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FORBIDDEN SUBGRAPHS AND FORBIDDEN SUBSTRUCTURES

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Abstract. The problem of the existence of a universal structure omitting a finite set of forbidden substructures is reducible to the corresponding problem in the category of graphs with a vertex coloring by two colors. It is not known whether this problem reduces further to the category of ordinary graphs. It is also not known whether these problems are decidable.

Introduction. There is an extensive graph theoretic literature which deals with the following problem. Consider a class K of graphs (finite or countably infinite) which is defined by the nonembeddability of specified *forbidden* subgraphs (e.g., the classes of triangle-free, or planar, or bipartite graphs). Is there a *universal* graph in K, into which all graphs in K embed? Usually there is no universal graph of the desired type, and so the goal is to isolate and analyze the unusual cases in which there is such a universal graph.

A variety of cases have been considered in which the set of constraints on K consists of a finite set of finite, connected forbidden subgraphs, and results of some reasonable generality have been obtained, such as [FK] for the case of a single 2-connected forbidden subgraph, or [KM,CS] for an arbitrary finite set of forbidden cycles. (See [KP] or [CSS] for further references.) It is worth raising the question explicitly as to whether it is reasonable to seek a complete solution in this case, and since this is a decision problem of a conventional type the question can at least be posed rigorously:

> *Is there an algorithm which determines for each finite set of finite connected "forbidden" graphs, whether the corresponding* universal graph *exists, for the class of graphs omitting the specified ones?*

However there are two slightly different questions here, because there are two notions of universality which have been considered: we may require embeddings of graphs belonging to the class K either as *subgraphs* or as *induced subgraphs* of the universal graph. Both variants are reasonable and the analysis tends to be remarkably similar in either case, though the latter variant is somewhat easier to deal with.

Work to date could be interpreted as the search for a clear-cut criterion that would make it evident that this problem is decidable. We are motivated here by the feeling that the problem may well be undecidable, which would put the results to date in a

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different light. There is a natural approach to a proof of undecidability. In the first place, the problem of the existence of universal graphs can be generalized to any class of combinatorial structures, and in that form appears to have some affinity with such problems as Wang's *domino problems*, which are shown to be undecidable by a direct encoding of the halting problem [KMW]. So the following program comes into consideration:

(A) Prove the undecidability of the generalized universal graph problem, preferably by some variant of the standard encoding methods.
(B) Reduce the generalized form of the problem to the original case involving graphs.

Unexpected difficulties arise, and we only managed to carry out a variant of (B), in which the generalized problem is reduced to the corresponding problem for graphs *with a vertex coloring*; it suffices to allow two colors to get a problem equivalent to the most general form.

On the other hand, in the proof of this result, we are able to show that the two different notions of universality lead to problems of the same complexity (for the general case, or the case of graphs with a vertex coloring).

Thus Problem (A) remains entirely open at this point, and Problem (B) amounts to the question whether the problem of the existence of universal graphs with forbidden subgraphs in the presence of a vertex coloring by two colors is harder than the corresponding question for uncolored graphs.

Concerning Problem (A), we would strongly suggest that the variant in which the universal graph is required to be $\aleph_0$-*categorical* is well worth considering. In the first place, known universal graphs tend to be either $\aleph_0$-categorical, or of the degenerate type illustrated by the case of regular graphs of degree 2. Secondly, the combinatorial content of this variant can be analyzed explicitly and this analysis can be turned to account in practice, as shown in [CSS]. Indeed it is this version which exhibits the closest affinity to a domino problem. For an elucidation of these remarks the reader is referred to [CSS]. We believe that this case goes to the heart of the matter.

We will give more precise statements of the results in §2, after laying out the terminology appropriate to the most general case of our decision problem in §1. The remaining three sections give the proof of the main theorem.

§1. Terminology. Since we deal with combinatorial structures of a general type, we use the language of model theory. We fix a *finite relational language* L; we allow L to impose (a)symmetry and (ir)reflexivity conditions on the relations as well. Thus by varying L we may consider the classes of graphs, tournaments, directed graphs, and the variants involving vertex colorings or edge colorings, as well as hypergraphs and the like. Equality is viewed as a logical relation rather than a structural relation, and is incorporated into the language L, though on one occasion we will treat it as a relation on much the same footing as the others. We can consider forbidden substructures and universal structures at this level of generality, as we will now make explicit. All structures considered will be either finite or countably infinite from this point on.

DEFINITION 1. Let $\mathscr{M}$ be a relational structure.

1. $\tilde{\mathscr{M}}$ is the graph whose vertices are the elements of $\mathscr{M}$, with two elements a, b adjacent if they are involved in some relation holding in $\mathscr{M}$: that is, we require that $R(c_1, \dots, c_k)$ should hold for some sequence $\bar{c}$ containing a and b.
2. $\mathscr{M}$ is *connected* if $\tilde{\mathscr{M}}$ is.

DEFINITION 2. Let $\mathscr{M}$ and $\mathscr{N}$ be relational structures of the same type, with underlying sets M and N, and suppose $M \subseteq N$.

1. $\mathscr{M}$ is a *weak substructure* of $\mathscr{N}$ if the inclusion map is a homomorphism (every relation holding in $\mathscr{M}$ holds in $\mathscr{N}$).
2. $\mathscr{M}$ is a *strong substructure* of $\mathscr{N}$ if the inclusion map is an isomorphism (every relation holding in $\mathscr{M}$ holds in $\mathscr{N}$, and conversely).

This is a deviation from standard model theoretic terminology, to avoid a conflict with established graph theoretic terminology.

DEFINITION 3. Let L be a finite relational language, $\mathscr{C}$ a set of L-structures, K a class of L-structures, and $\mathscr{M}$ an L-structure.

1. $\mathscr{M}$ is *$\mathscr{C}$-free* if no weak substructure of $\mathscr{M}$ is isomorphic with a structure in $\mathscr{C}$.
2. $\mathscr{M}$ is weakly (resp. strongly) *universal* for K if every structure in K is isomorphic with a weak (resp. strong) substructure of $\mathscr{M}$.
3. A weakly (resp. strongly) universal $\mathscr{C}$-free structure is a $\mathscr{C}$-free structure which is universal for the class of $\mathscr{C}$-free structures.

A remark on connectivity: If $\mathscr{C}$ is a class of connected structures then the class of $\mathscr{C}$-free structures is closed under the formation of disjoint unions (in the natural sense), and it is only under the latter assumption that it is natural to look for a universal structure. Komjáth and Pach point out in [KP] that the question can be rephrased so as to make sense more generally, without changing its meaning in the case of connected constraints. We could easily follow their lead here, with some adjustments in terminology.

§2. Statement of results. We show that the following decision problems are equivalent:

I. For L a finite relational language and $\mathscr{C}$ a finite set of finite L-structures, determine whether there is a weakly universal $\mathscr{C}$-free L-structure.

II. For L a finite relational language and $\mathscr{C}$ a finite set of finite L-structures, determine whether there is a strongly universal $\mathscr{C}$-free L-structure.

III. For $\mathscr{C}$ a finite set of finite $\{0, 1\}$-colored graphs, determine whether there is a weakly universal $\mathscr{C}$-free $\{0, 1\}$-colored graph.

IV. For $\mathscr{C}$ a finite set of finite $\{0, 1\}$-colored graphs, determine whether there is a strongly universal $\mathscr{C}$-free $\{0, 1\}$-colored graph.

Furthermore, if in each case we require the elements of $\mathscr{C}$ to be *connected* (as is natural), then the four restricted problems are also mutually equivalent.

A $\{0, 1\}$-*colored graph* is taken to be a graph with a partition of its vertex set V into two disjoint sets V_0 and V_1, or equivalently, a function $f : V \to \{0, 1\}$. Thus the set of colors is held fixed.

The method of proof involves two reductions:

1. reduction of II to I.
2. reduction of I to both III and IV (by a single construction).

This shows that all four problems are equivalent.

It is an open question whether we can reduce further, from $\{0, 1\}$-colored graphs to ordinary graphs. The problem would be to find an analog of our §6 for this case.

The fact that the two problems III and IV are equally difficult, at the level of graphs with a vertex coloring, confirms what experience would suggest in the case of ordinary graphs as well. As yet however that equivalence has not been proved for ordinary graphs.

§3. The first reduction.

THEOREM 1. *There is an effective procedure which associates to a finite relational language L and a finite set $\mathscr{C}$ of finite L-structures, another finite relational language L^*, and a finite set $\mathscr{C}^*$ of finite L^*-structures so that the following are equivalent*:

1. There is a strongly universal $\mathscr{C}$-free L-structure.
2. There is a weakly universal $\mathscr{C}^*$-free L^*-structure.

Furthermore, if the structures in $\mathscr{C}$ are connected then the structures in $\mathscr{C}^*$ are connected.

PROOF. Given L and $\mathscr{C}$ we perform the following construction.

1. $L^* = L \cup \{R' : R \in L\}$ where for each relation symbol R in L, R' is a new relation symbol with the same number of places whose preferred interpretation in a model is the complement of R. We will not be in a position to require this interpretation, however.

2. For R a relation symbol with n places in L, let $\mathscr{C}^*_R$ be the set of L^*-structures which satisfy the condition:

$$\exists x_1 \dots \exists x_n R(\bar{x}) \wedge R'(\bar{x})$$

and which are minimal with this property. Note that such a structure has cardinality at most n, and is connected.

3. $\mathscr{C}^* = \bigcup\{\mathscr{C}^*_R : R \in L\} \cup \mathscr{C}$.

The *canonical expansion* of an L-structure $\mathscr{M}$ to an L^*-structure $\mathscr{M}^*$ is formed by interpreting R' in $\mathscr{M}^*$ as the complement of R. ⊣

LEMMA 1. If $\mathscr{M}$ is a strongly universal $\mathscr{C}$-free L-structure, then $\mathscr{M}^*$ is a weakly universal $\mathscr{C}^*$-free L^*-structure.

PROOF. Clearly $\mathscr{M}^*$ is $\mathscr{C}^*$-free. Let $\mathscr{N}^*$ be any $\mathscr{C}^*$-free L^*-structure, and set $\mathscr{N} = \mathscr{N}^* \restriction L$. Then $\mathscr{N}$ is $\mathscr{C}$-free and hence there is a strong embedding $f : \mathscr{N} \to \mathscr{M}$. Viewed as a map from $\mathscr{N}^*$ into $\mathscr{M}^*$, this gives a weak embedding of L^*-structures. ⊣

LEMMA 2. If $\mathscr{M}^*$ is a weakly universal $\mathscr{C}^*$-free L^*-structure, then $\mathscr{M}^* \restriction L$ is a strongly universal $\mathscr{C}$-free L-structure.

PROOF. Let $\mathscr{M} = \mathscr{M}^* \restriction L$. Clearly this is $\mathscr{C}$-free. Now let $\mathscr{N}$ be a $\mathscr{C}$-free L-structure. Then $\mathscr{N}^*$ is a $\mathscr{C}^*$-free L^*-structure and by assumption there is a weak embedding $f : \mathscr{N}^* \to \mathscr{M}^*$. By the construction of $\mathscr{N}^*$, this amounts to a strong embedding of $\mathscr{N}$ into $\mathscr{M}$, as required. ⊣

Theorem 1 follows. Thus problem II is reducible to problem I.

§4. **The second reduction.** Let L_0 be the language of $\{0,1\}$-colored graphs, that is, graphs equipped with a coloring of the vertices by two fixed colors 0 and 1. We wish to show:

THEOREM 2. *There is an effective procedure which associates to any finite relational language L and finite set $\mathscr{C}$ of finite L-structures, a finite set $\mathscr{C}_0$ of finite $\{0,1\}$-colored graphs, so that the following are equivalent:*

1. *There is a weakly universal $\mathscr{C}$-free L-structure.*
2. *There is a weakly universal $\mathscr{C}_0$-free $\{0,1\}$-colored graph.*
3. *There is a strongly universal $\mathscr{C}_0$-free $\{0,1\}$-colored graph.*

Furthermore, if the structures in $\mathscr{C}$ are connected, then the $\{0,1\}$-colored graphs in $\mathscr{C}_0$ are connected.

Construction. We begin by defining certain $\{0,1\}$-colored graphs W_π^R which will be called *witness graphs*.

Let I be the set of triples (R,π,i) for which:
(i) R is a relation symbol of L, with n places,
(ii) π is a partition of the set $\{1,\dots,n\}$, and
(iii) $1 \le i \le n$.

Let $N = |I|$ and let $\rho\colon I \to \mathbb{N}$ be $1-1$ and satisfy: $\rho(i) > 1$ for all $i \in I$; for all $i_1, i_2, i_3 \in I$ we have $\rho(i_1) + \rho(i_2) \neq \rho(i_3) + 1$

For $d_1,\dots,d_k \ge 1$, a *star* of type $(d_1,\dots,d_k)$ is a $\{0,1\}$-colored graph formed by taking $k+1$ paths $P_0,\dots,P_k$, with P_i of length d_i, and identifying their initial vertices, then coloring the resulting graph as follows: the terminal vertex of each path is given color 0, and all other vertices have color 1. Thus a star is, in particular, a tree with a unique vertex of degree greater than 2, called the *center*; and the vertices of color 0 are the leaves of this tree. The $k+1$ leaves of the star will be denoted $a_1,\dots,a_k$, in order.

The *witness graph* W_R^π associated with a relation $R \in L$ having n places, and a partition π of $\{1,\dots,n\}$ with $|\pi|$ classes, is the star of type

$$(\rho(R,\pi,i_1),\dots,\rho(R,\pi,i_{|\pi|}),1,1)$$

where $i_j = min\{\pi_j\}$, $(\pi_j)_{j\le|\pi|}$ being an enumeration of the classes in π. In general the final entries 1, 1 are not needed, but we wish to ensure that $k \ge 2$ in all cases. Intuitively, W_R^π represents the assertion that $R(x_1,\dots,x_n)$ holds, where for $1 \le i \le n$, if $i \in \pi_j$ and $i^* = min\{\pi_j\}$, then x_i represents a_{i^*}.

Using these witness graphs, our construction proceeds as follows. We give a procedure which associates to an L-structure $\mathscr{M}$, a $\{0,1\}$-colored graph $G_{\mathscr{M}}$. We then list some $\{0,1\}$-colored graphs which are omitted by $G_{\mathscr{M}}$, if $\mathscr{M}$ omits $\mathscr{C}$. This gives us a candidate for the class $\mathscr{C}_0$. The proof that the class $\mathscr{C}_0$ has the required properties involves considerable checking.

For $\mathscr{M}$ an L-structure, the construction of $G_{\mathscr{M}}$ is as follows. Let V_0 be the underlying set of $\mathscr{M}$, i.e., the universe of the structure, $|\mathscr{M}|$. View V_0 as an *independent set* of vertices of color 0. For each relation $R(b_1,\dots,b_n)$ holding in $\mathscr{M}$, including the equality relation, consider the partition π of $\{1,\dots,n\}$ consulting of the equivalence classes for the relation

$$i \sim j \text{ iff } b_i = b_j$$

and attach the witness graph W_R^π freely to V_0, identifying its leaves $a_1, \dots, a_{|\pi|}$ with the given $b_i \in V_0$. Note that two of the leaves of W_R^π are not identified with elements of V_0.

Construction of $\mathscr{C}_0$. $\mathscr{C}_0$ will consist of certain finite $\{0,1\}$-colored graphs which in fact cannot embed in $G_{\mathscr{M}}$. We must restrict ourselves to a finite number of these.

NOTATION. Let G be a $\{0,1\}$-colored graph, where $V(G) = V_0(G) \cup V_1(G)$ gives the coloring.

For $v \in V_1(G)$, let C_v be the connected component of v in the graph $G \restriction V_1(G)$, and set

$$C_v^+ = C_v \cup \{\text{the immediate neighbors of } C_v\}$$

We now list some properties of $G_{\mathscr{M}}$ which follow at once from our construction. These properties can all be expressed by the omission of some finite set of $\{0,1\}$-colored graphs, and these forbidden graphs will form the set $\mathscr{C}_0$.

1. For any $v \in V_1$, C_v^+ always embeds in one of the witness graphs W_R^π.

This property of $G_{\mathscr{M}}$ can be expressed by saying that $G_{\mathscr{M}}$ is $\mathscr{C}_0^{(1)}$-free, where $\mathscr{C}_0^{(1)}$ consists of the minimal graphs which violate this condition. Observe that these minimal counterexamples are of bounded size, and hence $\mathscr{C}_0^{(1)}$ is finite (with the usual proviso that $\mathscr{C}_0^{(1)}$ contains only one representation of each isomorphism type).

2. For each R, π, and $v_1, \dots, v_{|\pi|}$ of color 0, there is at most one copy of the witness graph W_R^π attached to $\bar{v}$.

Again, this can be expressed as: $G_{\mathscr{M}}$ is $\mathscr{C}_0^{(2)}$-free, for a suitable finite set $\mathscr{C}_0^{(2)}$ of $\{0,1\}$-colored graphs.

3. V_0 is an independent set.

Here $\mathscr{C}_0^{(3)}$ should consist of a single edge, with endpoints of color 0.

4. (a) If a vertex of color 0 lies at distance 1 from a vertex of degree at least 3, then it has degree 1.
   (b) If two vertices of color 0 are connected by a path of vertices of color 1, of total length $\rho(R, \pi, i) + 1$, then at least one of the two vertices has degree 1.
   (c) If two vertices of color 0 are connected by a path of length 2, then each has degree 1.

This corresponds to a finite set of constraints $\mathscr{C}_0^{(4)}$.

5. $\mathscr{C}_0^{(5)} = \{G_A : A \in \mathscr{C}\}$.

Let $\mathscr{C}_0 = \mathscr{C}_0^{(1)} \cup \mathscr{C}_0^{(2)} \cup \mathscr{C}_0^{(3)} \cup \mathscr{C}_0^{(4)} \cup \mathscr{C}_0^{(5)}$. By our construction it is clear that if $\mathscr{M}$ is $\mathscr{C}$-free, then $G_{\mathscr{M}}$ is $\mathscr{C}_0$-free.

With this choice, we claim that Theorem 2 holds. We require $2 \Longrightarrow 1 \Longrightarrow 3$. The proof of this claim occupies the next two sections.

§5. Theorem 2, $2 \Longrightarrow 1$. The notation was established in the preceding section.

Given a weakly universal $\mathscr{C}_0$-free $\{0,1\}$-colored graph G, we wish to extract from it a weakly universal $\mathscr{C}$-free L-structure $\mathscr{M}(G)$.

CONSTRUCTION. Let G be a $\{0,1\}$-colored graph. Set

$M(G) = \{v \in V_0(G)$: v does not lie at distance 1 from any vertex of degree at least 3, and is not at distance $\rho(R,\pi,i)+1$ from any vertex of color 0 and degree at least 2 along any path whose internal vertices have color 1, for any $R,\pi,i\}$.

We impose on the set $M = M(G)$ an L-structure $\mathscr{M}$ as follows. For $R \in L$ an n-place relation (other than equality), π a partition of $\{1,\dots,n\}$, $\bar{v} = (v_1,\dots,v_{|\pi|})$ a sequence of elements of M, and $x_1,\dots,x_n$ an enumeration of $\bar{v}$ with repetitions corresponding to π, we take $R(x_1,\dots,x_n)$ to hold in $\mathscr{M}$ if and only if the colored witness graph W_R^π embeds into G over $\bar{v}$.

The resulting structure is called $\mathscr{M}(G)$. Our claim is

PROPOSITION 1. If G is a weakly universal $\mathscr{C}_0$-free $\{0,1\}$-colored graph, then $\mathscr{M}(G)$ is a weakly universal $\mathscr{C}$-free L-structure.

LEMMA 5.1. Let G be a $\mathscr{C}_0$-free $\{0,1\}$-colored graph. Then $G_{\mathscr{M}(G)}$ embeds canonically into G over $M(G)$.

PROOF. $G_{\mathscr{M}(G)}$ is obtained from $M(G)$ by freely attaching appropriate witness graphs W_R^π over subsets of $M(G)$. We claim that G, being $\mathscr{C}_0$-free, has the following properties:

1. For any $R \in L$, an n-place relation symbol, any partition π of $\{1,\dots,n\}$, and any sequence $\bar{v} = (v_1,\dots,v_{|\pi|})$ in $M(G)$, there is at most one copy of the witness graph W_R^π attached to $v_1,\dots,v_{|\pi|}$, and if there is one then it meets $M(G)$ only in $\bar{v}$.

2. For any two distinct witness graphs W_1, W_2 attached to sequences $\bar{v}^{(1)}, \bar{v}^{(2)}$ in $M(G)$, the attachment is free over $M(G)$. That is, $V(W_1) \cap V(W_2)$ is the intersection of $\bar{v}^{(1)}$ and $\bar{v}^{(2)}$ (as sets of vertices).

The uniqueness claim in (1) is not critical, but serves to clarify the picture. In any case it is taken care of by consideration of $\mathscr{C}_0^{(2)}$.

To complete the proof of (1), we argue that any copy of W_R^π attached to $v_1,\dots,v_{|\pi|}$ meets $M(G)$ only in $\bar{v}$. This is taken care of for the most part by the coloring. The two "extra" leaves of W_R^π cannot be in $M(G)$, as they are adjacent to vertices of degree at least 3.

Now (2) could be ensured by imposing an additional requirement on $\mathscr{C}_0$, and this is entirely reasonable. However we claim that this is unnecessary. Suppose that $W_1 = W_{R_1}^{\pi_1}$ and $W_2 = W_{R_2}^{\pi_2}$ are attached to $\bar{v}^{(1)}, \bar{v}^{(2)}$ respectively. These are stars. As G is $\mathscr{C}_0^{(4)}$-free, if W_1 meets W_2 in a vertex outside the intersection of $\bar{v}^{(1)}$ and $\bar{v}^{(2)}$, then they meet in a vertex v of color 1. In this case, C_v^+ embeds in some W_R^π. In particular $W_1 \cup W_2$ is a star, with a unique center c. Any path of length greater than 1 beginning at c and terminating at a leaf is of length $\rho(R,\pi,i)$ for some unique triple (R,π,i); and R,π are independent of the path chosen. So $R_1 = R_2 = R$, $\pi_1 = \pi_2 = \pi$, and the degree of c is at most $|\pi|+2$, it follows easily that $W_1 = W_2$. ⊣

COROLLARY. If G is a $\mathscr{C}_0$-free $\{0,1\}$-colored graph, then $\mathscr{M}(G)$ is $\mathscr{C}$-free.

PROOF. As $G_{\mathscr{M}(G)}$ embeds in G, it is $\mathscr{C}_0^{(5)}$-free. ⊣

LEMMA 5.2. For any L-structure $\mathscr{M}_0$, $\mathscr{M}(G_{\mathscr{M}_0}) = \mathscr{M}_0$.

PROOF. Evidently $M(G_{\mathscr{M}_0})$ is the underlying set of $\mathscr{M}_0$, and it should be clear that the construction of $G_{\mathscr{M}_0}$ creates no unintended witnesses. ⊣

Now the proof of Proposition 1 will be reduced to one additional lemma. Let G be a weakly universal $\mathscr{C}_0$-free $\{0, 1\}$-colored graph. Our claim is that $\mathscr{M}(G)$ is a weakly universal $\mathscr{C}$-free L-structure. Let $\mathscr{M}_0$ be a $\mathscr{C}$-free L-structure. Then $G_{\mathscr{M}_0}$ is a $\mathscr{C}_0$-free $\{0, 1\}$-colored graph. Fix a weak embedding $f : G_{\mathscr{M}_0} \to G$. We wish to derive from this a weak embedding of $\mathscr{M}_0$ into $\mathscr{M}(G)$. Thus the following suffices in view of Lemma 5.2.

LEMMA 5.3. Let G be a $\mathscr{C}_0$-free $\{0, 1\}$-colored graph, $\mathscr{M}_0$ an L-structure, and $f : G_{\mathscr{M}_0} \to G$ a weak embedding. Then

1. $f[M(G_{\mathscr{M}_0})] \subseteq M(G)$.
2. As a map from $\mathscr{M}(G_{\mathscr{M}_0})$ into $\mathscr{M}(G)$, f is a weak embedding.

PROOF. 1. In $G_{\mathscr{M}_0}$, every vertex in the underlying set of $\mathscr{M}_0$ has a witness to "$v = v$" attached to it. Taking into account that G is $\mathscr{C}_0^{(4)}$-free, our claim follows easily.

2. Given (1), this is immediate. ⊣

§6. **Theorem 3,** $1 \implies 3$. Given a weakly universal $\mathscr{C}$-free L-structure $\mathscr{M}$, we wish to construct a strongly universal $\mathscr{C}_0$-free $\{0, 1\}$-colored graph. This graph will be a certain extension of $G_{\mathscr{M}}$ which we will call $G^*_{\mathscr{M}}$.

CONSTRUCTION. Let G be a $\{0, 1\}$-colored graph. A *partial* witness is any *proper* $\{0, 1\}$-colored subgraph of a $\{0, 1\}$-colored graph of the form W^{π}_R.

The graph G^* is constructed from G by attaching, freely, infinitely many copies of all partial witnesses in all possible ways. More precisely, for each sequence $\bar{v}$ from $M(G)$, and for each partial witness W of an appropriate form, attach infinitely many copies of W to $\bar{v}$, freely. To determine whether W is appropriate requires that W is given explicitly as a subgraph of some W^{π}_R. In this case, all but two of the leaves of W^{π}_R represent potential elements of $M(G)$. We require that the number of such leaves which lie in W should equal the length of $\bar{v}$, so that the attachment process is sensible. Note that possibly $\bar{v}$ is empty and we take a disjoint union in this case.

PROPOSITION 2. If $\mathscr{M}$ is a weakly universal $\mathscr{C}$-free L-structure, then $G^*_{\mathscr{M}}$ is a strongly universal $\mathscr{C}_0$-free $\{0, 1\}$-colored graph.

LEMMA 6.1. If $\mathscr{M}$ is $\mathscr{C}$-free then $G^*_{\mathscr{M}}$ is $\mathscr{C}_0$-free.

PROOF. Let $G = G_{\mathscr{M}}$. We use only the following facts:
(1) G is $\mathscr{C}_0$-free.
(2) No connected component of G is a path.

Now the only constraints that present any difficulties are those in $\mathscr{C}_0^{(4)}$, and here our primary concern is with one constraint: if two centers v, w of color 0 are connected by a path of vertices of color 1, with the total path length of the form $\rho(R, \pi, i) + 1$, then at least one of the vertices has degree 1.

In the case of $G_{\mathscr{M}}$ this is rather evident. We argue as follows in the slightly more general case in which only conditions $(1, 2)$ are assumed. Firstly, all new vertices of color 0 added in passing from G to G^* will have degree 1 and therefore pose no problems. Secondly, between pairs of vertices of color 0 in G, no new paths of the relevant lengths will be created. Thus we need only concern ourselves with the following case: v, w are vertices of color 0 in G, and with a path of the specified type connecting them.

If v has degree greater than 1, then w has degree 1 and lies outside $M(G)$. Hence in G^* the degree of w remains 1.

Thus we need only consider the case in which both v and w have degree 1, and lie in $M(G)$. In such a case G^* would indeed violate our constraint. However we claim that in this case the connected component of v in G would reduce to a single path connecting v to w. Let P be the given path connecting v to w. If P is not the whole of this connected component, then it contains a vertex u of degree at least 3. Then C_u^+ should embed in a witness graph W_R^π. The structure of these witness graphs forces u to be adjacent to v and w. But then correspondingly v or w lies outside $M(G)$, a contradiction. ⊣

For the proof of Proposition 2 we take $\mathscr{M}$ a weakly universal $\mathscr{C}$-free L-structure and we consider $G^*_{\mathscr{M}}$. Take G a $\mathscr{C}_0$-free $\{0, 1\}$-colored graph. We wish to show that G embeds strongly in $G^*_{\mathscr{M}}$. We may take G existentially complete in the class of $\mathscr{C}_0$-free $\{0, 1\}$-colored graphs. By assumption we have a weak embedding $f : \mathscr{M}(G) \to \mathscr{M}$. To conclude we prove the following three lemmas.

Lemma 6.2. Let $f : \mathscr{M}_1 \to \mathscr{M}_2$ be a weak embedding of L-structures. Then f extends canonically to a strong embedding $f^* : G_{\mathscr{M}_1} \to G_{\mathscr{M}_2}$ of $\{0, 1\}$-colored graphs.

Proof. Immediate. ⊣

Lemma 6.3. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be L-structures and let $f : G_{\mathscr{M}_1} \to G_{\mathscr{M}_2}$ be a strong embedding of $\{0, 1\}$-colored graphs. Then f extends to a strong embedding of $G^*_{\mathscr{M}_1}$ into $G^*_{\mathscr{M}_2}$.

Proof. It suffices to show that $f[M(G_{\mathscr{M}_1})] \subseteq M(G_{\mathscr{M}_2})$. This is rather straightforward given that equality is taken as a relation of R and therefore every $v \in M(G_{\mathscr{M}_1})$ has at least one witness graph attached. ⊣

Lemma 6.4. Let G be existentially closed in the class of $\mathscr{C}_0$-free $\{0, 1\}$-colored graphs. Then G embeds strongly in $G_{\mathscr{M}(G)}$.

Proof. We showed in Lemma 5.1 that $G_{\mathscr{M}(G)}$ embeds canonically in G over $M(G)$. We claim first that this is a strong embedding. For $v \in V(G_{\mathscr{M}(G)})$ we consider C_v and C_v^+ as defined in §2; these are to be computed in $G_{\mathscr{M}(G)}$. We claim that $G \restriction V(C_v^+) = C_v^+$ and that the edges of $G \restriction V(C_v^+) \cup V(C_w^+)$ are internal to C_v^+ or C_w^+. In both cases our claim follows by considering C_v^+ as computed in G, and recalling the significance of $\mathscr{C}_0^{(1)}$, which forces the result to be the same whether the computation is made in G or in $G_{\mathscr{M}(G)}$. Since there are no edges between vertices of color 0, it follows that the embedding of $G_{\mathscr{M}(G)}$ in G is strong. So we will view $G_{\mathscr{M}(G)}$ as an induced subgraph of G.

Now let $V' = V(G) \setminus V(\mathscr{M}(G))$ and $V_i' = V' \cap V_i(G)$ for $i = 0, 1$. For $v \in V_1'$ we consider C_v^+ as computed in G. We make the following claims.

(1) $V' = \bigcup\{V(C_v^+) \colon v \in V_1'\}$.
(2) C_v^+ is a partial witness for $v \in V_1'$.
(3) $C_v^+ \restriction [V' \cap V(C_v^+)]$ is the connected component of v in $G \restriction V'$ for $v \in V_1'$.
(4) $V(C_v^+) \cap V(G_{\mathscr{M}(G)}) \subseteq M(G)$ and this intersection consists of all vertices of C_v^+ which are intended to represent vertices of $\mathscr{M}(G)$, relative to some embedding of C_v^+ in a witness graph W_R^π, for $v \in V_1'$.

Note that these claims suffice to complete the proof, since jointly they mean that the various C_v^+ are partial witnesses which are freely attached to appropriate subsets of $M(G)$, and which cover both the vertices and edges of G not in $G \restriction V(G_{\mathscr{M}(G)})$.

1. $V' = \bigcup\{V(C_v^+) \colon v \in V_1'\}$.

The claim here is that each vertex $v \in V_0'$ is adjacent to a vertex in V_1'. Now if v is adjacent to some vertex w, then $w \in V_1(G)$ and easily $w \notin V_1(G_{\mathscr{M}(G)})$, as otherwise $v \in V(C_w^+) \subseteq V(G_{\mathscr{M}(G)})$. So it suffices to show that v is not isolated in G. However this is immediate by existential completeness: if v were isolated, then the extension G' of G in which v is given one neighbor of color 1 would again be $\mathscr{C}_0$-free, and this contradicts existential completeness.

2. C_v^+ is a partial witness for $v \in V_1'$.

In view of the nature of $\mathscr{C}_0^{(1)}$, C_v^+ embeds in a witness graph W_R^π. If $C_v^+ \simeq W_R^\pi$ then tracing through the definition it is easy to see that $C_v^+ \subseteq G_{\mathscr{M}(G)}$, using the constraint set $\mathscr{C}_0^{(2)}$ as well. This contradicts the choice of v, so C_v^+ is a partial witness.

3. For $v \in V_1'$, $C_v^+ \restriction V' \cup V(C_v^+)$ is the connected component of v in $G \restriction V'$.

One may check easily that for $v \in V_1(G_{\mathscr{M}(G)})$, all immediate neighbors of v lie in $V(G_{\mathscr{M}(G)})$, using the constraint set $\mathscr{C}_0^{(1)}$. Thus for $v \in V_1'$ we have $V(C_v) \subseteq V_1'$. It follows that $V' \cap V(C_v^+)$ is contained in the connected component of v in $G \restriction V'$.

If $V' \cap V(C_v^+)$ is not the whole of the connected component of v in $G \restriction V'$, then there are adjacent vertices $w_1 \in V' \cap V(C_v^+)$, $w_2 \in V'(C_v^+)$. If $w_1 \in V_1'$ then $w_1 \in C_v$ and hence $w_2 \in C_v^+$. Accordingly we must have $w_1 \in V_0'$. As $w_1 \notin M(G)$, we have one of the following:

(i) w_1 is adjacent to a vertex of degree 3.
(ii) w_1 is at distance $\rho(R, \pi, i) + 1$ from a vertex of color 0, and degree at least 2, along a path whose internal vertices have color 1, for some R, π, i.

However in both cases w_1 has degree 1 and is adjacent to a vertex in C_v, so $w_2 \in V(C_v)$.

4. For $v \in V_1'$, we have $V(C_v^+) \cap V(G_{\mathscr{M}(G)}) \subseteq M(G)$, and this intersection consists of all vertices of C_v^+ which are intended to represent vertices of $\mathscr{M}(G)$, relative to some embedding of C_v^+ into a witness graph W_R^π.

The first of these two points is straightforward. $V_1(C_v^+) = V_1(C_v)$ and this is disjoint from $V(G_{\mathscr{M}(G)})$, as already noted. So $V(C_v^+) \cap V(G_{\mathscr{M}(G)}) \subseteq V_0(G_{\mathscr{M}(G)})$. If $v \in V_0(C_v^+) \cap V_0(G_{\mathscr{M}(G)}) \setminus M(G)$ then as in the proof of (3) v has degree 1 and hence has a neighbor in $V_1(C_v) \cap V(G_{\mathscr{M}(G)})$, a contradiction.

For the second point, we first must make the content of the assertion more explicit. C_v^+ is a connected partial witness, and has at most one vertex of degree greater than

2, its center. When there is such a vertex then let S consist of the vertices of color 0 at distance greater than 1 from the center. Our claim in this case is

$$\text{(A)} \qquad V(C_v^+) \cap V(G_{\mathscr{M}(G)}) = S.$$

This is straightforward in view of the constraints on G.

There remains the case in which C_v^+ consists of a path through v whose vertices of color 0 (if there are any) appear at the ends. Here one must consider all possible embeddings into a witness graph. The critical case is that in which C_v^+ consists of a path with both endpoints of color 0, whose length is of the form $\rho(R, \pi, i) + 1$ for some R, π, i. In this case our claim becomes:

$$\text{(B)} \qquad |V(C_v^+) \cap V(G_{\mathscr{M}(G)})| = 1.$$

We will show that in G exactly one of the endpoints of C_v^* has degree 1. In particular this endpoint is not in $M(G)$, and the other one is in $M(G)$, by inspection of the definitions and the constraints in G. So this will suffice for the proof of (B).

One of the constraints on G states that at least one of the endpoints of C_v^+ has degree 1. We must eliminate the possibility that both endpoints have degree 1. Let w be one of these endpoints, and form G' by giving w a second neighbor of color 1. Then G' is again $\mathscr{C}_0$-free (the main concern here is with $\mathscr{C}_0^{(4)}$) and then the embedding of G into G' contradicts existential completeness. Our claim follows. ⊣

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