GRAPHS OMITTING A BUSHY TREE

BY

G. Cherlin[1]

Dept. Mathematics, Rutgers University, Hill Center, Busch Campus, New Brunswick, NJ 08903

AND

N. Shi[2]

Dept. Mathematics, East Stroudsburg University, East Stroudsburg, PA 18301

AND

L. Tallgren

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ABSTRACT

A tree is called *bushy* if it has no vertex of degree 2. Theorem: the class of countable graphs omitting a fixed bushy tree has no universal element.

Introduction

Rado observed [R] that there is a universal countable graph, that is a countable graph G such that every countable graph is isomorphic to an induced subgraph of G. Many similar problems have been considered in the literature, involving a wide range of classes of graphs, and frequently set theoretic complications are involved. We have chosen to focus on a form of the general problem that can be construed as an algorithmic problem: given a finite set $\mathcal{C}$ of finite connected graphs, does the class of countable graphs which omit $\mathcal{C}$ contain a universal element? Here we say that a graph G omits a class $\mathcal{C}$ of graphs if none of the graphs in $\mathcal{C}$ is isomorphic to a subgraph of G (a much more stringent condition than omission of these graphs as induced subgraphs). The problem in general is to characterize the classes $\mathcal{C}$ for which there is such a universal graph, or at least to establish that there is an algorithm which will produce the answer in each case. This problem remains open when $\mathcal{C}$ consists of a single element, though this is by far the best understood case: graphs omitting a single "forbidden subgraph".

It is an immediate consequence of Fraïssé's theory [F] that if $\mathcal{C}$ consists of a single complete graph then the corresponding class of countable graphs has a universal element. Füredi and Komjáth have shown [FK] that the class of countable graphs omitting a given *incomplete, 2-connected* graph A has no universal element. This remarkably general result was arrived at by stages; the case of a cycle of order 4 was treated in [P], the case of general cycles was handled in [CK] (and finite sets of finite cycles are handled completely by combining [KMP] and [CS]), while complete bipartite graphs are treated in [KP]. The method is more or

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less the same in all cases, going right back to the case of the cycle of order 4, but very substantial difficulties arise as one attempts to carry it out in general.

The general graph can be analyzed in terms of its blocks, which is to say, roughly speaking, as a tree of (mainly) 2-connected blocks meeting in single vertices. In view of the Füredi/Komjath result, this suggests that an analysis of graphs omitting a given tree would cast some complementary light on the situation. The simplest case of this, graphs omitting a given finite path, is elegantly analyzed in [KMP], and yields a positive result: there is a universal object in all such cases, because there is a structure theory for this class. Work of the third author, to be included in his doctoral dissertation, shows that this structure theory is valid somewhat more generally in a form which can be used to solve positively a number of related problems, and in particular to provide a positive answer for trees formed from a path by attaching one more vertex. There is a certain amount of evidence suggesting that in all other cases, there is no universal graph omitting the given tree.

Conjecture [Tallgren]

Let T be a tree of order n, and $\mathcal{G}$ the class of countable graphs omitting T. Then $\mathcal{G}$ has a universal element if and only if the tree T contains a path of order $n-1$.

This paper will provide some of the evidence for this conjecture; some additional evidence will be noted below. We call a tree *bushy* if it has no vertex of degree 2. Given a tree T, we say that a graph G is *T-free* if it omits T, that is T does not embed in G as a subgraph.

Theorem

Let T be a finite bushy tree with at least 5 vertices. Then the class of T-free countable graphs has no universal element. More specifically, there is a collection of $2^{\aleph_0}$ T-free graphs such that at most countably many of them embed into any single countable T-free graph.

Theoretical considerations which it would be too tedious to detail here suggest that the presence of vertices of degree 2 introduces an additional set of complications. The proof we give here involves certain subtleties that disappear if one is willing to sacrifice much of the generality of the result.

Scholium

Let T be a finite bushy tree, and let m be the minimal degree of a vertex of T, other than the leaves. If there is a vertex of T which has degree m and is adjacent to a leaf, then there is a collection of $2^{\aleph_0}$ T-free graphs H such that H is a connected component of any T-free graph containing it.

In our proof of the theorem it will be seen that we arrive at a situation strictly parallel to the situation described in the scholium, but at a slightly higher level of abstraction. The subtleties involved in dealing with vertices of degree 2 seem to be of a rather more serious nature, though we feel that the overall situation will probably turn out to be analogous, if these subtleties are mastered.

Other evidence for the conjecture stated above concerns generalized stars, which are trees containing only one vertex of degree greater than 2, and bridges, which are trees consisting of a single path which forks at both ends (so there are two vertices of degree greater than 2, both of which have degree 3 and are adjacent to two leaves). There is no universal countable graph omitting a given bridge [GK], and the result for generalized stars conforms to the stated conjecture; the proof is expected to be included in the third author's doctoral dissertation.

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The following notation will be fixed throughout.

T is a bushy tree with at least two nonleaves (vertices of degree greater than 1).

$n = |T|$.

T_1 is the set of nonleaves of T which are adjacent to at least one leaf of T. T_2 is the subtree of T whose vertices are the nonleaves of T.

$t = |T_1|$

m is the minimal degree of a nonleaf of T (so $m \geq 3$); m_1 is the minimal degree in T of a vertex of T_1.

$s = n - t - 1$

The argument that follows will not work if T_1 contains only one vertex. In this case T is a star and a T-free graph is a graph of maximal degree at most $n-2$. It is easy to see in this case that there is a universal countable T-free graph if and only if n is at most 4. We will assume therefore that $t \geq 2$.

We remark that for any tree T, the order of T equals:

$$2 + \sum_{v \in V(T)} (\deg(v) - 1),$$

and hence with the notation above:

Lemma 1

$n \geq t(m_1 - 1) + 2 \geq t(m - 1) + 2$

We use the notation $\Gamma(v)$ for the set of neighbors of v in a given graph, and occasionally we write $\Gamma(A)$ for $\bigcup_{v \in A} \Gamma(v) - A$, which we refer to as the set of neighbors of A. Cliques are the vertex sets of complete subgraphs of a specified graph.

§1. Tight embedding

We keep the notation established in the introduction. Our main tool is the following technical notion.

Definition 1

Let G be T-free, A a clique of order s. Then A is tightly embedded *in G if there is a partition of A into nonempty subsets $A_1, \ldots, A_k$ and a set of distinct vertices $w_1, \ldots, w_k$ in $G - A$ such that:*

1. *$A_i \subseteq \Gamma(w_i)$;*
2. *$k > t$.*

The significance of the parameter s can be seen in the proof of the following result, and again in the proof of Lemma 8.

Lemma 2

If G is T-free and A is a tightly embedded clique of order s in G, then the sets $\Gamma(v) \cap A$ for $v \in \Gamma(A)$ partition A. In other words, the vertices $w_1, \ldots, w_k$ referred to in the definition of tight embedding must exhaust the neighbors of A, and we have $A_i = \Gamma(w_i) \cap A$ for all $i \leq k$.

Proof:

If this is not the case then for some $i \leq k$ and for some $v \in A_i$, $\Gamma(v) - A$ contains a vertex $w \neq w_i$. As a matter of notation we may take $i = t + 1$ and we may suppose that $w \notin \{w_1, \ldots, w_{t-1}\}$. We will produce a contradiction by embedding T in G. We enumerate the vertices of T_1 as $v_1, \ldots, v_t$ in such a way that v_t is adjacent to at least two leaves of T. We define a partial embedding f of T into G whose domain consists of $v_1, \ldots, v_t$ together with one leaf adjacent to each of $v_1, \ldots, v_{t-1}$, and two leaves adjacent to v_t; the vertices v_i are mapped into A_i for $i \leq t-1$ and v_t maps to v, while the leaf selected adjacent to v_i maps to w_i for $i \leq t-1$, and the two leaves selected adjacent to v_t map to w and w_{t+1}. At this point the domain of f consists of $2t+1$ points and its range meets A in t points, so there remain $n-(2t+1)$ vertices in T to map into $s-t$ vertices of A, and since $s-t = n-(2t+1)$, there is such an extension of f. Since A is a clique, any such extension of f is an embedding of T into G. ∎

Corollary

Let G be a T-free graph, H a subgraph, and A a clique of order s which is tightly embedded in H. Then A is tightly embedded in G, $\Gamma(A)$ is the same whether computed in H or in G, and G and H contain the same edges joining A to $\Gamma(A)$.

Proof:

The first statement is immediate on the basis of the definition of tight embedding. The second then follows from Lemma 2. ∎

Lemma 3

Let G be a T-free graph, $A \subseteq G$ a tightly embedded clique of order s, and $C \subseteq G$ any clique of order s which meets A. Then $C = A$.

Proof:

As $C \subseteq \Gamma(A) \cup A$, Lemma 2 implies that $C - A$ contains at most one vertex. Now if $C - A$ contains a vertex v, then as $\Gamma(v)$ meets A in $s-1$ points, Lemma 2 implies that A has at most two neighbors, a contradiction as we have assumed $t \geq 2$ and $|\Gamma(A)| > t$. ∎

Lemma 4

Suppose that the bushy tree T has a vertex $w_0 \in T_1$ of degree m. Let G be a T-free graph, A and B cliques of order s with A tightly embedded in G, and w a vertex outside $A \cup B$ with $|\Gamma(w) \cap A| \geq m-2$ and $|\Gamma(w) \cap B| \geq 1$. Then $|\Gamma(w)| = m-1$.

Proof:

Assuming $\Gamma(w) \geq m$, we will embed T in G. Select $X \subseteq \Gamma(w) \cap A$ with $|X| = m-2$, $b \in \Gamma(w) \cap B$, and $v \in \Gamma(w) - (X \cup \{b\})$. Also fix a leaf v_0 of T adjacent to w_0. Let T' be a connected component of $T - \{v_0, w_0\}$ whose order is maximal subject to $|T'| \leq (n-1)/2$ (there is at most one component of $T - \{v_0, w_0\}$ not satisfying this condition). Note that if T' does not consist of just one leaf of T, then we have $|T'| \geq m$

In order to embed T into G we first send v_0, w_0 to v, w respectively. We will embed T' into B, using b for the neighbor of w_0 in T'. We will then embed the rest of T into $A \cup \Gamma(A)$. There are various points to be checked.

In the first place we must show that the embedding of T' into B is possible, or in other words that $|T'| \leq s$. As $|T'| \leq (n-1)/2$ and $s = n-t-1$ this reduces to: $n \geq 2t+1$. As $m \geq 3$ this follows from Lemma 1. Now fix such an embedding f_0 of $T' \cup \{v_0, w_0\}$ into $B \cup \{v, w\}$.

In the second place we must extend f_0 to the remaining vertices of T. The remaining neighbors of w_0 in T will be mapped into X. Let $t_0 = |T_1 - \Gamma(w_0)|$. Choose distinct vertices $w_1, \ldots, w_{t_0}$ in $\Gamma(A) - \{w\}$ and vertices $v_i \in \Gamma(w_i) \cap A$. Let f_1 be an extension of f_0 which sends the vertices in $T_1 - \Gamma(w_0)$ to the v_i

for $1 \le i \le t_0$ and which sends one leaf adjacent to each such vertex to the corresponding vertex w_i. The domain of f_1 has $2t_0 + (m-2) + 2 + |T'|$ elements, and the range meets A in $t_0 + (m-2)$ elements, so to extend f_1 to all of T we require:

$$n - (2t_0 + (m-2) + 2 + |T'|) \le s - (t_0 + (m-2)) = n - t - 1 - (t_0 + (m-2))$$

This reduces to $t \le t_0 + 1 + |T'|$. But $t_0 \ge t - m$ and if $|T'| > 1$ then $|T'| \ge m$, as noted earlier, and the desired inequality certainly holds. On the other hand if $|T'| = 1$, then the maximal choice of T' (with at most one exceptional component) implies that w_0 is adjacent to at most one nonleaf. In this case $t_0 \ge t - 1$ and again the required inequality holds. ∎

Dropping our special hypothesis on T, we get a weaker result.

Lemma 5

Let G be a T-free graph, A and B cliques of order s with A tightly embedded in G, and w a vertex outside $A \cup B$ with $|\Gamma(w) \cap A| \ge m - 2$ and $|\Gamma(w) \cap B| \ge 1$. Then $|\Gamma(w) \cap (A \cup B)| = m - 1$.

Proof:

If there is a vertex in T_1 of degree m in T then the result of the previous lemma is stronger. Suppose therefore that every vertex of T_1 has degree greater than m, and in particular has degree at least 4. Then Lemma 1 says:

$$n \ge 3t + 2 \tag{1}$$

We must show that if $|\Gamma(w) \cap B| \ge 2$ or $|\Gamma(w) \cap A| \ge m - 1$, then we can construct an embedding of T into G.

Suppose first that $|\Gamma(w) \cap B| \ge 2$. Then we choose w_0 a vertex of T of order m, and components T', T'' of $T - \{w_0\}$ of minimal order; then $|T'| + |T''| \le (2/3)(n-1)$ since there are at least three components. To embed T into G, first send w_0 to w and embed T', T'' into B with the vertices adjacent to w_0 mapped to two vertices of B adjacent to w. For this we need $|T'| + |T''| \le s$, which follows from (1). Now to complete the embedding we again use A together with t_0 neighbors of A, where $t_0 = |T_1 - \Gamma(w_0)|$. As in the previous lemma, to complete this we require the estimate:

$$n - (2t_0 + (m-2) + 1 + |T'| + |T''|) \le s - (t_0 + (m-2))$$

which reduces to: $t \le t_0 + |T'| + |T''|$. Now as $m < m_1$, the vertex w_0 is not adjacent to a leaf and T', T'' are not leaves, and since $t_0 \ge t - m$ our inequality holds.

Now suppose that $|\Gamma(w) \cap A| \ge m - 1$. In this case we embed one component T' of $T - \{w_0\}$ into B (respecting the adjacency with w_0), where $|T'| \le n/2$. With $t_0 = |T_1 - \Gamma(w_0)|$ the final estimate required to complete the embedding using $A \cup \Gamma(A)$ will be: $n - (2t_0 + (m-1) + 1 + |T'|) \le s - (t_0 + (m-1))$, or $t \le t_0 + |T'|$. This holds, as before, since $|T'| \ge m$. ∎

Lemma 6

Let G be a T-free graph, A and B two disjoint cliques of order s which are tightly embedded in G, and w a vertex in $\Gamma(A)$ with $|\Gamma(w) \cap A| = 1$, $|\Gamma(w) \cap B| = m - 2$. Then for any clique C of order s contained in G, if $w \in \Gamma(C)$ then $C = A$ or $C = B$.

Proof:

If C meets A or B then our result follows by Lemma 3. If $m = m_1$ the result follows by Lemma 4. If C is disjoint from $A \cup B$ and $m < m_1$ then we can easily embed T in G, by a variant of the construction in the proof of the previous lemma, in the case $|\Gamma(w) \cap B| \ge 2$. ∎

§2. A construction

As motivation for the construction that follows, we first indicate a "decoding" procedure that extracts some relevant structure from an arbitrary T-free graph (specifically, we are decoding some of the structure of the algebraic closure operator in the model theoretic sense, but it is not necessary to go into this).

Definition 3

Let G be a T-free graph.

X_G is the set of cliques of order s which are tightly embedded in G.

$N_G : X_G^2 \longrightarrow \mathbb{N}$ is defined as follows. For $C_1, C_2 \in X_G$, $N_G(C_1, C_2)$ is the cardinality of the following set:

$$\{w \in \Gamma(C_1 \cup C_2) : |\Gamma(w) \cap C_1| = m-2, |\Gamma(w) \cap C_2| = 1\}$$

Lemma 7

If G is a T-free graph and H is a subgraph (not necessarily an induced subgraph), then $X_H \subseteq X_G$ and N_H is the restriction of N_G to X_H.

Proof:

This is all contained in the corollary to Lemma 2. ∎

Now we describe the construction of a large number of T-free graphs for which the associated structures (X_G, N_G) will be seen to be incompatible in a sense that yields the main result.

Construction

Let $\varepsilon \in \{0,1\}^{\mathbb{N}}$.

1. We define a notion of t-acceptability. If $t > 2$ then any ε in $\{0,1\}^{\mathbb{N}}$ is t-acceptable. If $t = 2$ then ε is t-acceptable if and only if there is no $k \in \mathbb{N}$ with $\varepsilon(k) = \varepsilon(k+1) = 1$.

2. For ε a t-acceptable sequence, we construct a labeled rooted tree T_ε (inductively, by levels) as follows. The labels will be 0, 1, or 2. The root of the tree carries the label 0. For $k \in \mathbb{N}$, any vertex at level k with label l has $s - \varepsilon(k) - l(m-2)$ successors (a positive number, as we shall see momentarily) at level $k+1$. If $\varepsilon(k) = 0$ these successors all carry the label 1, and otherwise one of them carries the label 2 and the remainder carry the label 1. In particular when $t = 2$ we will never have $\varepsilon(k) = 1$ at the same time that the label is 2. Using Lemma 1 we find that

$$s - \varepsilon(k) - l(m-2) \geq t(m-2) + 1 - \varepsilon(k) - l(m-2) = (t-l)(m-2) + (1 - \varepsilon(k))$$

and this is positive since either $t > l$ or else $t = l = 2$, in which case $\varepsilon(k) = 0$.

3. From the tree T_ε we construct a T-free graph G_ε as follows. Let G_ε° be the disjoint sum of complete graphs K_v of order s, for $v \in V(T_\varepsilon)$, and $C_v = V(K_v)$. We extend G_ε° as follows. If u, v are successive vertices of T_ε and v carries the label l, then l vertices w are introduced satisfing: $|\Gamma(w) \cap C_u| = 1$; $|\Gamma(w) \cap C_v| = m-2$. Furthermore we require that whenever w_1, w_2 are neighbors of the clique C_v, then $\Gamma(w_1) \cap C_v$ and $\Gamma(w_2) \cap C_v$ are disjoint. Observe in this connection that if v has the label l, then C_v has l neighbors each of which is adjacent to $m-2$ vertices of C_v, and $s - l(m-2)$ neighbors each of which is adjacent to one vertex of C_v, so that we may take the sets $\Gamma(w) \cap C_v$ to partition C_v as w runs over the neighbors of C_v.

Lemma 8

If $\varepsilon \in \{0,1\}^{\mathbb{N}}$ is t-acceptable then the associated graph G_ε is T-free and the cliques C_v for v a vertex of T_ε are tightly embedded in G_ε.

Proof:

We show first that G_ε is T-free. The vertices not in the cliques C_v are of degree $m-1$. Hence in any embedding f of T into G_ε the vertices of T_2 must map into various cliques C_v, and since T_2 is connected they must map into a single clique C_v and T maps into $C_v \cup \Gamma(C_v)$. As the sets $\Gamma(w) \cap C_v$ $(w \in \Gamma(C_v))$ partition C_v, the range of f contains at most t vertices in $\Gamma(C_v)$. This means we have mapped n vertices injectively into at most $s+t=n-1$ vertices, a contradiction.

To check that C_v is tightly embedded, one point remains: we must check that C_v has at least $t+1$ neighbors. According to our construction the minimal number of neighbors is $2+[s-2(m-2)]=n-t-2m+5$, so our claim is: $n \geq 2t+2m-4$. Using Lemma 1 we see that it suffices to have $(m-1)t+2 \geq 2t+2m-4$, or: $(m-3)(t-2) \geq 0$, so our assumptions $m \geq 3$, $t \geq 2$ suffice. ∎

Lemma 9

Let $\varepsilon \in \{0,1\}^{\mathbb{N}}$ be t-acceptable, $H=G_\varepsilon$, and suppose H is a subgraph of the T-free graph G. If $C_1 \in X_H$ and $C_2 \in X_G$, with $N_G(C_1,C_2)>0$ or $N_G(C_2,C_1)>0$, then $C_2 \in X_H$.

Proof:

Let $w \in \Gamma(C_1) \cap \Gamma(C_2)$ in G. As C_1 is tightly embedded in H, $w \in H$. There is a tightly embedded clique C of order s in H with $\{|\Gamma(w) \cap C_1|, |\Gamma(w) \cap C|\} = \{1, m-2\}$ or $\{m-2, 1\}$. Lemma 6 applies in G and forces $C_2 = C \in X_H$. ∎

Lemma 10

Let G be a countable T-free graph. Then there are only countably many t-acceptable sequences $\varepsilon \in \{0,1\}^{\mathbb{N}}$ for which G_ε embeds in G as a subgraph.

Proof:

We make X_G into a graph by setting $E(C_1,C_2)$ iff $N(C_1,C_2)>0$ or $N(C_2,C_1)>0$. By the previous lemma if G_ε embeds into G as a subgraph then the image of X_{G_ε} in X_G is a connected component of this graph. Furthermore N_{G_ε} will be the restriction of N_G to this component, and ε can be recovered from N_{G_ε}. As X_G is countable, the claim follows. ∎

The theorem as stated in the introduction follows. As noted at the outset, the proof given is valid for $t \geq 2$, and the excluded case $t=1$ is completely straightforward.

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