Practice Test One, Math 477, Oct. 16, 2018

October 15, 2018

NAME: ____

Circle problems to be graded: 1 2 3 4 5

1.	
2.	
3.	
4.	
5.	
Total	

1. 10 balls are randomly placed into 3 urns, That is, the 10 balls are placed, one after another, independently, and with equal probability, into one of the 3 urns. What is the probability that every urn is occupied?

SOLUTION We may take the sample space to consisit of vectors $\omega = (x_1, \ldots, x_{10})$ where each $x_j \in \{1, 2, 3\}$. The value of x_j denotes the urn into which the jth ball is placed. For each outcome ω, we have $P(\omega) = 3^{-10}$.

For $k = 1, 2, 3$, let E_k be the event that urn k ends up empty. Let F be the event that every urn is occupied. Then

$$F = \bigcap_{k=1}^{3} E_k^c = \left(\bigcup_{k=1}^{3} E_k \right)^c ,$$

and so

$$P(F) = 1 - P\left(\bigcup_{k=1}^{3} E_k \right) .$$

We now apply the inclusion-exclusion formula. For each k, the kth urn is left empty if and only if all 10 balls go into the other 2 urns, and this can be done 2^{10} ways. Hence,

$$P(E_1) = P(E_2) = P(E_3) = \left(\frac{2}{3} \right)^{10} .$$

Next, for each $j \neq k$, both the jth and kth urns are left empty if and only if all 10 balls go into the other urn, and this can be done exactly 1 way. Hence,

$$P(E_1 \cap E_2) = P(E_2 \cap E_3) = P(E_3 \cap E_1) = \left(\frac{1}{3} \right)^{10} .$$

Since the balls must go into one of the urns, $E_1 \cap E_2 \cap E - 3 = \emptyset$, and so $P(E_1 \cap E_2 \cap E - 3) = 0$. By the incluision-exclusion formula,

$$P(F^c) = P\left(\bigcup_{k=1}^{3} E_k \right) = 3 \left(\frac{2}{3} \right)^{10} - 3 \left(\frac{1}{3} \right)^{10} ,$$

and so

$$P(F) = 1 - P(F^c) = 1 - 3 \left(\frac{2}{3} \right)^{10} + 3 \left(\frac{1}{3} \right)^{10} .$$

At this point, all probabilistic reasoning is done, and you should leave the answer in this form. However, the numerical answer turn out to be

$$P(F) = 0.9480262...$$

2. Suppose that 5 men out of 100 are colorblind, and 25 women out of $10,000$ are colorblind. Assume half the population is male and half is female. A person selected at random from the population turns out to be colorblind. What is the probability that this person is male?

SOLUTION Let E be the event that the randomly selected person is male. Let F be the event that the randomly selected person is colorblind. We are asked to compute $P(E|F)$. By the definitions,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)} \frac{P(E \cap F)}{P(E)} = \frac{P(E)}{P(F)} P(F|E) \ ,$$

and you could cut out the middle by citing Bayes' formula, We are given that $P(F|E) = 0.05$ and that $P(E) = 0.5$. It remains to compute $P(F)$:

$$\begin{aligned} P(F) &= P(F \cap E) + P(F \cap E^c) \\ &= P(E)P(F|E) + P(E^c)(P(F|E^c) \\ &= \frac{1}{2}(0.05 + 0.0025) = \frac{1}{2}0.0525 \ . \end{aligned}$$

Finally,

$$P(E|F) = \frac{0.05}{0.0525} = \frac{20}{21} \ .$$

3. An urn initially contains 5 white balls and 7 black balls. Each time a ball is selected, its color is noted and it is replaced in the urn along with two other balls of the same color. What is the probability that of the first 4 balls selected, exactly 2 are black?

SOLUTION There are $\binom{4}{2} = 6$ outcomes corresponding to the event that of the first 4 balls selected, exactly 2 are black, namely

$$(BBWW)\ ,(BWBW)\ ,(BWWB)\,(WWBB)\ ,(WBWB)\ ,(WBBW)\ ,$$

in the obvious notation. By independence,

$$P(BBWW) = \frac{7}{12}\frac{9}{14}\frac{5}{16}\frac{7}{18}\ ,$$

and

$$P(BWBW) = \frac{7}{12}\frac{5}{14}\frac{9}{16}\frac{7}{18}\ .$$

Now you see the pattern: We have the same four numbers in the nuerator and the same four numbers in the denominator every time. Hence the 6 outcomes that make up yhe event of interest all have the same probability. Hence the prbability in question is

$$6\frac{7}{12}\frac{9}{14}\frac{5}{16}\frac{7}{18}\ .$$

All probabilistic reasoning is now done, and you should leave your answer in this form. However, the answer simplifies to

$$\frac{35}{128} = 0.2734375\ .$$

4. Urn A contains 5 white balls out of 7 total balls, the rest black Urn B contains 3 white balls out of 12 total balls, the rest black. We flip a fair coin, and if the outcome is heads, we select a ball from urn A, while if the outcome is tails, we select a ball from urn B. Suppose a while ball is selected. What is the probability that result of the coin toss was heads? Define a random variable X to have the value 1 if the selected ball is white, and to have the value 2 if the selected ball is black. Compute the mean and variance of X

SOLUTION Let E be the event that the result of the coin toss is heads. Let F be the event that the selected ball is white. We are asked to compute $P(E|F)$. By the definitions,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)} \frac{P(E \cap F)}{P(E)} = \frac{P(E)}{P(F)} P(F|E) \ ,$$

and you could cut out the middle by citing Bayes' formula, We are given that $P(F|E) = \frac{5}{7}$ and that $P(E) = \frac{1}{2}$. It remains to compute $P(F)$:

$$\begin{aligned} P(F) &= P(F \cap E) + P(F \cap E^c) \\ &= P(E)P(F|E) + P(E^c)(P(F|E^c) \\ &= \frac{1}{2}\left(\frac{5}{7} + \frac{1}{4}\right) = \frac{27}{56} \ . \end{aligned}$$

Finally,

$$P(E|F) = \frac{28}{27}\frac{5}{7} = \frac{20}{27} \ .$$

It now follws that $P(\{X = 1\}) = P(F) = \frac{27}{56}$ and $P(\{X = 2\}) = P(F^c) = \frac{29}{56}$. Then

$$\mathrm{E}(X) = 1\frac{27}{56} + 2\frac{29}{56} = \frac{85}{56}$$

and

$$\mathrm{E}(X^2) = 1\frac{27}{56} + 4\frac{29}{56} = \frac{143}{56}$$

Hence

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \frac{781}{3136} \ .$$

5. Toss a fair coin m times. Let X denote the number of heads minus the number of tails. Compute $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

SOLUTION Ley Y denote the number of heads. Then the number of tails is $m - Y$, and so $X = 2Y - m$. The random variable Y has a binomial dostribution with parameters m and $p = \frac{1}{2}$. Hence $\mathrm{E}(Y) = \frac{m}{2}$ and $\mathrm{Var}(Y) = \frac{m}{4}$. By the linearity of the expectation,

$$\mathrm{E}(X) = \mathrm{E}(2Y - m) = 2\mathrm{E}(Y) - m = 0 \ .$$

By the properties of the variance,

$$\mathrm{Var}(X) = \mathrm{Var}(2Y - m) = \mathrm{Var}(2Y) = 4\mathrm{Var}(Y) = m \ .$$