Solutions for Practice Test Two, Math 477

November 26, 2018

1. If you buy a ticket in each of 50 lotteries, and you have a 1 in 100 chance of winning a prize in each of them, what is the (approximate) probability that you will win:

(a) at least once

(b) exactly once

(c) at least twice

SOLUTION Done in class. This is a Poisson Limit problem. An exact answer involving lots of large binomial coefficients would get very little credit. Wherever you are asked for an approximate probability, what is required is something that can easily be evaluated.

2. A factory produces two types of coin. One is fair, and the other is biased; it comes up heads 55% of the time. We have a coin from the factory, but do not know what type it is. To test this, we toss the coin $1,000$ times. We will decide the coin is fair if the resulting numbers of heads is less than 525, and we will decide the coin is biased is the resulting numbers of heads is at least 525. There are two types of errors we could make: (1) Deciding the coin is fair when actually it is biased. and (2) deciding the coin is biased when actually it is fair. Determine the (approximate) probability of both types of error.

SOLUTION Done in class. This is a DeMoivre-Laplace Theorem problem. An exact answer involving lots of large binomial coefficients would get very little credit. Wherever you are asked for an approximate probability, what is required is something that can easily be evaluated.

3. Let X be a random variable whose distribution is exponential with parameter λ. Define $Y := [X]$, where $[x]$ denotes the *integer part* of x; i.e., the largest integer that is not greater than x. Define $Z = X - Y = X - [X]$; this is the *fractional part* of X. For n a non-negative integer and $x \in (0, 1)$, compute each of

$$P(Y = n) \ , \qquad P(Z \leq x) \quad \text{and} \quad P(Y = n \text{ and } Z \leq x) \ .$$

Are Y And Z independent? Also, compute the variance of Z.

SOLUTION Done in class. Remember, you should know everything about the exponential distribution.

4. Three trucks break down on a road of length L, and the locations of the breakdowns are independent and uniformly distributed. For $0 < d < L/2$. compute the probability that no two trucks are within a distance d of one another.

SOLUTION The probability density function for the location of the breakdown of each truck is

$$f(x) = \begin{cases} \frac{1}{L} & x \in [0, L] \\ 0 & x \notin [0, L] \end{cases} .$$

Let X_j, $j = 1, 2, 3$, be the location of the breakdown of the jth truck. Let X_j^*, $j = 1, 2, 3$, be the corresponding order statistics. That is, $X_1^* = \min\{X_1, X_2, X_2\}$, $X_3^* = \max\{X_1, X_2, X_2\}$, X_3^*, and X_2^* is the value in the middle.

Then, by the basic result on order statistics for independent identically distributed random variables, the joint density of (X_1^*, X_2^*, X_3^*) is

$$f(x_1, x_2, x_3) = \begin{cases} 3! f(x_1) f(x_2) f(x_3) & 0 < x_1 < x_2 < x_3 < L \\ 0 & \text{otherwise} \end{cases} .$$

No two are within a distance d of one another if and only if $X_2^* > X_1^* + d$ and $X_3^* > X_2^* + d$. This correspond to the outcomes

$$\begin{aligned} x_2 + d &< x_3 < L \\ x_1 + d &< x_2 < L - d \\ 0 &< x_1 < L - 2d \end{aligned}$$

Hence the probability P we seek is

$$P = \int_0^{L-2d} \left(\int_{x_1+d}^{L-2d} \left(\int_{x_2+d}^{L} f(x_1, x_2, x_3) \mathrm{d}x_3 \right) \mathrm{d}x_2 \right) \mathrm{d}x_1$$

You would get almost all the credit for setting up this triple integral with the right limits, and having the correct formula for the joint density. However, let's compute the answer so we can check our result to see if it makes sense:

$$\begin{aligned} P &= \frac{6}{L^3} \int_0^{L-2d} \left(\int_{x_1+d}^{L-2d} \left(\int_{x_2+d}^{L} 1 \mathrm{d}x_3 \right) \mathrm{d}x_2 \right) \mathrm{d}x_1 \\ &= \frac{6}{L^3} \int_0^{L-2d} \left(\int_{x_1+d}^{L-2d} (L - d - x_2) \mathrm{d}x_2 \right) \mathrm{d}x_1 \\ &= \frac{(L-2d)^3}{L^3} . \end{aligned}$$

The answer makes sense: If $d = L/2$, we must have $x_1 = 0$, $x_2 = L/2$ and $x_3 = L$ exactly to have the trucks spaced at a distance d, and this has probability zero, in agreement with our general formula. At the other extreme, if $d = 0$, there is no restriction on the positions of the trucks and so the probability is 1 in this case, again in agreement with our formula.

5. Let X and Y be the Cartesian coordinates of a point chosen at random from the right half of the centered unit circle. That is, the joint probability density function of X and Y is

$$f(x, y) = \begin{cases} \frac{2}{\pi} & x^2 + y^2 \le 1 \ , x > 0 \\ 0 & \text{otherwise} \end{cases} .$$

Let $U = \sqrt{X^2 + Y^2}$ and $V = \arctan(Y/X)$. Also, compute the joint probability density function of U and V. Compute the probability density function of $U + V$.

SOLUTION Define functions $u(x, y) = \sqrt{x^2 + y^2}$ and $v(x, y) = \arctan(y/x)$. The transformation is one-to-one since $x > 0$. (Otherwise, we would have to take into account that $y/x = (-y)/(-x)$.) The inverse transformation is evidently

$$x(u, v) = u \cos v \qquad \text{and} \qquad y(u, v) = u \sin v \ .$$

As (x, y) ranges over the right half of the centered unit circle, $(u(x, y), v(x, y)$ ranges over the rectangle $\Omega := [0, 1] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. Er compute the Jacobian:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{bmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{bmatrix}\right| = u$$

(Note that u is non-negative by definition, so we can drop the absolute value sign.)

By the change of variables formula, the joint density $g(u, v)$ of U and V is given by

$$g(u,v) = \begin{cases} \frac{2}{\pi} u & (u,v) \in \Omega \\ 0 & \text{otherwise} \end{cases} .$$

Define

$$g_U(u) := \begin{cases} 2u & u \in [0,1] \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad g_V(v) := \begin{cases} \frac{1}{\pi} & v \in [\pi/2, \pi/2] \\ 0 & \text{otherwise} \end{cases} .$$

Then $g(u, v) = g_U(u)g_V(v)$, and hence U and V are independent. The density of $U + V$ is then given by the convolution

$$g_U * g_V(t) \ ,$$

which is

$$\begin{aligned} \int_{\mathbb{R}} g_U(t-v)g_V(v)\mathrm{d}v &= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} g_U(t-v)\mathrm{d}v \\ &= \frac{2}{\pi}\int_{t-\pi/2}^{t+\pi/2} g_U(s)\mathrm{d}s \end{aligned}$$

You would get most of the credit for this much. To evaluate the convolution integral, we must take into account that $g_U(s) = 0$ is $s < 0$ or if $s > 0$. The point $s = 0$ lies inside the region of integration if $t - \pi/2 < 0 < t + \pi/2$, and the point $s = 1$ lies inside the region of integration if $t - \pi/2 < 1 < t + \pi/2$. Hence the form of the integral charges at each of $t = -\pi/2$, $t = 1 - \pi/2$, $t = \pi/2$ and $t = 1 + \pi/2$. By the formula for $g_U(s)$:

(1) if $t < -\pi/2$, $g_U * g_V(t) = 0$.

(2) For $-\pi/2 < t < 1 - \pi/2$, $g_U * g_V(t) = \frac{1}{\pi}(t + \pi/2)^2$.

(3) For $1 - \pi/2 < t < \pi/2$, $g_U * g_V(t) = \frac{1}{\pi}$.

(4) For $\pi/2 < t < \pi/2 + 1$, $g_U * g_V(t) = \frac{1}{\pi}(1 - (t - \pi/2)^2)$.

(5) For $t > 1 + \pi/2$, $g_U * g_V(t) = 0$.

As you can check, $\int_{\mathbb{R}} g_U * g_V(t)\mathrm{d}t = 1$, as must be the case.