Infinite Imprimitive Homogeneous 3-edge-colored Complete Graphs

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Introduction

We will refer to a complete graph with a 3-edge coloring as a *3-graph* in the present paper, for lack of a better name; equivalently, a 3-graph is a structure in a language consisting of three nontrivial symmetric 2-types. A 3-graph is said to be *homogeneous* if every isomorphism between finite induced 3-subgraphs extends to an automorphism of the whole 3-graph. The finite homogeneous 3-graphs are known [7,8], and in fact there is a decent theory of finite homogeneous structures for relational languages [5, 10] which provides a very rough classification in general. The present paper classifies the infinite imprimitive homogeneous 3-graphs, as part of an ongoing project to explore the ramifications of a combinatorial method due primarily to Lachlan. There is no systematic theory of infinite homogeneous structures, even for binary languages. It is not clear whether the infinite homogeneous 3-graphs are classifiable. The classification of the infinite homogeneous directed graphs is known, and the case of 3-graphs appears harder but in some ways similar.

We expect that an attempt to apply Lachlan's method to the classification of all infinite homogeneous 3-graphs will either lead to a considerable development of that method, or to some completely unfamiliar phenomena. The case treated here is one of the obvious special cases that needs to be dealt with first. The classification of imprimitive homogeneous directed graphs was dealt with in a similar spirit in [1].

In §1 we will present a catalog of the examples found with a brief discussion of their properties. With one exception, the five families listed are analogs of imprimitive directed graphs; the exceptional family is the class of product 3-graphs. Homogeneous product structures exist quite generally but are not encountered in the context of directed graphs, as the type structure is too limited. We will see that the first three families in our catalog are rather special in character, while the last two families are of more or less generic type. The bulk of the analysis is directed toward identification theorems for the structures lying in these last two families. In particular the fifth and last family has only two members, each of which needs its own identification theorem. As a result, about a third of our analysis will be devoted to these two structures.

The paper is organized as follows. In §1 we describe our five families of infinite imprimitive homogeneous 3-graphs, which we refer to as: *composite* 3-graphs; *product* 3-graphs; *double covers*; *restricted generic*; and *generic* type. The first four families contain infinitely many examples, while the last family consists of just two examples. In §2 we establish some useful preliminary results which hold uniformly for homogeneous 3-graphs not lying in the first three families. In §§3-4 we carry out the classification of the restricted generic homogeneous 3-graphs, which includes all those which have only finitely many equivalence classes. §5 contains the characterization of the semigeneric 3-graph, and §6 contains the characterization of the generic imprimitive 3-graph. A version of Lachlan's method is used at the end of §4 and in §6. The other cases are handled by *ad hoc* methods.

We assume a considerable degree of familiarity with the methods presently in use, including the use of Fraissé's theory of amalgamation classes, and our discussion of Lachlan's method is rather brief as this has been developed more fully on a number of other occasions, for example in [1-3,9]. We move back and forth rather often between homogeneous structures and the corresponding amalgamation classes, especially toward the end of the paper.

Quite a few elementary consequences of homogeneity in the category of 3-graphs will be used without explicit mention, such as transitivity of the automorphism group, and homogeneity of the structure induced on the set of realizations of some 1-type over a finite set. Later sections of the paper make fairly explicit use of the classification results of earlier sections; the operation of restriction to a 1-type may produce a structure of the type considered in an earlier section, or a new structure of the type under consideration in a given section. In fact, a similar consideration is one of the main reasons for wishing to have the classification given here, prior to launching into the analysis of the primitive case.

The following notation will be used throughout. The three edge relations are denoted R_1, R_2, R_3. We usually write "$\mathrm{tp}(ab) = i$" rather than "$R_i(a, b)$", and we read this: "The edge (ab) has type (or color) i."; or in a more model-theoretic vein: "The pair (a, b) has 2-type i." Note that a pair a, b with $a = b$ also has a 2-type, referred to as the *trivial 2-type*.

The relation E denotes some nontrivial equivalence relation on an imprimitive homogeneous 3-graph H, which after some brief preliminary analysis will always be taken to be the relation defined by: $R_1(x, y)$ or $x = y$. The requirement that the relation so defined is in fact an equivalence relation may be expressed as follows: there are no triangles with exactly two sides of type 1. Once this normalization of E is made, our argument will take place inside the category of 3-graphs satisfying this constraint, and it will generally be tacitly assumed to apply to all 3-graphs under consideration. Occasionally this requirement will be mentioned specifically, frequently in the following considerably less precise manner: "The type 1 defines an equivalence relation."

Amalgamation classes are always assumed to be classes of finite 3-graphs, and this requirement is also noted specifically on occasion. Types may be thought of as isomorphism types, but this terminology will be used with the usual degree of flexibility occurring in model theory. If A is a finite subset of a 3-graph H and p is a 1-type over A, then A^p denotes the set of realizations of p in H. When $A = \{a\}$ is a singleton and $p = i \in \{1, 2, 3\}$, the notation a^i is used for this set.

We note that the term "3-graph" is more commonly used in another way, to refer to 3-regular hypergraphs, and in particular the article [12] refers to that context. As there is no efficient general terminology for categories of combinatorial structures of general type, we have expropriated that term for our present use.

We are grateful to the referee for an attentive reading of the paper, which has led to the insertion of a little more connective tissue in the body of the text.

1 A catalog

We present the five types of infinite imprimitive homogeneous 3-graphs in detail, under the following headings:

I.	Composites
II.	Products
III.	A Double Cover
IV.	Restricted generic
V.	Generic type

Within each category, after describing the graphs in what we consider to be the most natural terms, we will highlight the characteristic features of the graphs involved in terms which are more useful for their identification in the course of the proof of the classification theorem. The proof that this catalog does indeed give a full classification of the imprimitive homogeneous 3-graphs will occupy the remaining sections of the paper.

I. *Composites* $K[G]$ *and* $G[K]$

The operation of *composition*, also called the wreath product, and denoted $A[B]$, takes as input two structures A and B and produces a third whose underlying set is the cartesian product $A \times B$ and whose automorphism group is $\mathrm{Aut}\,(B) \wr \mathrm{Aut}\,(A)$. The structure is thought of as the result of replacing each point of A by a copy of the structure B, while using the structure of A to determine the relations between points lying in distinct copies of B. This is particularly transparent when A and B are binary homogeneous structures with k and l nontrivial 2-types. Then $A[B]$ is binary homogeneous with $k+l$ nontrivial 2-types and its language may be thought of as the disjoint union of the languages of the component structures. The cases relevant to 3-graphsare: $k=1$, $l=2$ or $k=2$, $l=1$ in which case we are dealing with the composition in some order of a complete graph K with an ordinary graph G, denoted $K[G]$ or $G[K]$ as the case may be. In a more explicit notation we would write:

$$K^i_m[G^{j,k}] \qquad \text{or} \qquad G^{i,j}[K^k_m]$$

where the superscript lists the labels of the nontrivial 2-types and the subscript indicates the order of the complete graph. Here, according to our conventions for 3-graphs, we will take $\{i,j,k\} = \{1,2,3\}$. To make all of this completely explicit it suffices to consult the list of homogeneous graphs, which in any case will be presented in connection with family III, as the analysis in that case involves some consideration of individual examples.

In the structures $K[G]$ the nontrivial definable equivalence relation is the union of two nontrivial 2-types, together with equality, while in $G[K]$ it involves one nontrivial 2-type. These examples are characterized by the property that:

(∗) the 3-graph under consideration carries a nontrivial definable equivalence relation which is a *congruence*,

or in other words: the type of a pair of points lying in two distinct classes is determined by the two classes involved. It is clear that if such a 3-graph is homogeneous then it is composite (all of the classes are isomorphic) and the two factors are themselves homogeneous.

We will soon be interested exclusively in the case in which the relation E given by the union of one of the relations R_i and the equality relation is an equivalence relation on our 3-graph H. In this case the condition that E is a congruence can be formulated as follows:

(∗) the 3-graph contains no triangle whose edges are of three distinct types.

When we pass from the homogeneous 3-graph H to the associated amalgamation class we will find it convenient to work with such characterizations.

II. *Products* $K \times K$

The *product* $A \times B$ of two structures is again given set theoretically by the cartesian product, but the automorphism group is $\mathrm{Aut}\, A \times \mathrm{Aut}\, B$. The type of a pair of elements of $A \times B$ can be thought of as a pair: the type of the pair of first coordinates in A, and the type of the pair of second coordinates in B. Accordingly if A and B have k and l nontrivial 2-types respectively, $A \times B$ has $(k+1) \cdot (l+1) - 1$ nontrivial 2-types. The first nontrivial case, $k = l = 1$, with symmetric types, consists of products $K_m \times K_n$ of complete graphs, viewed as 3-graphs.

The characteristic feature of these examples is that:

(∗) the 3-graph under consideration carries two nontrivial definable equivalence relations which are *transversal* in the sense that any two classes meet in one point.

It is easy to see that in a homogeneous 3-graph with this property each equivalence relation must be defined by a single nontrivial 2-type, the classes for either relation are then complete graphs of constant size, and the whole structure is the product of these two complete graphs.

III. *The double cover* $\hat{\Gamma}_\infty$

The third construction is less well-behaved: it does not always give rise to homogeneous structures, and in fact produces only one infinite example, up to a permutation of the language. However the same or a very similar construction is important in other contexts, such as imprimitive homogeneous directed graphs or finite homogeneous 3-graphs, and we will give a fairly systematic discussion of it in the present setting. Let G be any graph, and for definiteness let us label the edge and nonedge types in G as types 1 and 2 respectively, and let us also momentarily consider equality as type 0. We define an associated graph $\hat{G}$ by first extending G by one more vertex $*$, adacent to each vertex of G, and then forming the *double cover* $\hat{G} = G^* \times \{0,1\}$, a sort of twisted composite, so that $G^* \times \{0\}$ and $G^* \times \{1\}$ are copies of G^*, while the type structure between these two copies of G^* is essentially the complement of the structure in G^*: $\mathrm{tp}((x,0),(y,1)) = 3 - \mathrm{tp}(x,y)$.

This behaves somewhat like a double cover of G in two related senses: first, the involution interchanging $(x,0)$ and $(x,1)$ is a central involution of $\mathrm{Aut}\,(\hat{G})$, and the quotient is an extension of $\mathrm{Aut}\,G$; second, over the point $(*,0)$, the structure $\hat{G}$ is equivalent to the structure given by two copies of G with a definable isomorphism between them – the point $(*,1)$ can be neglected as it is definable from $(*,0)$. The characteristic features of this family are that

$(*)$ the equivalence classes are of order 2; and the equivalence relation is not a congruence.

The double cover of a complete graph can also be viewed as a product.

It is fairly easy to see that a homogeneous 3-graph H in which one of the types defines an equivalence relation with classes of order 2, which is not a congruence, is the double cover $\hat{G}$ associated to some graph G, and that the graph G itself must be homogeneous. The main point is this: since some pair of equivalence classes realizes both available cross types, by homogeneity the same applies to any pair of distinct equivalence classes. Now to recognize H as $\hat{G}$ it suffices to consider the structure of H over any one of its points. By the same token, the graph G must be homogeneous. The only point which requires attention is the identification of the homogeneous graphs G for which the 3-graph $\hat{G}$ is again homogeneous. Certainly $\hat{G}$ becomes homogeneous when the point $(*,0)$ is fixed; so by general principles, homogeneity of $\hat{G}$ is equivalent to the following conditions:

$$G \text{ is homogeneous;}$$

$$\mathrm{Aut}\,\hat{G} \text{ is transitive on } \hat{G}.$$

Now we analyze the content of this last condition. For $x \in G$ we will write $\Gamma(x)$ for the set of neighbors of x in G, and $\Delta(x)$ for the set $G - (\Gamma(x) \cup \{x\})$. We write G^x for the graph obtained from G by converting edges and nonedges linking $\Gamma(x)$ and $\Delta(x)$ into nonedges and edges respectively. The point of this construction is that for $x \in G$, the set of neighbors of $(x,0)$ in $\hat{G}$ has the structure of G^x, with $(*,0)$ replacing $(x,0)$ and $\Delta(x) \times \{1\}$ replacing $\Delta(x) \times \{0\}$. From this we find:

Lemma 1 *Let G be a graph. The following are equivalent:*

1. $\mathrm{Aut}\,\hat{G}$ *is transitive on* $\hat{G}$.
2. *For* $x \in G$, $G^x \simeq G$.

We are interested in this condition when G is homogeneous, and hence has a transitive automorphism group; so it suffices to examine G^x for any single

$x \in G$. To work out the content of condition (2) in a more explicit form it seems to be necessary to know the homogeneous graphs explicitly. They are as follows up to complementation (which is simply a relabeling of types, and is of no significance here):

1. The pentagon C_5 and the line graph $E_{3,3}$ associated with the complete balanced bipartite graph $K_{3,3}$. This has nine vertices corresponding to the edges of $K_{3,3}$, with two of these vertices linked if the corresponding edges have a common vertex.
2. $m \cdot K_n$ which is $K_m[K_n]$ viewed as a graph.
3. Γ_n generic omitting K_n. Here $n \geq 3$ as otherwise this graph falls under group (2).
4. Γ_∞ generic, which is isomorphic to Rado's graph, also called the random graph.

Here we have included the finite examples (the two graphs listed under (1) and the graphs in (2) with $m, n < \infty$). By inspection the graphs in groups $(1, 4)$ satisfy our criterion. The graphs $G = m \cdot K_n$ for $m, n > 1$ are disconnected, but have connected transforms G^x. For m or n equal to 1 in group (2) the graph G or its complement is complete, and satisfies our condition since $G^x = G$, but the corresponding example is a product, as we have noted. In group (3) our condition fails since $2 \cdot K_{n-1}$ embeds in Γ_n and the transform of this graph contains K_{2n-3}, whereas $2n - 3 \geq n$. The 3-graphs associated with the two examples in group (1) are known from the classification of the finite homogeneous 3-graphs, and the example associated with Γ_∞ is the only one relevant here.

IV. *Restricted generic 3-graphs* $\Gamma^{i,j}_{m,n}$

The last two families involve graphs with a less straightforward structure which are best presented in terms of amalgamation classes. We define amalgamation classes $\mathcal{A}^{i,j}_{m,n}$ for distinct types $i, j \in \{1, 2, 3\}$ and for parameters m, n satisfying:

$$m < \infty; \quad m \leq n + 1 \leq \infty$$

The class $\mathcal{A}^{i,j}_{m,n}$ consists of all finite 3-graphs on which the type i defines an equivalence relation with at most n classes (which is no restriction if $n = \infty$) and which does not embed K^j_m (which is no restriction if $m = n + 1$). This is easily seen to be an amalgamation class, if one bears in mind that the only amalgamation problems which need to be considered are those which require amalgamation of two one-point extensions Aa, Ab of a common substructure A, for which it is sufficient to determine $\mathrm{tp}(a, b)$, which we do by taking $\mathrm{tp}(a, b) = i$ if possible and $\mathrm{tp}(a, b) \neq i, j$ otherwise. This construction never increases the number of equivalence classes and never involves the type j.

The amalgamation class $\mathcal{A}^{i,j}_{m,n}$ corresponds to a unique countable homogeneous 3-graph which we call $\Gamma^{i,j}_{m,n}$. The characteristic features of these examples are that:

(∗) the equivalence relation is defined by a single type i; the 3-graph omits some K^j_m with m finite; and it is not of types I-III.

It is not at all clear *a priori* that these conditions force a homogeneous 3-graph to have the precise form exhibited here, and this will be one of the main things that we need to prove.

V. *Generic type:* Γ^i_∞ *and* $\infty * K^i_\infty$
Finally we consider homogeneous 3-graphs in which

(∗) the type i defines an equivalence relation, not of the first three special types, and embedding K^j_n for all n and all types $j \neq i$.

We will find two examples. One is the completely generic 3-graph relative to the basic constraint that type i defines an equivalence relation. We denote this Γ^i_∞ where the subscript is intended to suggest the genericity. The other, which we call the semigeneric example, denoted $\infty * K^i_\infty$, is subject to a constraint we call the *parity constraint* (relative to the type i) which reads as follows, writing E for the equivalence relation given by the union of the type i and the equality relation:

> Between any two E-equivalent pairs, there are an even number of edges of each type

In terms of substructures, this means that:

(∗) the 3-graph under consideration omits two specific 3-subgraphs of order 4, in which one cross-type occurs 3 times, and the other once.

It is of course necessary to check that one can amalgamate finite 3-graphs satisfying the parity constraint, and again one considers amalgamations of one-point extensions Aa and Ab over A. There is no difficuly unless a and b are equivalent to elements a' and b' in A which lie in distinct E-classes, and in this case the type of ab is uniquely determined by the parity constraint. One must check that another choice of a' and b' cannot lead to a contradictory determination of the type, which amounts to the addition law for parity.

This concludes our presentation of the examples. The structure of our analysis follows the structure of this catalog, though as we have sufficiently indicated the necessary verifications of the classification in the first three families as the examples were presented, we will assume throughout that we are not dealing with 3-graphs of those three types. Accordingly, in the next section we will derive a few general properties of imprimitive homogeneous 3-graphs not in families I-III, and then turn in following sections to the detailed analysis of families IV and V.

2 Preliminaries

H is an infinite imprimitive homogeneous 3-graph and E is a nontrivial definable equivalence relation on H. By homogeneity, the relation E is the union of one or two nontrivial 2-types with the equality relation. If E involves two nontrivial 2-types then as there is only one remaining 2-type it follows that E is a congruence and H is composite of type $K[G]$. Therefore we may assume throughout:

E involves only one nontrivial 2-type, which we designate as type 1.

As mentioned in the introduction, this requirement applies to all 3-graphs from this point on.

We will take as a second standing hypothesis throughout that:

$$H \text{ does not belong to families } I - III.$$

At a later stage we will consider certain homogeneous 3-subgraphs derived from H and our standing hypothesis will not always be inherited in such cases, but as we already know these exceptional homogeneous 3-graphs explicitly this will not present any notable difficulties. The present section contains three generally useful lemmas relating to this situation; the last of these lemmas will allow inductive arguments to run smoothly.

Lemma 2 *If C is an E-class in H and $x \in H - C$ then for any type $i \neq 1$, the set $x^i \cap C$ is infinite.*

Proof:

Let $K = |C| \leq \infty$. If for some choice of x, C, and i the set $x^i \cap C$ is empty it follows easily that E is a congruence and H is composite, a contradiction. Suppose now that $|x^i \cap C| = k$ with $1 \leq k < K$. Replacing i if necessary by another type, we may suppose $2 \cdot k \leq K$. Then $|x^i \cap C| = k$ for any choice of x, C with $x \notin C$.

Fix two E-classes C_1 and C_2. If $x^i \cap C_2 = y^i \cap C_2$ for all $x, y \in C_1$ then for $z \in C_2$ we find $z^i \cap C_1 = \emptyset$ or C_1, a contradiction. Suppose next that for $x, y \in C_1$ we have $x^i \cap C_2$, $y^i \cap C_2$ always equal or disjoint. Pick x and y in C_1 so that these sets are disjoint. As $C_2 - x^i$ is indiscernible over x, this set is indiscernible over C_1, and every k-subset of $C_2 - x^i$ must occur as $z^i \cap C_2$ for some $z \in C_1$. It follows that any two k-subsets of $C_2 - x^i$ are equal or disjoint, and hence either $k = |C_2 - x^i|$ or $k = 1$.

If $k = 1$ and $K = 2$ then $H = \hat{G}$ for some G, a contradiction. If $k = 1$ and $K > 2$ then we claim that H is a product. If this fails, then we have a triple x, y, z of inequivalent points with $\mathrm{tp}(xz) = \mathrm{tp}(yz) = i \neq \mathrm{tp}(xy)$ and hence there is a z-definable bijection of $x/E - \{x\}$ with $y/E - \{y\}$ given by composing the bijections $x/E \leftrightarrow z/E \leftrightarrow y/E$ defined by the type i, and this bijection is given by some type i'. But then it follows easily that for $x \notin C$, $|x^{i'} \cap C| = 1$, and this forces $K = 2$.

We now know that there are $x, y \in C_1$ with $x^i \cap C_2$ and $y^i \cap C_2$ distinct but not disjoint. In particular $k > 1$. Let $X_1 = x^i \cap C_2$ and $X_2 = C_2 - x^i$, and set $k_l = |y^i \cap X_l|$ for $l = 1, 2$. The sets X_1, X_2 are mutually indiscernible over x and hence over C_1, and it follows that any k-set which meets X_1, X_2 in sets of size k_1 and k_2 respectively will occur as $z^i \cap C_2$ for some $z \in C_1$. Since $k_1 < k$ there are k-sets of this type meeting $y^i \cap C_2$ in $k - 1$ points. Thus we may take $k_1 = k - 1$ and we conclude that any k-set in C_2 meeting $x^i \cap C_2$ in $k - 1$ elements is again of the form $y^i \cap C_2$ for some $y \in C_1$. By repeated application of this principle, all k-subsets of C_2 are of this form. In particular C_2 realizes only one k-type over C_1, and similarly C_1 realizes only one k-type over C_2. As $k \geq 2$, we find that $|x^i \cap y^i \cap C_2|$ is independent of the choice of distinct $x, y \in C_1$. However this can be any number in the range $0, \ldots, k - 1$ and hence $k = 1$, a contradiction. ∎

Notice that conversely Lemma 2 implies our initial hypothesis: H is not composite, a product, or a double cover.

Lemma 3 *Let I be a finite subset of an E-class C_1 of H, and p a 1-type over I realized in a second E-class C_2. Then $I^p \cap C_2$ is infinite.*

Proof:

We suppose that I is a minimal counterexample and we fix $I_\circ \subseteq I$ with $|I - I_o| = 1$. Let $p_o = p \restriction I_o$. Then $I^{p_\circ} \cap C_2$ is infinite and $I^p \cap C_2$ is finite. By the previous lemma I_o is nonempty. Let $k = |I^p \cap C_2|$. As $I^{p_\circ} \cap C_2$ is indiscernible over I_o, any k-subset will occur as $I'^p \cap C_2$ for some $I' = I_o \cup \{a\}$ with $a \in C_1$. As in the previous proof it follows that $k = 1$.

Thus over I_o and the class C_2 we have a definable map from $C_1 - I_o$ onto $I_o^{p_\circ} \cap C_2$. We claim that this map is 1-1. If not, then we have a nontrivial equivalence relation on $C_1 - I_o$ definable from any parameter in C_2. This would mean that any two elements of C_2 impose the same partition into two sets on $C_1 - I_o$, and hence C_2 realizes only finitely many types over C_1. Thus C_2 has a nontrivial equivalence relation definable from any parameter in C_1, and this means that any element of C_1 imposes the same partition into two sets on C_2. But then the nonempty set $I^p \cap C_2$ must coincide with a set of the form $x^i \cap C_2$, which is infinite.

We have a bijection of $C_1 - I_o$ with $I_o^{p_\circ} \cap C_2$ which is definable from I_o and any parameter in C_2. Accordingly it must be defined by a 2-type i. Then for $x \in C_2$, $|x^i \cap C_1| \leq |I_o| + 1$, contradicting the previous lemma. ∎

We note that in the notation of the previous lemma every 1-type p over I is indeed realized in H, but not necessarily in every equivalence class, as we see in the case of the semigeneric 3-graph.

The next lemma has a fairly long proof, with quite explicit amalgamation arguments used to deal with the composite case.

Lemma 4 *Assume that i is the type 2 or 3 and that K_3^i embeds in H. Then for $x \in H$ the 3-graph x^i is not a composition, product, or double cover.*

Proof:

1. By Lemma 2 the E-classes of x^i are infinite and hence x^i is not a double cover.

2. Suppose that x^i is a product. Then there is a type j which defines a bijection between $x^i \cap C_1$ and $x^i \cap C_2$ for C_1, C_2 E-classes not containing x.

We show first:

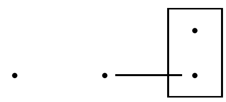
$$\text{(1)} \qquad \begin{array}{l} \text{Any two elements of } C_1 - x^i \text{ realize} \\ \text{the same 1-type over the pair } (x, \{C_2\}). \end{array}$$

Given elements $y_1, y_2 \in C_1 - x^i$, we want elements z_1, z_2 realizing the same type over xy_1 and xy_2 respectively. For this it suffices to find one element $y \in C_1 - x^i$ realizing both possible types over $x^i \cap C_2$, or over $C_2 - x^i$. If this is not possible then each element $y \in C_1 - x^i$ realizes one type over $x^i \cap C_2$, and the other type over $C_2 - x^i$. In particular there will be distinct elements of C_1 which realize the same type over C_2. There are also elements of C_1 which do not realize the same type over C_2, so the relationship of having the same type over C_2 is a nontrivial $\{C_2\}$-definable equivalence relation on C_1. In particular it is definable from any parameter $y \in C_2$ and hence coincides with the partition of C_1 into 1-types over any such y. However as x^i is a product, not all elements of C_2 give rise to the same partition of C_1. This proves (1).

Now for $y \in C_1 - x^i$ we consider the set $y^j \cap (x^i \cap C_2)$, which is definable from $(x, y, \{C_2\})$. By (1) this is always nonempty. This set corresponds bijectively to a subset of $x^i \cap C_1$. On the other hand we claim that $x^i \cap C_1$ realizes only one type over $(x, y, \{C_2\})$, since for any elements $y_1, y_2 \in x^i \cap C_1$ and any type k for which $y^k \cap x^i \cap C_2$ is infinite, there is $z \in y^k \cap x^i \cap C_2$ with $\mathrm{tp}(zy_1) = \mathrm{tp}(zy_2) \neq j$. It follows that $y^j \cap x^i \cap C_2$ corresponds bijectively to all of $x^i \cap C_1$, or in other words that $x^i \cap C_2 \subseteq y^j$ for $y \in C_1 - x^i$, with a similar statement holding when C_1, C_2 are interchanged.

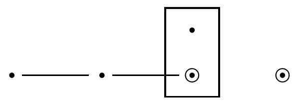
Fix $y_1, y_2, y_3 \in C_2$ with $y_1, y_2 \in x^i$, $y_3 \notin x^i$. Take $z \in y_2^j \cap (x^i \cap C_1)$. Then $zy_1y_2 \simeq zy_1y_3$, so y_1y_2 and y_1y_3 have the same type over $\{C_1\}$. Take $z' \in x^i \cap C_1 - (y_1^j \cup y_2^j)$. Then $\mathrm{tp}(z'y_1) = \mathrm{tp}(z'y_2) \neq j$ and thus there is $z'' \in C_1$ with $\mathrm{tp}(z''y_1) = \mathrm{tp}(z''y_3) \neq j$. But if $\mathrm{tp}(z''y_1) \neq j$ then $z'' \in x^i$ while if $\mathrm{tp}(z''y_3) \neq j$ then $z'' \notin x^i$, a contradiction.

3. We show that x^i is not composite. We represent type i by a nonedge and the third type (type $5 - i$) by an edge. Our claim is that H embeds the configuration:

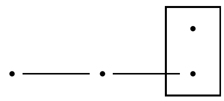
(∗) 

where the box represents an E-class. We will use explicit amalgamation arguments.

The desired configuration (∗) is forced by the amalgamation:

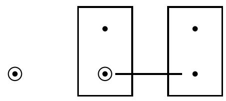


in which the circled vertices are those whose 2-type is to be determined in making the amalgam. The factors of this amalgamation are:

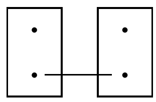
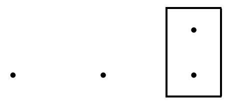
(1)  (2)

So it suffices to show that H contains copies of (1) and (2), or $(*)$; to do so, we will have to bring in another amalgamation problem below.

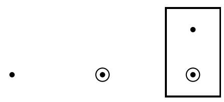
We get (1) or $(*)$ by the following amalgamation:



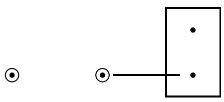
with the factors:

(3)  (4) 

If (4) does not embed in H we succeed with

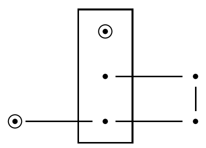


For (3) we use

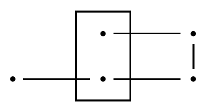
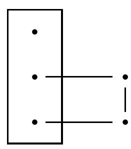


If this fails, then it yields (1) or $(*)$.

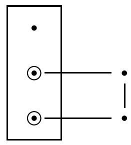
So we may suppose that we have (1) in H. A second attempt at $(*)$ is given by the following amalgamation problem:



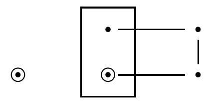
The factors are:

(5)  (6) 

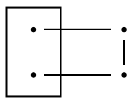
(6) is obtained from (1) using



For (5) we use



with factors

(7)  and (8)

Here (7) is contained in (6).

At this point either (2) or (8) would be sufficient to force $(*)$ by amalgamation; in other words, our supposed composition x^i must not contain either of the following:

(2′) (8′)

This forces x^i to be complete of type i or i' where $\{i, i'\} = \{2, 3\}$. As H embeds K_3^i by hypothesis, x^i omits $K_2^{i'}$. Thus if $\mathrm{tp}(xy) = i$ then for $C \in H/E - (x/E, y/E)$ we have $x^i \cap C \subseteq y^i \cap C$, hence $x^i \cap C = y^i \cap C$, and x and y have the same type over $H - (x/E \cup y/E)$. If follows easily that E is a congruence, a contradiction. ∎

After these preliminaries we divide the rest of the analysis into two major parts. We deal first with the *restricted case* in which some K_m^i is omitted. This of course includes all cases in which H/E is finite. After this we will be left with the analysis of the two cases of *generic type*, with or without the parity constraint, leading in each case to a unique example.

3 The restricted case: preparation

We have assumed that H carries an equivalence relation E which is the union of type 1 and the trivial 2-type. We now assume that H is *restricted*, and specifically: H omits some K^2_m or K^3_m with m finite. We take m minimal, and we may assume without loss of generality that H omits K^2_m. We set $n = |H/E| \leq \infty$. It is convenient to formulate our hypothesis as follows:

(∗) H omits K^2_m, and embeds K^2_{m-1}; if $m = n+1$ then H embeds K^3_n

Thus $m-1 \leq n$. There are some variations in the analysis according as n is finite or infinite, and for n finite the cases $m \leq n$, $m = n+1$ should be distinguished. The hypothesis according to which H is not composite, a product, or a double cover remains in effect, and accordingly Lemmas 2-4 apply. The result is then that H is generic for the data (m, n) in the sense described in the discussion of family IV in §1. The proof of this claim goes by induction on n for n finite, followed by the treatment of the case $n = \infty$. In checking the applicability of the induction hypothesis, one should make use of the formulation exactly as given in (∗). For a fixed value of n the proof proceeds by induction on m. For $n = 2$ our claim follows easily from Lemmas 2 and 3.

In this and the next section we work with a fixed value of m and n, under the inductive hypothesis that the classification of restricted graphs has been achieved for pairs (n', m') preceding the pair (n, m) lexicographically. It is convenient to use the notation $n-1$ systematically, with the convention that $n-1 = n$ for $n = \infty$. We confine ourselves in this section to relatively general aspects of the analysis, which will be applied in the following section.

Lemma 5 *Assume $m \leq n$. Then for $x \in H$, x^2 is generic omitting K^2_{m-1}, with $n-1$ classes.*

Proof:

That x^2 has $n-1$ classes follows from Lemma 2. By Lemma 4 and induction the claim follows. ∎

Lemma 6 *Let $x \in H$.*

1. *If $m \leq n$ then x^3 embeds K^2_{m-1}.*
2. *If $m \leq n < \infty$ then x^3 is generic omitting K^2_m, on $n-1$ classes.*

Proof:

By Lemma 2 x^3 meets each E-class of $H - (x/E)$, and by Lemma 4 x^3 is not a composition, product, or double cover. So given (1), induction yields (2).

For the proof of (1), if $m = 3$ our claim follows from Lemma 4. Assume now that $m > 3$.

We will use amalgamations to embed $I_1 +^3 K^2_{m-1}$ in H, where I_1 denotes a single vertex and $+^3$ is the operation of disjoint sum, with connecting edges of type 3.

We begin with the configurations $aA \simeq I_1 +^3 K^2_{m-3}$ and $B \simeq K^2_{m-2}$, and we adjoin elements c_1, c_2 with:

$$\begin{array}{c} \mathrm{tp}(c_1/A), \mathrm{tp}(c_2/AB) \equiv 2 \\ \mathrm{tp}(c_1/B), \mathrm{tp}(a/c_1c_2) \equiv 3 \end{array}$$

We pair the elements of aA and B and we put each pair in a single E-class. The types in $A \cup B$ which are not yet determined are set equal to 3. The types $\mathrm{tp}(a/B)$ and $\mathrm{tp}(c_1c_2)$ are not yet determined.

Our idea is to amalgamate $aABc_1$ with $aABc_2$ over aAB. If we assume these factors are present in H, and that the result of the amalgamation assigns to the pair c_1, c_2 the type 2 or 3, then we will have, respectively, $a +^3 c_1c_2A$ or $c_1 +^3 c_2B$ as the desired copy of $x +^3 K^2_{m-1}$. So our problem is to ensure $\mathrm{tp}(c_1c_2) \in \{2, 3\}$.

Although it is clear that c_1, c_2 remain distinct, it is not clear that they remain inequivalent. The first problem is to prevent this, and this then raises (or aggravates) a second problem, which is to find suitable factors inside H. To dispose of the first problem we adjoin an element c_1' with $\mathrm{tp}(c_1c_1') = 1$, $\mathrm{tp}(c_1'c_2) = 2$, $\mathrm{tp}(c_1'/AB) = 3$, and we amalgamate $aABc_1'c_1$ with $aABc_1'c_2$ over $aABc_1'$. The problem now is to embed $aABc_1'c_i$ in H for $i = 1, 2$.

The element a is paired with some $b \in B$. We will form $aABc_1c_2$ by amalgamating $aAbc_1'c_2$ and $ABc_1'c_2$ over $Abc_1'c_2$. These two factors are found in H by the previous lemma since the first one omits K^2_{m-1} and the second one has the form $c_2 +^2 ABc_1'$ where ABc_1' omits K^2_{m-1}.

This produces a suitable version of $aABc_1'c_2$ and determines the structure of the factor $aABc_1'c_1$, which also omits K^2_{m-1} as $m > 3$: no copy of K^2_{m-1} can contain c_1' or c_1 and the remaining elements occupy $m - 2$ distinct E-classes.

After this, we can carry out the amalgamation of $ac_1c_1'AB$ and $ac_1'c_2AB$ over $ac_1'AB$ as indicated at the outset. ∎

Lemma 7 *If $n < \infty$ and H embeds both K^2_n and K^3_n then x^2 and x^3 are generic on $n - 1$ E-classes.*

Proof:

By induction on n it suffices to show for example that x^3 embeds both K^2_{n-1} and K^3_{n-1}; the latter is trivial.

Suppose on the contrary that $k \leq n$ is minimal such that x^3 omits K^2_m. Then by induction and Lemma 4 x^3 is generic on $n - 1$ classes, omitting K^2_k. In particular for $y \in x^3$ it follows that $y^2 \cap x^3$ embeds K^3_{n-2} and hence x^2 embeds K^3_{n-2}. Thus x^2 can be identified, by induction, as the homogeneous graph on $n - 1$ classes, either omitting K^3_{n-1}, or embedding K^3_{n-1} and omitting K^3_n. In any case the contruction used in the previous Lemma again works. ∎

The next lemma is noteworthy in that it is false for the semigeneric graph, and hence relies essentially on the hypothesis that H is restricted.

Lemma 8 *For H restricted, any finite 3-graph with two classes embeds in H.*

Proof:

By making use of amalgamations whose solutions are uniquely determined (as E is an equivalence relation) we can reduce the problem to the following two cases:

(1) One E-class C_1 is indiscernible over the other E-class C_2.

(2) Both E-classes contain at most 2 elements.

In case (2) we may suppose that both E-classes do contain two elements, and that each pair realizes two distinct types over the other pair. Using Lemma 5 we reduce by induction to the case in which H omits K_3^2. It is easy to see that every 1-type over a pair of equivalent points is realized in H, but we need to show something stronger: if the equivalent pair $x, y \in H$ is fixed, then we claim that

(2′) Each 1-type over xy is realized in every class of $H - (x/E)$.

Then Lemma 2 implies (2).

If $n = 2$ then (2′) is evident. Assume $n > 2$. Fix $a \in H$ and consider a^2, which omits K_2^2, has infinite E-classes, and meets every E-class of $H - (a/E)$. In particular if we take $x, y \in a^2$ equivalent, then any E-class of $H - (x/E, a/E)$ is represented by some element $b \in a^2$, and $\mathrm{tp}(b/xy) = 3$. In particular any two E-classes in $H - (a/E, x/E)$ are conjugate by an automorphism which fixes a, x, and y.

If the type of a over x, y (i.e., $\equiv 2$) is realized in two distinct E-classes, then any two classes of $H - x/E$ are conjugate over x, y, since for $n = 3$ these two realizations are conjugate over x, y, and for $n > 3$ the elements b found in the previous paragraph can be used. So (2′) follows in this case. There remains the possibility that this type is realized in a unique class of H. In this case there will be disjoint pairs (x_1, y_1) and (x_2, y_2) for which this E-class is the same, and there will be other such disjoint pairs for which the corresponding E-classes are different. This means that each E-class of H carries a nontrivial 4-place relation, which is impossible. ∎

4 The restricted case: conclusion

After these preliminaries we can carry out the analysis of the restricted case. The notation of the previous section is retained. In particular m, n are defined with $m \leq n+1 \leq \infty$ and $m < \infty$, and

$(*)$ $\quad H$ omits K^2_m, and embeds K^2_{m-1}; if $m = n+1$ then H embeds K^3_n

We dispose first of the case in which n is finite.

Lemma 9 *If $n < \infty$ then H is generic with n classes, omitting K^2_m.*

Proof:

We have to embed every finite 3-graph on n classes, omitting K^2_m, into H. We first reduce to the case in which $n-1$ of the classes are singletons and the remaining class is indiscernible over these $n-1$ points.

So let G be any finite 3-graph with n classes, omitting K^2_m, and let C be one particular E-class of G. Adding points to C if necessary we may suppose that any two points of $G - C$ realize distinct types over C. We may also suppose that G actually has n distinct classes. We may suppose that all the classes in $G - C$ are singletons, since for any class $C_1 \neq C$, G is the unique amalgam of the various 3-graphs obtained by replacing C_1 by any one of its points. Finally we may replace C successively by its various indiscernible subsets defined over $G - C$.

Now suppose that G has $n-1$ singleton classes and an indiscernible class C (which may also be a singleton). Let $x \in G - C$. If x realizes only one type over $G - \{x\}$ then G embeds in H by Lemmas 5-7 and induction. Now let $\mathrm{tp}(x/C) = i$, $i' = 5 - i$, and suppose that $\mathrm{tp}(xy) = i'$ for some $y \in G - \{x\}$. We will give an explicit amalgamation forcing G into H.

Let G_o be a copy of $G - (C \cup \{x\})$ and for $c \in C$ let G_c be a copy of $G - \{x, y\}$. Let

$$G^* = G_o \cup \bigcup_{c \in C} G_c,$$

organized so that G^* occupies only $n-2$ E-classes, and let all types which are not otherwise fixed be taken to be 3. Adjoin an auxiliary element c' to C. We intend to perform an amalgamation of $G^*c'x$ with $G^*c'C$ over G^*c', with the following idea: if $\mathrm{tp}(x/C) \equiv i$ then $xCG_o \simeq G$, while if $\mathrm{tp}(xc) = i'$ with $c \in C$, then $xcG_c \simeq G$. This requirement determines $\mathrm{tp}(x/G^*)$, $\mathrm{tp}(c/G_c)$, and $\mathrm{tp}(c/G_o)$ but leaves $\mathrm{tp}(c'/G^*x)$ undetermined. We will take $\mathrm{tp}(c'/G^*x) \cong 3$. Then G^*x embeds in a^3 for $a \in H$. As $G^*c'C$ has only $n-1$ classes and omits K^2_{m-1}, it also embeds in H. ∎

When H/E is infinite the analysis is less direct. We begin with a slight extension of Lemma 8.

Lemma 10 *If $n = \infty$ then every finite 3-graph G with 3 classes, omitting K^2_m, in which at least one class is a singleton, embeds in H.*

Proof:

Write $G = G_o \cup \{x\}$ with x a singleton E-class of G. We may suppose that each equivalence class C of G_o is either indiscernible over $G - C$ or has two elements, as G can be obtained as the unique solution to an amalgamation problem involving structures of this type. If x realizes only one type over G_o then Lemmas 5,6,8 apply unless $m = 3$ and x realizes type 2 over G_o, in which case our claim is trivial. So we will assume that x realizes types 2 and 3 over G_o.

We deal first with the following case:

x realizes only one type over each equivalence class of G_o.

Write $G_o = C_2 \cup C_3$ where $\mathrm{tp}(x/C_i) \cong i$. We perform the following amalgamation. Let I be an E-class with $k_1 + k_2 - 1$ elements. For $J \subseteq I$ of order k_i, let $D_{i,J}$ be a copy of $G_o - C_i$ and let D be the union of all $D_{i,J}$, for $i = 2, 3$ and $|J| = k_i$, viewed as a single E-class. Adjoin another element c to the class I. We now describe the structure of $D \cup Ic$. Here D and Ic are distinct E-classes and:

$$\mathrm{tp}(c/D) \cong 3$$
$$(J, D_{i,J}) \simeq (C_i, G_o - C_i) \text{ for suitable } i, J$$

Our plan is to amalgamate DIc with Dxc over Dc where:

$$\mathrm{tp}(xc) = 3$$
$$\mathrm{tp}(x/D_{i,J}) = \mathrm{tp}(x/G_o - C_i)$$

Now xcD embeds in H since xD emebeds in c^3 and cID embeds in H. After amalgamating there will be some $i = 2$ or 3 and some $J \subseteq I$ of order k_i so that $\mathrm{tp}(x/J) \cong i$. Then $xJD_{i,J} \simeq G$.

Now we suppose that

There is an equivalence class C of G_o of order 2
over which x realizes types 2 and 3.

Let $C = \{c_2, c_3\}$ with $\mathrm{tp}(x/c_i) = i$. Let $D = G_o - C$ which we take to be either indiscernible over x or of order 2, realizing two distinct types over x.

We adjoin an element to C with $\mathrm{tp}(xc) = 3$, and with $\mathrm{tp}(c/D)$ determined by the amalgamation of xcc_2 with xcD over xc. Then the amalgamation of xcc_2D and xcc_3D over xcD will force G into H. We still have to deal with the factor xcc_3D, but the effect of this is to change the class C into an indiscernble set over x. If D is indiscernible over x we are in a case treated above, and otherwise $|D| = 2$ and we repeat the argument to replace D by an indiscernible set over x, of order 2. ∎

Now we arrive at the main point in the treatment of restricted homogeneous 3-graphs. We introduce a set of "generators" (in a very rough sense) for the amalgamation class corresponding to H.

Definition 1 *For $m \geq 3$ let $\mathcal{A}(m)$ consist of the following 3-graphs:*

1. *All triangles having exactly one side of type 1;*
2. *K_n^1 and K_n^3 for all n;*
3. *K_{m-1}^2.*

We work with the amalgamation class of finite 3-graphs $\mathcal{A}$ corresponding to H. This class satisfies the following positive and negative constraints:

$$(+_m) \qquad \mathcal{A}(m) \subseteq \mathcal{A}$$

$$(-_m) \qquad K_m^2 \notin \mathcal{A}; \text{ no triangle with exactly two sides of type 1 is in } \mathcal{A}.$$

Our objective is to show that the conditions $(\pm_m)$ determine $\mathcal{A}$ uniquely. We will call a 3-graph G *m-constrained* if the class of its finite substructures satisfies $(-_m)$. The following result will evidently complete the classification of the restricted homogeneous 3-graphs.

Proposition 1 *If $\mathcal{A}$ satisfies condition $(\pm_m)$ then $\mathcal{A}$ consists of the finite m-constrained 3-graphs.*

The use of amalgamation classes allows us to reformulate this as follows. (This is a subtle and flexible idea introduced by Lachlan.)

Definition 2 *Let $\mathcal{A}$ be an amalgamation class satisfying $(\pm_m)$. Then $\mathcal{A}^*$ is the set of $A \in \mathcal{A}$ satisfying:*

Any m-constrained extension $A \cup I$ by one more E-class belongs to $\mathcal{A}$

Proposition 2 *If $\mathcal{A}$ is an amalgamation class satisfying $(\pm_m)$ then $\mathcal{A}^*$ is an amalgamation class satisfying $(\pm_m)$.*

To derive Proposition 1 from Proposition 2, we show by induction on n that for any amalgamation class satisfying $(\pm_m)$ and any finite 3-graph G with $|G/E| = n$, G embeds in $\mathcal{A}$. For any n, the induction hypothesis applies to $\mathcal{A}^*$ and $n-1$ (using Proposition 2) and this amounts to our claim for $\mathcal{A}$ and n. This trivial inductive argument is the point of Lachlan's idea.

Much of the proof of Proposition 2 is formal and straightforward. That $\mathcal{A}^*$ satisfies the negative constraints is immediate. That $\mathcal{A}^*$ is an amalgamation class is relatively straightforward, using the fact that an amalgamation problem in $\mathcal{A}^*$ has only finitely many possible solutions. One argues that if none of the possible solutions lie in $\mathcal{A}^*$ then some amalgamation problem in $\mathcal{A}$ would have no solution.

The main point is of course to prove that $\mathcal{A}(m)$ is contained in $\mathcal{A}^*$. This is the core of the classification argument. Some of the work has already been done:

by Lemmas 8 and 10 any finite 3-graph which consists either of a single class or of two classes, with one of them a singleton, will belong to $\mathcal{A}^*$. Accordingly to complete the classification of the restricted imprimitive homogeneous 3-graphs, what remains to be proved is just:

Lemma 11 *Let $\mathcal{A}$ be an amalgamation class of finite 3-graphs which satisfies conditions $(\pm_m)$. Then*

1. $K^2_{m-1} \in \mathcal{A}^*$;
2. $K^3_n \in \mathcal{A}^*$ *for all* n.

Proof:
Set $A = K^i_k$ with $i = 2$, $k = m-1$ or $i = 3$, $k = n$. Consider an m-constrained extension AI of A by a further E-class. By the usual reductions we may suppose that I is either (i) indiscernible over A, or (ii) of order 2.

If some $a \in A$ satisfies $\mathrm{tp}(a/I) \cong i$ we conclude by induction applied to a^i. To complete the analysis of case (i) we deal with the case: $\mathrm{tp}(a/I) \cong i'$ where $i' = 5 - i$. We perform the following amalgamation.

Let A_1, A_2 be disjoint copies of K^i_{k-1} and K^i_{k-2} occupying $k-1$ E-classes all together, and let $aA_1 \simeq K^i_k$, $aA_2 \simeq K^i_{k-1}$. Let I_2 be a copy of I with $\mathrm{tp}(aA_2/I_2) \cong i'$ and with $A_1 \cup A_2 \cup I_2$ again occupying a total of $k-1$ E-classes. We introduce a further equivalence class bI_2 with:

$$\begin{array}{ll} \mathrm{tp}(b/aA_1A_2I_2) \cong 3; & \mathrm{tp}(I_1/A_1I_2) \cong i' \\ \mathrm{tp}(I_1/A_2) \cong i & \end{array}$$

Amalgamate $A_1A_2I_2ab$ with $A_1A_2I_2I_1b$ over $A_1A_2I_2b$. If $\mathrm{tp}(a/I_1) \cong i'$ then $A_1aI_1 \simeq AI$ and if $\mathrm{tp}(ay) = i$ for some $y \in I_1$) then $A_2ayI_2 \simeq AI$. We need to find the factors $A_1A_2I_2ab$ and $A_1A_2I_2I_1b$ in H; in particular we should make them m-constrained.

We may suppose inductively that every m-constrained 3-graph on $k-1$ classes is in $\mathcal{A}^*$, since if $A \notin \mathcal{A}^*$ then the the homogeneous 3-graph corresponding to $\mathcal{A}^*$ has already been classified. In other words every m-constrained 3-graph on k classes is in $\mathcal{A}$. This disposes of the second factor. As the first factor has the form $(b) +^3 A_1A_2I_2a$, a similar argument, using Lemma 6 as well, disposes of this factor as well.

Now suppose that $|I| = 2$ and I is not indiscernible over A. Fix a copy of A in H. We claim that every 1-type over A is realized in every E-class of H outside A/E. Using homogeneity, if this is true for at least one 1-type over A, it is true for all 1-types over A which are realized in H – but they are all realized, by the first part of the argument.

Now consider any 1-type over A which involves the type i. Working in a^i for $a \in A$, it follows by induction that any such type is realized in all E-classes of a^i outside $(A - \{a\})/E$, and hence in all E-classes of H outside A/E. ∎

To sum up, if H is a homogeneous imprimitive 3-graph in which the equivalence relation E is given by the 2-type labelled 1 together with the trivial 2-type, and H omits some K_m^2 or K_m^3, then without loss of generality condition $(*)$ holds, and if H has finitely many equivalence classes, it is identified by Lemma 9, while otherwise it is covered by Proposition 1, which follows by a simple formal argument from Proposition 2, which in turn is covered by Lemmas 8, 10, and 11.

5 The semigeneric 3-graph

As always, H is an imprimitive homogeneous 3-graph, and more specifically H carries an equivalence relation E which is the union of the 2-type labelled 1 and the trivial 2-type. Furthermore we may now assume that H embeds all 3-graphs of the form K_m^2 and K_m^3, with m finite. These hypotheses remain in force to the end of the paper. It will be useful to express these conditions in terms of the following class $\mathcal{A}(\infty)$ of "generators":

(1) All triangles involving exactly one edge of type 1;

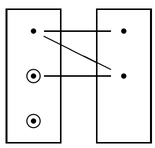
(2) K_n^i for all n and for all $i = 1, 2, 3$.

Thus we assume that H embeds all 3-graphs in $\mathcal{A}(\infty)$; the condition that E is an equivalence relation corresponds to the *omission* of triangles containing exactly two edges of type 1.

In addition, to deal with the semigeneric case, we must consider the 3-graphs O^i for $i = 2, 3$ which consist of two E-classes of order 2, with exactly one edge of type i, and hence three edges of the other cross type. In the present section we show that if H omits one of these two graphs then it is semigeneric; the remaining – generic – case is the subject of the following section. Omission of both of these graphs will be called the *parity constraint* (which however we take to include the condition that the type 1 defines an equivalence relation). The parity constraint simplifies the analysis of amalgamation considerably.

Lemma 12 *H omits O^2 if and only if it omits O^3.*

Proof:



∎

For the remainder of this section the parity constraint is imposed on H:

(3^-) H omits O^2 and O^3

We now undertake to show that H is semigeneric, or in other words:

Every finite 3-graph G which satisfies the parity constraint embeds in H.

To this end we fix some additional notation. The finite 3-graph G in question is fixed throughout, and we set $n = |G/E|$. The argument proceeds by induction on n. Let G^* be a section of G, that is a subgraph consisting of n inequivalent elements of G. The only possible amalgam of the 3-graphs $G^* \cup I$ for I varying over the E-classes of G, is the 3-graph G itself, in view of the parity constraint. Accordingly if all these subgraphs embed in H then G will also embed in H. Therefore we may suppose $G = G^* \cup I$ for one E-class I, or in a more convenient notation: $G = G_o I$ with G_o a set of $n-1$ inequivalent elements, and I an additional E-class. As usual we may take I to be either indiscernible over G_o, or of order 2. We prefer to organize the analysis under the following two headings:

$$(A) \qquad I \text{ is indiscernible over some point } a \in G_o$$

$$(B) \qquad |I| = 2$$

The second case reduces to the first via an amalgamation of $G_o cc_2$ with $G_o cc_3$ where c, c_2, c_3 are equivalent and $G_o c_2 c_3 \simeq G_o I$. We fix an element $x \in G_o$ and choose notation so that $\mathrm{tp}(xc_i) = i$. We will take $\mathrm{tp}(xc) = 3$ and determine $G_o cc_2$ by amalgamating $G_o c_2$ with xcc_2 over xc_2. As c, c_3 are indiscernible over x, case (A) applies to $G_o cc_3$.

Thus it suffices to deal with case (A): I is indiscernible over $a \in G_o$, with type i. If a realizes only type i over the rest of G_o then we conclude by induction applied to a^i. We do not know the precise structure of a^i but Lemma 4 applies and hence a^i is either restricted, of known type, or satisfies the same hypotheses as H, in which case induction on n applies. We assume therefore that $\mathrm{tp}(ab) = i' \neq i$ for some $b \in G_o$.

We perform the following amalgamation. Let G_1 be a copy of $G_o - \{a\}$ and for $c \in I$ let G_c be a copy of $(G_o - \{a, b\}) \cup I$. Let $A = G_o \cup \bigcup_{c \in I} G_c$, arranged so as to occupy $n-2$ E-classes, and satisfying the parity constraint. Let further AI be taken so that

$$G_1 I \simeq (G_o - \{a\}) I; \quad cG_c \simeq (G_o - \{a\}) I \quad \text{for } a \in I$$

Now the parity constraint allows a unique completion of the type of AI. More explicitly: each class of A contains some $g_1 \in G_1$ and some $g_c \in G_c$ for each $c \in I$. Then $\mathrm{tp}(c' g_c)$ is determined, for $c' \neq c$, by:

$$\mathrm{tp}(c' g_1); \quad \mathrm{tp}(c g_1); \quad \mathrm{tp}(c g_c)$$

Our intent is to amalgamate AI and Aa so that either $aG_1 I \simeq G_o I$ or $caG_c \simeq G_o I$ for some $c \in I$. The main problem is to ensure that $a/E \neq I/E$. For this purpose we adjoin elements x_1, x_2 to one of the E-classes of A and we require that the parity of a over x_1, x_2 differs from the parity of the elements of I over x_1, x_2. Then we amalgamate AIx_1x_2 and Aax_1x_2 over Ax_1x_2. Each of the factors has $n-1$ E-classes and can be taken to satisfy the parity constraint.

Thus if H contains the 3-graphs in $\mathcal{A}(n)$ and satisfies the parity constraint, it is semigeneric.

6 The generic imprimitive 3-graph

We now assume that H embeds the 3-graphs in $\mathcal{A}(\infty)$, listed under $(1, 2)$ at the beginning of the previous section, and in addition:

(3^+) $$H \text{ embeds } O^2 \text{ and } O^3$$

(there is no parity constraint). We want to show that any finite 3-graph on which the type 1 defines an equivalence relation can be embedded into H.

We again follow the line introduced by Lachlan, as we did at the end of §4. We associate to any amalgamation class $\mathcal{A}$ another amalgamation class $\mathcal{A}^*$:

$$\{A \in \mathcal{A} : \text{Any extension } AI \text{ of } A \text{ by one more } E\text{-class } I \text{ belongs to } \mathcal{A}\}$$

and we claim that if $\mathcal{A}$ contains $\mathcal{A}(\infty) \cup \{O^2, O^3\}$, then so does $\mathcal{A}^*$. This then shows that there is only one such amalgamation class, in the same way that Proposition 2 was used to prove Proposition 1 in §4; and applying this to the amalgamation class of induced 3-subgraphs of H, it follows that H is generic.

Assume accordingly, for the remainder of the section, that $\mathcal{A}$ is a fixed amalgamation class containing $\mathcal{A}(\infty) \cup \{O^2, O^3\}$, and that H is the corresponding homogeneous 3-graph. After some preliminaries, we will show in Lemmas 15-17 that $\mathcal{A}(\infty) \cup \{O^2, O^3\} \subseteq \mathcal{A}^*$.

Lemma 13 *Every finite 3-graph with at most two E-classes embeds in H.*

Proof:

We let $G = C_1C_2$ and we may assume that C_2 is indiscernible over C_1, or both classes have order 2. If C_2 is indiscernible over C_1 then Lemma 3 applies: wherever the type of C_2 over C_1 is realized, there are infinitely many realizations.

Accordingly we may suppose that C_1 and C_2 have order 2. There is little difficulty using O^2 and O^3 to force any other configuration of this type into H. Cf. Lemma 12. ∎

Another way of expressing this is: $K^1_n \in \mathcal{A}^*$ for all n.

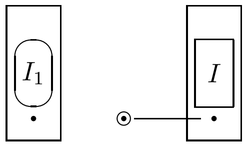
Lemma 14 *Every finite 3-graph with three E-classes, two of which are singletons, embeds in H.*

Proof:

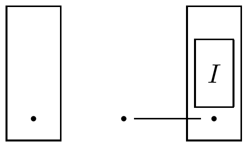
It suffices to show that every 1-type over K^2_2 or K^3_2 is realized infinitely many times in every other class of H. We work with K^2_2. It suffices to show:

1. Some 1-type over K^2_2 is realized in every other class of H;
2. Every 1-type over K^2_2 is realized infinitely many times in some class of H.

If the 1-type over K_2^2 involves the type 2 we can conclude by induction, working in x^2. In this case the type is realized infinitely many times in every class. It reamins to show that $I +^3 K_2^2$ can be embedded in H for every finite E-class I. The following amalgamation suffices (we use edges to represent the type 2):



The factor:



embeds into H by embedding and extending by I. ∎

Lemma 15 *All triangles with exactly one edge of type 1 lie in $\mathcal{A}^*$.*

Proof:

Let the triangle be ca_2a_3 with $\mathrm{tp}(a_2a_3) = 1$. We must embed an extension of the form ca_2a_3I into H, where I is an additional E-class.

Suppose first that $\mathrm{tp}(ca_i) = i$ for $i = 2, 3$. Amalgamate caa_2I with caa_3I over caI where a, a_2, a_3 are E-equivalent, $\mathrm{tp}(ca) = 2$, and $= tp(a/I)$ is determined by amalgamating ca_3I and ca_3a over ca_2. This is possible by the previous lemma. It remains to show that the factor caa_2I embeds in H. This however is an extension of a triangle with one edge of type 1 and the remaining edges of the same type, so it will suffice for the proof of the Lemma to treat this case.

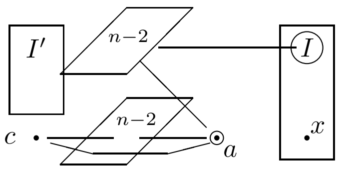
Suppose $\mathrm{tp}(ca_i) = 2$ for $i = 2, 3$. Let I_2, I_3 be two copies of I. Amalgamate $caa_2I_2I_3$ and $caa_3I_2I_3$ over caI_2I_3 where aa_2a_3 and I_2I_3 form two E-classes, $ca_2a_3I_2 \simeq ca_2a_3I$, $\mathrm{tp}(ca) = 3$, and $a_iaI_i \simeq a_2a_3I$. The onlyu difficulty is to force the factors $caa_iI_2I_3$ into H. If we amalgamate $ca_iI_2I_3$ with $aa_iI_2I_3$ over $a_iI_2I_3$, we either get $\mathrm{tp}(ca) = 3$ or we get a copy of ca_2a_3I. The factors $ca_iI_2I_3$ and $aa_iI_2I_3$ are covered by the previous two lemmas. ∎

Lemma 16 *K_n^2 and K_n^3 are in $\mathcal{A}^*$ for all n.*

Proof:

Let $A \subseteq H$ with e.g. $A \simeq K_n^2$. It suffices to show that every 1-type over A has infinitely many realizations in every E-class of $H - A/E$. For a-types involving type 2 this follows by considering x^2.

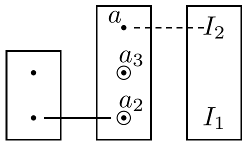
It now suffices to show that the remaining configuration $K^2 +^3 I$ with I a finite E-class can be embedded in H. We use the following amalgamation:



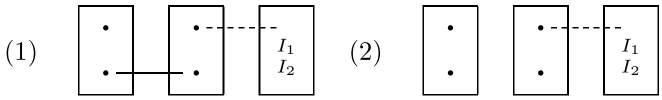
The factor omitting a has n classes, hence embeds in H. The factor omitting I has the form $(x) + A$ where A has n classes, and hence embeds in H. ∎

Lemma 17 $O^2 \in \mathcal{A}^*$

Proof:
We use an amalgamation of the following form:



where the dotted edge indicates that the type of a over $I_1 I_2$ is determined in the course of the construction, namely in the construction of the factor omitting a_3. We exhibit the two factors required:



The factor (2) and the subfactors needed to form (1) are easily obtained on the basis of the information so far. We know that the 3-graph corresponding to $\mathcal{A}^*$ is either generic or semigeneric. In the case of (2), for example, the first two classes satisfy the parity constraint and hence certainly belong to $\mathcal{A}^*$. Thus (2) belongs to $\mathcal{A}$, as claimed. The amalgamation producing (1) uses O^2 and another factor known to be in $\mathcal{A}$. ∎

With this lemma we have completed the verification that $\mathcal{A}^*$ contains $\mathcal{A}(\infty) \cup \{O^2, O^3\}$. As we have explained, it follows by induction that $\mathcal{A}$ contains all 3-graphs on which the relation E given by the 2-type 1 and the trivial 2-type is an equivalence relation; equivalently, H is generic.

This completes the classification of the imprimitive homogeneous 3-graphs.

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