)$ is

$$g(u_1, \ldots, u_k) = \begin{cases} \lambda^k e^{-\lambda u_k} & 0 \le u_1 \le u_2 \le \cdots \le u_k \\ 0 & \text{otherwise} \end{cases} .$$

Finally, computing the marginal distribution,

$$g_k(u) = \lambda^k e^{-\lambda u} \int_{0 \le u_1 \le u_2 \le \cdots \le u_{k-1} \le u} 1 \mathrm{d}u \cdots \mathrm{d}u_{k-1} \ .$$

Doing the $k-1$ successive integrations yields the factor of $u^{k-1}/(k-1)!$. □

0.2 LEMMA. *For each $k \in \mathbb{N}$, and each $t > 0$,*

$$P(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \ .$$

Proof. By (0.1) and Lemma 0.1,

$$P(N_t = k) = \int_0^t \frac{(\lambda u)^{k-1}}{(k-1)!} \lambda e^{-\lambda u} \mathrm{d}u - \int_0^t \frac{(\lambda u)^k}{k!} e^{-\lambda u} \lambda \mathrm{d}u \ .$$

But, integrating by parts,

$$\begin{aligned} \int_0^t \frac{(\lambda u)^{k-1}}{(k-1)!} \lambda e^{-\lambda u} \mathrm{d}u &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{(\lambda u)^k}{k!}\right) e^{-\lambda u} \mathrm{d}u \\ &= \left(\frac{(\lambda u)^k}{k!}\right) e^{-\lambda u} \Big|_0^t + \int_0^t \frac{(\lambda u)^k}{k!} \lambda e^{-\lambda u} \mathrm{d}u \\ &= \frac{(\lambda t)^k}{k!} e^{-\lambda t} + \int_0^t \frac{(\lambda u)^k}{k!} \lambda e^{-\lambda u} \mathrm{d}u \ . \end{aligned}$$

□

Now recall the "memoryless" property of the exponential distribution: For each j,

$$P(T_j \ge t + s | T_j \ge t) = P(T_j \ge s) \ .$$

This means that if we know that $N_t = k$, the distribution of the alarm-bell rings that occur in the interval $[t, t+s]$ is exactly the same as the distribution of the alarm-bell rings that occur in the interval $[0, s]$, and moreover, provided $t > s$ so that there is no overlap of $[0, s]$ and $[t, t+s]$, these two random numbers are independent: The Poisson process "starts afresh" at time t, independent of what has happened before time t be the memoryless property of the exponential distribution. This same reasoning yields:

0.3 THEOREM. *For all* $0 \leq a < b \leq c < d$*,* $N_b - N_a$ *and* $N_d - N_c$ *are independent random variables with*

$$P(N_b - N_a = k) = \frac{(\lambda(b-a))^k}{k!} e^{-\lambda(b-a)} \quad \text{and} \quad P(N_b - N_a = \ell) = \frac{(\lambda(d-c))^\ell}{\ell!} e^{-\lambda(d-c)} \ .$$

The theorem says that the number of "alarm bell rings" in disjoint intervals are independent, and that these numbers depend only on the length of the interval and not on its location; the distribution is Poisson with parameter being the product of the length of the interval and the underlying exponential rate λ.

Note also that for small $h > 0$,

$$P(N_h \geq 2) = \sum_{k=2}^{\infty} 1 - e^{-\lambda h} - \lambda h e^{-\lambda h} \ ,$$

and by L'Hospital's Rule,

$$\lim_{h \downarrow 0} \frac{P(N_h \geq 2)}{h} = 0 \ .$$

That is, $P(N_h \geq 2) = o(h)$.