Test One Solutions, Math 477, Oct. 16, 2018

October 19, 2018

1. A town has 3 newspapers, I, II and III. $10,000$ people subscribe to paper I, $30,000$ people subscribe to paper II, and $5,000$ people subscribe to paper III. Moreover, $8,000$ people subscribe to papers I and II, $2,000$ people subscribe to papers I and III, and $4,000$ people subscribe to papers II and III. Finally, $1,000$ people subscribe to all three papers, and the population of the town is $100,000$. How many people in the town do not subscribe to any newspaper?

SOLUTION Let E_1 be the set of people subscribing to newspaper I, E_2 be the set of people subscribing to newspaper II, and E_3 be the set of people subscribing to newspaper III.

By the inclusion-exclusion formula and the given information,

$$\begin{aligned} \#\left(\cup_{j=1}^{3} E_j\right) &= (\#(E_1) + \#(E_2) + \#(E_3)) - (\#(E_1 \cap E_2) + \#(E_2 \cap E_3) + \#(E_3 \cap E_1)) + \#(E_1 \cap E_2 \cap E_3) \\ &= 45,000 - 14,000 + 1,000) = 32,000 \ . \end{aligned}$$

Therefore, $68,000$ people do not subscribe to any newspaper.

2. Given 20 unrelated people, what is the probability that among the 12 months in the year, there are 4 months containing exactly 2 birthdays, and 3 months containing 4 birthdays? (Assume that any randomly chosen person is equally likely to be born in any of the 12 months. Discuss how you use the information that the people are unrelated in your solution.)

SOLUTION Since the people are unrelated, we may suppose that the birth months of each are indpeendent. (This would not be the case if there were twins included.) The sample space will consist of all vectors $\omega = (m_1, \ldots, m_{20})$ where each m_j denotes the birth month of the jth person. We have $P(\omega) = 12^{-20}$ for each $\omega \in S$.

Now let E be the event that there are 4 months containing exactly 2 birthdays, and 3 months containing exactly 4 birthdays. There are $\binom{12}{4}$ ways to choose the 4 months that will have 2 birthdays. There are then 8 months left, and so there are $\binom{8}{3}$ ways to then choose the 3 months that will have 4 birthdays each. Given any choice of the 7 months, we want to count the number of $(m_1, \ldots, m_{20})$ in which each m_j is one of the 7 months, and each month shows up the required number of times. This is the same as counting the number of "words" that can be formed with 7 letters reppeated the specified number of times:

$$\frac{20!}{(2!)^4(4!)^3} \ .$$

Therefore,

$$P(E) = \frac{\binom{12}{4}\binom{8}{3}\frac{20!}{(2!)^4(4!)^3}}{12^{20}} \ .$$

You were to leave the answer in this form. but this works out to be

$$P(E) = 0.0000795315...$$

3. An urn contains 5 white balls and 5 black balls. A fair die is rolled, and the resulting number of balls is drawn from the urn. What is the probability that all of the balls selected are white? What is the expected number of white balls?

SOLUTION Let X be the number of white bals selected. On the first draw, the probability of selecting a white ball is $\frac{5}{10} = \frac{1}{2}$. After this, there are 9 balls in the urn, 4 of them white, so the prbability that the second ball is selcted white, given that the first one is white, is $\frac{4}{9}$. Hence the probabily that both of 2 balls selected will be white is $\frac{1}{2}\frac{4}{9}$. In the same way we see that if p_k dnotes the probability that all k of k balls selected are white,

$$p_1 = \frac{1}{2}\ , \quad p_2 = \frac{1}{2}\frac{4}{9}\ , \quad p_3 = \frac{1}{2}\frac{4}{9}\frac{3}{8}\ , \quad p_4 = \frac{1}{2}\frac{4}{9}\frac{3}{8}\frac{2}{7}\ , \quad p_5 = \frac{1}{2}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{1}{6}\ .$$

Let Y be the result of the toss of the die. We are interested in the event E for which $X = Y$. We have

$$\begin{aligned} P(X = Y) = \sum_{k=1}^{6} P(\{X = k\} \cap \{Y = k\}) &= \sum_{k=1}^{6} P(Y = k)P(X = k|Y = k) \\ &= \frac{1}{6}\sum_{k=1}^{5} P(X = k|Y = k) = \frac{1}{6}\sum_{k=1}^{5} p_k\ . \end{aligned}$$

since $X \leq 5$, and $P(Y = k) = \frac{1}{6}$ for each k. Hence the answer is

$$\frac{1}{6}\left(\frac{5}{10} + \frac{5}{10}\frac{4}{9} + \frac{5}{10}\frac{4}{9}\frac{3}{8} + \frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7} + \frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{1}{6}\right)$$

This formula suffices for full credit, but computing the numbers we find

$$P(Y = X) = \frac{5}{36}\ .$$

For the second part, as before, let X be the number of white balls selected, let Y be the result of the die toss, and let Z be the number of black balls selected. Then $X + Z = Y$, and so

$$\mathrm{E}(X) + \mathrm{E}(Z) = \mathrm{E}(X + Z) = \mathrm{E}(Y) = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6}\ .$$

By symmetry – nothing favors black or white – $\mathrm{E}(X) = \mathrm{E}(Z)$. hence $2\mathrm{E}(X) = \frac{21}{6}$. Finally,

$$\mathrm{E}(X) = \frac{21}{12} = \frac{7}{4}\ .$$

4. A sample of 3 items is selected at random from a box containing 20 items, of which 3 are defective. What is the expected number of defective items in the sample?

SOLUTION When make the selection, we are forming a random subset of 3 elements from the set of 20. There are $\binom{20}{3}$ such subsets, all equally likely. Let X be the number of defective items selected. There is exactly one way to form a subset of 3 defective items – take all 3. Thus,

$$P(X = 3) = 1\binom{20}{3}^{-1} .$$

To form a subset with two defective elements, there are $\binom{3}{2} = 3$ choices for the 2 defective elements, and then $\binom{17}{1}$ choices for the remaining element. Hence

$$P(X = 2) = \binom{3}{2}\binom{17}{1}\binom{20}{3}^{-1} .$$

In the same way, we see

$$P(X = 1) = \binom{3}{1}\binom{17}{2}\binom{20}{3}^{-1} \quad \text{and} \quad P(X = 0) = \binom{3}{0}\binom{17}{3}\binom{20}{3}^{-1} .$$

(We do not need this last probability to answer the question, but here it is.)

Then by the definition,

$$\mathrm{E}(X) = \sum_{j=0}^{3} jP(X = j) = \left[3 + 2\binom{3}{2}\binom{17}{1} + 1\binom{3}{1}\binom{17}{2}\right]\binom{20}{3}^{-1} .$$

You could have left your answer in the form above, but carrying out the computations, one finds

$$\mathrm{E}(X) = \frac{9}{20} .$$

5. 20 cards are selected at random from a 52 card deck. The top card is turned over, and it is an ace. Let X be number of aces in the 20 card sample. For $k = 1, 2, 3, 4$, find the probability that there are k aces in the sample of 20 cards.

SOLUTION Let F be the event that the top card is an ace. For $k = 0, 1, 2, 3, 4$, let E_k be the event that the stack of 20 cards contains k aces. We need to compute $P(E_k|F)$. By Bayes' formula,

$$P(E_k|F) = \frac{P(E_k)}{P(F)}P(F|E_k) .$$

It is clear that $P(F|E_k) = \frac{k}{20}$. It is also clear that since the top card is randomly selcted from the whole deck, and there are 4 aces out of 52 cards, $P(F) = \frac{1}{13}$. It remans to compute $P(E_k)$ for $k = 1, 2, 3, 4$. (Clearly, $k = 0$ is irrelevant.)

There are $\binom{4}{k}$ ways to choose the k aces that will go into the stack of 20, and then there are $\binom{48}{20-k}$ ways to choose a set of $20 - k$ cards that are not aces. There are of course $\binom{52}{20}$ ways to choose a set of 20 of the cards. Thus

$$P(E_k) = \frac{\binom{4}{k}\binom{48}{20-k}}{\binom{52}{20}} .$$

Altogether,

$$P(E_k|F) = \frac{\binom{4}{k}\binom{48}{20-k}}{\binom{52}{20}} \frac{13k}{20} .$$

Explicitly,

$$P(E_1|F) = \frac{4960}{20825} \ , \quad = P(E_2|F) = \frac{9424}{20825} \ , \quad P(E_3|F) = \frac{5472}{20825} \quad \text{and} \quad P(E_4|F) = \frac{969}{20825} \ .$$

You can check that these probabilities sum to 1 as they must.