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# On groups of finite Morley rank of even type

Tuna Altınel [a,*] and Gregory Cherlin [b,1]

[a] *Institut Girard Desargues, Université Claude Bernard Lyon-1, Domaine de la Doua, Bâtiment Braconnier, 21, avenue Claude Bernard, 69622 Villeurbanne cedex, France*

[b] *Department of Mathematics, Rutgers University, Hill Center, Piscataway, NJ 08854, USA*

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## 1. Introduction

The Cherlin–Zilber algebraicity conjecture states that infinite simple groups of finite Morley rank are isomorphic to algebraic groups over algebraically closed fields. On the basis of the 2-Sylow theory it is possible to divide these groups *a priori* into four classes, said to be of even, odd, degenerate, and mixed type, where even and odd types should be algebraic groups over fields of characteristic respectively equal to 2 or unequal to 2, and degenerate and mixed type groups conjecturally do not exist. It has been shown that a *minimal counterexample* to the algebraicity conjecture (more formally, a nonalgebraic simple $K^*$-group) cannot be of mixed type, and the more elaborate classification of simple $K^*$-groups of even type appears now to be approaching completion. This raises the question whether we are in a position to verify the algebraicity conjecture for *all* simple groups of finite Morley rank of even type.

The study of $K^*$-groups follows an *inductive* approach analogous to the *revisionist* approach of finite group theory. Consequently, the successful completion of the $K^*$-classification project in the even type case can produce at best the following result: a simple group of finite Morley rank of even type *with no simple definable section of degenerate type* is algebraic. Given the real possibility that nonalgebraic simple groups of finite Morley rank of degenerate type may exist, we look for an approach which profits from the lessons learned to date in the study of $K^*$ groups of even type, but which, if successful, could yield the full algebraicity conjecture in the even type case. The present paper is devoted to the presentation of an appropriate inductive framework, which we call $L^*$-groups, and some results which provide grounds for cautious optimism in this regard. The following theorem

* Corresponding author.
*E-mail addresses:* altinel@desargues.univ-lyon1.fr (T. Altınel), cherlin@math.rutgers.edu (G. Cherlin).
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is an analog of a result proved in the early stages of the analysis of simple $K^*$-groups of finite Morley rank of even type [1,4].

**Theorem 1.** *Let $G$ be a simple $L^*$-group of even type with a weakly embedded subgroup $M$. Then $M^\circ/O_2^\circ(M^\circ)$ is of degenerate type.*

In the $K^*$-case, the corresponding result is that $M^\circ$ is solvable. We shall explain the connection between these results more fully toward the end of Section 2.

As a second application of the $L^*$-theory we show that the treatment of the mixed type case can be reduced to the even type case.

**Theorem 2.** *Let $G$ be a simple group of finite Morley rank, all of whose proper definable infinite simple sections of even type are algebraic. Then $G$ is not of mixed type.*

**Corollary 2.1**. *If all simple groups of finite Morley rank of even type are algebraic, then there is no simple group of finite Morley rank and of mixed type.*

It was known from the $K^*$ theory that the elimination of mixed type groups would follow from the algebraicity conjecture for both even and odd types [18]. The characteristic feature of our approach consists of methods for dealing with degenerate and odd type sections whose structure is left completely arbitrary. We intend to pursue the matter in [7].

This project was initiated in [2]. There are two simple principles which are essential to our present enterprise. The first of these is largely responsible for the initiation of the project in [2], where it occurs as Proposition 2.8.6:

**Proposition 3.2**. *Let $G = XU$ be a group of finite Morley rank with $U, X$ connected definable, $U$ 2-unipotent, $X \lhd G$, and $X$ of either degenerate or odd type. Then $U$ centralizes $X$.*

Our second general principle shows that possible degenerate sections of an $L$-group are not so intricately involved with the structure as might be feared, and allows the direct transfer of a considerable body of $K$-group theory to our setting.

**Proposition 3.4**. *Let $G$ be an $L$-group. Then $B(G)$ is a $K$-group.*

Here $B(G)$ is the (definable) subgroup of $G$ generated by all its 2-unipotent subgroups. This proposition, as stated, depends on a slight extension of a result of Poizat in [24], whose proof in general requires the classification of the finite simple groups:

**Fact 4.19**. *If $K$ is a field of finite Morley rank of characteristic $p \neq 0$ then every simple definable section $G$ of $\mathrm{GL}_n(K)$ is definably isomorphic to an algebraic group over $K$.*

This is stated by Poizat for simple subgroups. One must bear in mind here that the field $K$ comes equipped with arbitrary additional structure, so there is no obvious reason for definable subgroups of $\mathrm{GL}_n(K)$ to be at all well-behaved (technically, this is tied up with the so-called "bad field" problem).

In practice one can manage with considerably less (the Feit–Thompson theorem suffices), and we will indicate in the body of the paper how Proposition 3.5 and certain related arguments can be recast to accomplish this. The essential point will be the replacement of Fact 4.19 by a similar Fact 4.21 in applications, particularly in the proof of Lemma 3.11 in Section 3.

We also make essential use of part of the theory of Suzuki 2-groups developed in [13].

Another point which may be useful in the further development of the theory was pointed out by Borovik (personal communication):

**Fact 1.1.** *Let $G = U \rtimes X$ be a group of finite Morley rank with $U, X$ connected and definable, $U$ an abelian 2-group, and $X$ solvable and containing no involutions, and acting faithfully on $U$. Then $X$ is abelian.*

We do not need this result here, but it is of importance for the further development of these ideas [7].

The paper is organized as follows. In the next section we review the main definitions relating to the "type" of a group, which are based on results of the 2-Sylow theory, as well as the definitions relating to inductive approaches to the algebraicity conjecture: the notions of $K$-groups and $K^*$-groups, used to date, and the parallel but more general notions of $L$-groups and $L^*$-groups, for which at least some of the previous theory can be pushed through, as we will show. In Section 3 we develop the essential points of the general theory of $L$-groups, Propositions 3.2 and 3.5, which have been discussed above. In Section 4, included for ease of reference, we collect essential points of the general theory of groups of finite Morley rank (a few of these are applied in Section 3). In Section 5 we present our main technical result, the proof of Theorem 1 in the strongly embedded case, which deviates quite strongly from the line of argument in the $K^*$ case. In Section 6, for the sake of completeness, we outline the proof in the weakly but not strongly embedded case, where one follows the line of arguments given in [4]. Finally in Section 7 we prove Theorem 2, following the line of argument in [18], and paying some further attention to the treatment of degenerate sections (in Lemma 7.21).

## 2. Definitions

In recent years considerable attention has been paid to the elucidation of the possible 2-Sylow structure in a minimal counterexample to the algebraicity conjecture, an enterprise which makes considerable use of ideas exploited in the "revisionist" (i.e., highly inductive) approach to the classification of the finite simple groups.

### 2.1. Types of groups

There is a general 2-Sylow theory valid in groups of finite Morley rank, due to Borovik and Poizat. A Sylow 2-subgroup is a maximal 2-subgroup, and a Sylow° 2-subgroup is the connected component of a Sylow 2-subgroup.

**Fact 2.1** [12]. *Let $G$ be a group of finite Morley rank. Then*

(1) *Any two Sylow* 2*-subgroups of $G$ are conjugate.*
(2) *If $S$ is a Sylow°* 2*-subgroup, then $S$ has the form*

$$U * T \quad (\textit{central product, finite intersection})$$

*with $U$* 2-unipotent (*a definable connected* 2*-group of bounded exponent*) *and $T$ a* 2*-torus* (*a divisible abelian* 2*-group*).

In algebraic groups only one of these factors $U$, $T$ can be nontrivial, depending on whether the characteristic is, or is not, two; and if the group $G$ is simple, then *exactly* one of the two is nontrivial. In groups of finite Morley rank we have, a priori, four cases to deal with:

(1) $U, T \neq 1$: mixed type;
(2) $U \neq 1$, $T = 1$: even type;
(3) $U = 1$, $T \neq 1$: odd type (in algebraic groups, this includes characteristic 0);
(4) $U = T = 1$ (i.e., $S$ is finite): degenerate type.

The algebraicity conjecture predicts that mixed and degenerate types do not occur, and that even type and odd type correspond respectively to algebraic groups in characteristic 2 and characteristic not 2.

### 2.2. Inductive frameworks

The standard inductive framework for considering the algebraicity conjecture has been the following.

**Definition 2.2.** Let $G$ be a group of finite Morley rank.

(1) A *section* of $G$ is a quotient $H/K$ with $K \lhd H \leqslant G$; the section is *definable* if $H$ and $K$ are definable, and *proper* unless $H = G$ and $K = 1$.
(2) $G$ is a *$K$-group* if every definable connected simple section is isomorphic to an algebraic group over an algebraically closed field.
(3) $G$ is a *$K^*$-group* if every proper definable connected simple section is isomorphic to an algebraic group over an algebraically closed field.

Thus a simple $K^*$-group of finite Morley rank is either algebraic, or is a minimal counterexample to the algebraicity conjecture.

We will work in a setting which allows degenerate (and in some cases, nonalgebraic odd type) sections.

**Definition 2.3.**

(1) An *$L$-group* is a group of finite Morley rank in which every infinite definable simple section is either an algebraic group over an algebraically closed field, or of odd or degenerate type; in other words, we exclude definable simple sections of mixed type, and we require definable simple sections of even type to be algebraic.
(2) An *$L^*$-group* is a group of finite Morley rank in which every *proper* infinite definable simple section is either an algebraic group over an algebraically closed field, or of odd or degenerate type.

In the context of $K^*$-groups, the elimination of the mixed type case was completed in [18], building on the special case treated in [3], in which it is assumed, among other things, that the definable connected solvable sections of the group in question which contain no involutions are nilpotent.

As far as even type is concerned, there has been a very active classification project aimed at verifying the algebraicity conjecture in the case of simple $K^*$-groups of finite Morley rank of even type, a project which appears now to be reaching completion.

However in [2] the possibility of weakening the $K^*$-hypothesis substantially in the even type case was investigated, and the $L^*$-theory was explored. The definition we adopt here reduces to the one given in [2] in the even type case, and also serves well in our treatment of the mixed type case in Section 7. The proposal is to adapt the methods used for $K^*$-groups to $L^*$-groups. The ideal target would be:

**$L^*$-conjecture.** *A simple $L^*$-group of finite Morley rank and of even type is algebraic.*

There are some grounds for cautious optimism that the $L^*$-conjecture may be approachable by means closely related to those which have been used in the $K^*$-context. The advantage of this version over the $K^*$ version would be that this conjecture is actually *equivalent* to the full algebraicity conjecture for arbitrary simple groups of finite Morley rank and even type, and this in turn would also dispose of the *mixed type* case (Theorem 2).

### 2.3. Weak solvability

In the case of $K^*$-groups, the analysis of the even type case begins with the study of weakly embedded subgroups (Definition 4.36), namely:

**Fact 2.4** [4,19]. *Let $G$ be a simple $K^*$-group of even type with a weakly embedded subgroup. Then $G \cong \mathrm{PSL}_2(K)$ for some algebraically closed field $K$ of characteristic* 2.

A similar result for $L^*$-groups would be of major importance.

The proof of Fact 2.4 begins with an analysis of the structure of a weakly embedded subgroup $M$; it is shown that $M^\circ$ is solvable. In the $K^*$-context Fact 2.4 was proved in

[19], building on [1,4]. Again, the initial work assumed among other things the nilpotence of connected sections without involutions, a hypothesis lifted in [19] by use of the theory of solvable groups, notably the work of Frécon in [14].

In the $L^*$ context one does not expect to obtain solvability of $M^\circ$ directly at the outset. There are however two natural analogs of that result in the $L^*$ context:

(1) $M^\circ/O_2^\circ(M)$ is of degenerate type;
(2) $B(M)$ is solvable.

Using standard results in the theory of groups of finite Morley rank one shows that these two results are equivalent; yet a third form of this condition is: $B(M) = O_2^\circ(M)$. (Cf. Section 5.) It is the first form which is most useful for the intended applications. In the $K^*$-context any proper connected subgroup of degenerate type is solvable, so in that setting our Theorem 1 is equivalent to the solvability of $M^\circ$.

In the proof of Theorem 1 we will have to distinguish two cases: the group $M$ in question either is, or is not, *strongly* embedded (Definition 4.29). If it is *not* strongly embedded, then the analysis runs closely parallel to that given in the $K^*$-case, which we will therefore only summarize. If $M$ *is* strongly embedded, however, then we have to deviate substantially from the earlier approach, and this constitutes the technical core of the present article.

## 3. $L$-groups

We now take up our subject in earnest.

As we have mentioned, the analysis of the 2-Sylow structure of $K^*$-groups begins (logically, at least) with:

**Fact 3.1** [18]. *There exists no simple $K^*$-group of mixed type.*

The classification of simple $K^*$-groups of finite Morley rank and even type is also well under way, and it appears that the algebraicity conjecture will be confirmed in that case, using the amalgam method from finite group theory to perform the final identification.

Very difficult problems remain in the analysis of the other two types, which will not concern us here. Our aim is to weaken the reliance on the $K^*$ hypothesis in the favorable cases: even and mixed types; as explained in the introduction (and proved in Section 7), the mixed-type case reduces to the even-type case. Now the *even-type* case is quite intricate, even in the $K^*$-case. The present paper is intended to provide a case study in the adaptation of arguments occurring very early in the analysis, and which have depended on the solvability of degenerate sections, to the case of general groups of even type, in which nonsolvable degenerate sections may occur.

The remaining part of this section is devoted to the preparation of a suitable general theory of $L$-groups, much like the theory of $K$-groups as it has been used to date. The first step is perhaps the most important: we claim that a unipotent 2-group can only act trivially on a group of odd or degenerate type; when the latter is assumed *solvable*, this follows from standard results (cf. [11, Exercise 2, p. 175] and [1]).

**Proposition 3.2.** *Let $G = XU$ be a group of finite Morley rank where $U$ and $X$ are definable, and $X \lhd G$. If $U$ is a unipotent 2-group, and $X$ is of odd or degenerate type, then the action of $U$ on $X$ is trivial.*

**Proof.** The argument is by induction on the rank and degree of $X$. Note that $X \cap U$ is finite, by Fact 2.1(2).

Considering the quotient by $C_U(X)$ we may assume toward a contradiction that

(1) the action of $U$ on $X$ is faithful and $U$ is nontrivial.

Passing to a suitable connected normal subgroup of $U$, we may also suppose that

(2) $U$ is elementary abelian.

We show that

(3) $X$ is connected.

If $X^\circ < X$ then by induction on degree, $U$ centralizes $X^\circ$. As $X/X^\circ$ is finite, $U$ acts trivially on $X/X^\circ$. Hence for $x \in X$ the map $\gamma_x : U \to X^\circ$ defined by $\gamma_x(u) = [u, x]$ is a homomorphism, whose image is then a unipotent 2-subgroup of $X$, which must be trivial. So $U$ centralizes $X$, a contradiction.

We show now that $Z(X)$ is finite. If $X$ is abelian, then $G$ is solvable and by Fact 4.10 we find that $U \leqslant F(G)$. As $U$ is a maximal unipotent 2-subgroup of $F(G)$, and $F(G)$ is nilpotent, $U$ is normal in $G$. Hence, $[U, X] \leqslant U \cap X$. Hence $[U, X]$ is finite, as remarked above, and is connected by Fact 4.2. Thus $[U, X] = 1$, a contradiction. So $Z(X) < X$ and hence, by induction on the rank and degree of $X$, we find that $U$ centralizes $Z(X)$.

If $Z(X)$ is infinite then induction on rank applies also to $X/Z(X)$, and hence for $x \in X$ the map $\gamma_x : U \to Z(X)$ defined by $\gamma_x(u) = [x, u]$ is a homomorphism, whose image is then a unipotent 2-subgroup of $X$, which must be trivial. Thus in this case $U$ centralizes $X$, a contradiction. We conclude that $Z(X)$ is finite.

We show that $Z(G)$ is finite as well. If $Z(G)$ is infinite then since $Z(G) \cap X$ is finite using the last paragraph, $Z(G)X/X$ is infinite. This implies that $Z(G)$ contains a nontrivial unipotent 2-subgroup. Since $U$ is a maximal unipotent 2-subgroup of $G$, any unipotent 2-subgroup of $Z(G)$ is contained in $U$ as well. This contradicts the assumption that $U$ acts faithfully on $X$.

We let $\overline{G} = G/Z(G)$. $\overline{G}$ is centerless by Fact 4.14. $\overline{U}$ is a maximal unipotent 2-subgroup and $\overline{X}$ is connected of odd or degenerate type. If $[\overline{U}, \overline{X}] = 1$, then $[U, X] \leqslant Z(G)$. Since $U$ is connected, so is $[U, X]$. Then it follows from the finiteness of $Z(G)$ that $U$ centralizes $X$, a contradiction. Thus $[\overline{U}, \overline{X}] \neq 1$.

The discussion of the last paragraph shows that $\overline{G}$ is also a counterexample, with $\mathrm{rk}(X) = \mathrm{rk}(\overline{X})$. So we still have a minimal counterexample. Moreover $\overline{U}$ is elementary abelian. Now, using the finiteness of $Z(G)$ and Fact 4.2 again, $[\overline{u}, \overline{X}] = 1$ if and only if $[u, X] = 1$. Since $U$ acts faithfully on $X$, $\overline{U}$ acts faithfully on $\overline{X}$ as well. Thus $\overline{G}$ has all the properties (1)–(3) we have proven for $G$. In addition $Z(\overline{G}) = 1$. We may therefore assume that $Z(G)=1$. Since, as shown above, $Z(X)$ is finite, we have $Z(X)\leqslant Z(G)$, and hence we have $Z(X)=1$ as well.

If $U\lhd G$ then $[X,U]\leqslant U\cap X$. As above this implies that $[U,X]=1$, a contradiction. So $N_G(U)<G$. In particular, $N_X(U)<X$. As $U$ acts on $N_X(U)$, by induction $U$ centralizes $N_X(U)$, and hence $N_G(U)=C_G(U)$.

We claim now that $N_G(U)=C_G(U)$ is strongly embedded in $G$. Let $S$ be a Sylow 2-subgroup of $G$ containing $U$. Then $U=S^\circ$ and hence $N(S)\leqslant N(U)$. In view of Fact 4.30(2), it suffices now to check that $C_G(i)\leqslant C_G(U)$ for $i\in I(N_G(U))$. As $i\in C(U)$ and $Z(G)=1$, the group $C_X(i)<X$ is $U$-invariant, so by induction on the rank and degree of $X$, we find that $U$ centralizes $C_X(i)$ and hence $C_G(i)=C_X(i)U\leqslant C_G(U)$.

Now it follows that $I(U)$ is a single conjugacy class (Fact 4.32), which contradicts the fact that $G/X$ is abelian. □

Before stating a useful corollary of Proposition 3.2, we need the following definition:

**Definition 3.3.** Let $G$ be a group of finite Morley rank. $B(G)$ is the subgroup of $G$ generated by its 2-unipotent subgroups. (If there are none, this is 1.)

By Fact 4.2(1), $B(G)$ is always definable and connected. Of course, $B(B(G))=B(G)$. A group $G$ of finite Morley rank is said to be of $B$-*type* if $G=B(G)$.

**Corollary 3.4.** *Let $H$ be a group of finite Morley rank of $B$-type and $X$ a definable connected normal subgroup of degenerate type. Then $X\leqslant O(Z(H))$.* (*We refer the reader to Definition* 4.16 *for $O$ and some remarks related to this notion.*)

**Proof.** By assumption, $H=B(H)$. For any unipotent 2-subgroup $U$ of $H$, $[U,X]=1$ by Proposition 3.2. Therefore, $X\leqslant Z(H)$. By Fact 4.12, $X$ has no involutions. □

According to [1, Fact 2.51], the structure of connected $K$-groups is remarkably straightforward (compared, for example, to the structure of finite groups): modulo its solvable radical, any connected $K$-group is a direct sum of finitely many simple groups. This is due largely to the fact that the simple groups in question are algebraic, and their automorphism groups are well under control (Fact 4.22). One cannot expect to control the structure of $L$-groups to the same degree, since there is no limit to the potential complexity of the degenerate sections. However, these difficulties can frequently be evaded by consideration of the group $B(G)$.

We will now prove that if $G$ is an $L$-group of finite Morley rank, then $B(G)$ is a $K$-group; which is more neatly phrased as:

**Proposition 3.5.** *Let $G$ be an $L$-group of finite Morley rank and of $B$-type. Then $G/\sigma(G)$ is a finite direct product of simple algebraic groups of even type.*

**Proof.** Replacing $G$ by $G/\sigma(G)$, we may suppose that $\sigma(G)=1$. By Fact 4.17, $\mathrm{soc}(G)$ is a direct sum of definable simple subgroups. As $G$ is an $L$-group, these factors are either algebraic groups, or odd type or degenerate, and as $G$ is connected they are normal in $G$. Suppose there is a degenerate or odd type factor $K$. Then by Proposition 3.2, the unipotent subgroups of $G$ centralize $K$, and as $G = B(G)$, we find that $K$ is central in $G$, a contradiction. Thus $\mathrm{soc}(G)$ is a finite sum of simple algebraic groups (of even type), and in view of Fact 4.22, as $G$ is connected, it follows that $G = \mathrm{soc}(G)C_G(\mathrm{soc}(G))$. If $C_G(\mathrm{soc}(G)) \neq 1$ then $Z(G) = C_G(\mathrm{soc}(G)) \cap \mathrm{soc}(G) \neq 1$ by Fact 4.17, hence $\sigma(G) \neq 1$, a contradiction. So $G = \mathrm{soc}(G)$. □

**Corollary 3.6.** *Let $G$ be an $L$-group of finite Morley rank and of $B$-type. Then $G$ is a $K$-group.*

This follows using Fact 4.19; thus this corollary makes indirect use of the classification of the finite simple groups. The corollary has the advantage of facilitating the *direct* application of $K$-group theory to our subject. In most such cases Proposition 3.5 already suffices for the results in question, and where that is not the case it seems one can manage with Fact 4.21.

We will now give the consequences of Corollary 3.6 used in the present paper. We give references to the original $K$-group facts from which they are derived. These also follow from Proposition 3.5 by inspection of the original proofs, except in the case of Lemma 3.11, where Fact 4.21 is used.

**Lemma 3.7** [1, Fact 2.51]. *Let $H$ be a $B$-type $L$-group. Then $H/\sigma^\circ(H) = H_1/\sigma^\circ(H) * \cdots * H_k/\sigma^\circ(H)$, where the $H_i/\sigma^\circ(H)$ are quasisimple algebraic groups over algebraically closed fields.*

This is found in [1] in the more restricted form of a description of $H/\sigma(H)$; for the preceding formulation, we incorporate Fact 4.18 as well.

**Lemma 3.8** [6, 2.26]. *Let $H$ be a connected $L$-group of even type. Then $O_2(B(H))$ is connected.*

**Lemma 3.9** [5, Fact 2.33]. *Let $H$ be a connected $L$-group of even type with $O_2^\circ(H) = 1$. Then $B(H) = E(B(H))$.*

**Proof.** We may assume $H = B(H)$. Then $O_2(H) = 1$ by Lemma 3.8, and in particular, $O_2(F(H)) = 1$; so $F(H)$ contains no involutions. By Fact 4.10, $\sigma^\circ(H)/F^\circ(H)$ is divisible abelian, hence contains no involutions as $H$ is of even type. It follows that $\sigma^\circ(H)$ contains no involutions.

Let $\overline{H} = H/\sigma^\circ(H)$. By Lemma 3.7, $\overline{H}$ is a finite product of quasisimple algebraic groups $\overline{L}_i$ with $L_i \lhd H$. It suffices to show that each $L_i^{(\infty)}$ is quasisimple. So we may suppose that $\overline{H}$ is a quasisimple algebraic group.

Now as $\sigma(H)/\sigma^\circ(H)$ is finite, we have $[H, \sigma(H)] \leqslant \sigma^\circ(H) \leqslant Z(H)$. So by the three subgroups lemma, $\sigma(H^{(\infty)}) \leqslant Z(H^{(\infty)})$. By Fact 4.18, $H^{(\infty)}$ is a quasisimple algebraic group. □

**Lemma 3.10** [4, Proof of Lemma 5.5]. *Let $H$ be a connected $L$-group of $B$-type. Then $H^{(\infty)}$ is generated by its definable, connected $2^{\perp}$-subgroups.*

**Proof.** Let $K = H^{(\infty)}$. Since $H/O_2(H)$ is of $B$-type and $O_2(H/O_2(H)) = 1$, we have the following, using Lemma 3.9:

$$H/O_2(H) = E\bigl(H/O_2(H)\bigr) = \bigl(H/O_2(H)\bigr)^{(\infty)} = H^{(\infty)}O_2(H)/O_2(H) \cong K/O_2(K).$$

As a result, $\overline{K} = K/O_2(K)$ is a central product of quasisimple algebraic groups. Let $T(K)$ be the subgroup generated by definable connected $2^{\perp}$-subgroups of $K$. It suffices to show that $T(K)$ covers the maximal tori of $\overline{K}$, since $T(K)$ then will cover $\overline{K}$, so that $K/T(K)$ is solvable, hence trivial.

Let $S$ be a Sylow 2-subgroup of $K$. Then $S/O_2(K)$ has a complement $T/O_2(K)$ in $N(S/O_2(K))$, and these complements generate $N(S)$. So it suffices to show that $T/O_2(K)$ is covered by a definable connected $2^{\perp}$-subgroup of $N(S)$. $T/O_2(K)$ is itself a $2^{\perp}$-group and $O_2(K)$ is a 2-unipotent group. By Fact 4.11 (Schur–Zassenhaus), $T$ splits over $O_2(K)$ and is therefore covered by a connected $2^{\perp}$-subgroup. $\square$

**Lemma 3.11** [4, Proposition 3.4]. *Let $X \rtimes Y$ be a group of finite Morley rank where $X$ and $Y$ are definable and connected, $X$ is an $L$-group of even type, and $Y$ is a $2^{\perp}$-group. Then $Y$ normalizes a Sylow$^{\circ}$ 2-subgroup of $X$.*

The reduction to [4] involves replacing $X$ by $B(X)$ and applying a $K$-group fact to $B(X) \rtimes Y$. As this requires Corollary 3.6, the argument depends indirectly on the classification of the finite simple groups. Because $Y$ is a $2^{\perp}$-group, inspection of the proof would show that only Feit–Thompson is required here (namely, one argues that $Y$ acts on $B(X)/\sigma(B(X))$ as a $2^{\perp}$-group of inner automorphisms, and the point is that this forces the image of $Y$ in the quotient to be solvable).

**Lemma 3.12.** *Let $H$ be a connected $L$-group of even type with a weakly embedded subgroup $M$. Then*

$$H = L \times D$$

*where $L = B(H) \simeq \mathrm{SL}_2(K)$, with $K$ algebraically closed of characteristic 2, and $D = C_H(L)$ is a subgroup of degenerate type. $M^{\circ} \cap L$ is a Borel subgroup of $L$ and $D \leqslant M$.*

**Proof.** Let $L = B(H)$ and $D = C_H(L)$. Let $S$ be a Sylow$^{\circ}$ 2-subgroup of $M$. Then $S \leqslant L$ and, by a Frattini argument, $H \leqslant L \cdot N(S)$. If $L \leqslant M$, we get $H \leqslant M$, a contradiction. It follows that $M \cap L$ is weakly embedded in $L$. Hence by Fact 4.40, $L \simeq \mathrm{PSL}_2(K)$ for some algebraically closed field $K$ of characteristic 2. As $M \cap L$ is weakly embedded in $L$, $M \cap L$ is a Borel subgroup of $L$.

Now Fact 4.22 shows that $H = LC_H(L)$. Since $L$ is simple, $D \cap L = 1$. Hence $H = L \times D$, and $D$ is of degenerate type. As $D \leqslant C(L) \leqslant C(S)$, we have $D \leqslant M$. $\square$

## 4. Background material

We review material from the general theory of groups of finite Morley rank. Our main reference is [11].

### *4.1. Definable closure*

Groups of finite Morley rank satisfy the *descending chain condition on definable subgroups*, and in particular any definable subgroup $H$ of a group $G$ of finite Morley rank has a "connected component" $H^\circ$, the smallest definable subgroup of $H$ of finite index in $H$. One can also define the *definable closure* $d(X)$ of an arbitrary subset $X$ of $G$ as the smallest definable subgroup of $G$ that contains $X$. One can then define the connected component an arbitrary subgroup $H$ of $G$, not necessarily definable, as $H^\circ = H \cap d(H)^\circ$. With this notation, one has:

**Fact 4.1** [11, Lemmas 5.35 and 5.36]. *Let $G$ be a group of finite Morley rank.*

(1) *For any $x \in G$, $C_G(x) = C_G(d(\langle x\rangle))$.*
(2) *If $B$ is a definable normal subgroup of $G$ and $X \subseteq G$ such that $B \subseteq X$ then $d(X/B) = d(X)/B$.*

### *4.2. Zilber indecomposability*

A fundamental result on groups of finite Morley rank is *Zilber's Indecomposability Theorem*. We state two special cases:

**Fact 4.2** [11, Section 5.4]. *Let $G$ be a group of finite Morley rank.*

(1) *The subgroup of $G$ generated by a family of definable connected subgroups of $G$ is itself definable and connected.*
(2) *If $H$ is a definable connected subgroup of $G$ and $X$ any subset of $G$, then $[H, X]$ is a definable connected subgroup of $G$.*

These results will be used freely in the sequel.

### *4.3. Sylow 2-subgroups*

We have mentioned the Sylow 2-theory already:

**Fact 4.3** [12].

(1) *The Sylow 2-subgroups of a group of finite Morley rank are conjugate.*
(2) *If $S$ is a Sylow 2-subgroup of a group of finite Morley rank then $S^\circ$ is the central product of a definable connected nilpotent subgroup of bounded exponent* (*a* unipotent 2-group) *and a divisible abelian 2-subgroup* (*a* 2-torus). *These two groups are uniquely determined.*

The connected component of a Sylow 2-subgroup is called a *Sylow*$^\circ$ 2-*subgroup*.

**Fact 4.4** [25, Corollary 1.5.5]. *Let $G$ be a group of finite Morley rank and $N$ a definable normal subgroup of $G$. Then the Sylow* 2*-subgroups of $G/N$ are exactly the images of those of $G$.*

In the analysis of groups with strongly embedded subgroups, it is essential to have information about so-called *Suzuki* 2*-groups*, which were analyzed in the case of finite Morley rank by Davis and Nesin [13].

**Definition 4.5.** A *Suzuki* 2*-group* is a pair $(S,T)$ where $S$ is a nilpotent 2-group of bounded exponent and $T$ is an abelian group that acts on $S$ by automorphisms and which is transitive on the involutions of $S$.

A Suzuki 2-group is said to be *free* if $T$ acts on $S$ freely: for any $g \in S$ and $t \in T$, $g^t = g$ implies either $g = 1$ or $t = 1$.

A Suzuki 2-group is said to be *abelian* if $S$ is abelian.

A Suzuki 2-group is said to be of *finite Morley rank* if the structure $(S,T)$ is of finite Morley rank.

**Fact 4.6** [13]. *A free Suzuki* 2*-group of finite Morley rank is abelian.*

*4.4. Nilpotent groups*

We will need the following structure theorem for nilpotent groups of finite Morley rank.

**Fact 4.7** [23]. *Let $H$ be a nilpotent group of finite Morley rank. Then $H = D * B$ where $D$ and $B$ are definable characteristic subgroups, with $D$ divisible, $C$ of bounded exponent, and $D \cap C$ finite.*

Similarly:

**Fact 4.8** [11, Exercise 10, p. 93]. *The definable closure of a cyclic subgroup of a group of finite Morley rank is the direct sum of a finite cyclic group with a divisible abelian group.*

**Fact 4.9** [11, Exercise 1, p. 97]. *Let $p$ be a prime number. Then an infinite nilpotent $p$-group of finite Morley rank and of bounded exponent has infinitely many central elements of order $p$.*

For a group $G$ of finite Morley rank $F(G)$ will denote its *Fitting subgroup*, the subgroup of $G$ generated by its normal nilpotent subgroups, which is definable and nilpotent [22].

In a group $G$ of finite Morley rank, $O_2(G)$ denotes the largest normal 2-subgroup. This is not always definable. However, $O_2(G) = O_2(F(G))$, so if $G$ is of even type then $O_2(G)$ is definable.

*4.5. Solvable groups*

**Fact 4.10** [21]. *Let $G$ be a connected solvable group of finite Morley rank. Then $G/F^\circ(G)$ is divisible and abelian. In particular, $G'$ is nilpotent.*

Hall subgroups can be defined in the context of solvable groups of finite Morley rank with respect to any set of primes $\pi$, and the Hall $\pi$-subgroups are conjugate [9,11].

**Fact 4.11** [10, Proposition C]. *Let $G$ be a solvable group of finite Morley rank and $H$ a normal Hall $\pi$-subgroup of $G$ of bounded exponent. Then any subgroup $K$ of $G$ with $K \cap H = 1$ is contained in a complement to $H$ in $G$, and the complements of $H$ in $G$ are definable and conjugate to one another.*

**Fact 4.12** ([11, Theorem 9.29], [14]). *The Hall $\pi$-subgroups of a connected solvable group of finite Morley rank are connected.*

**Fact 4.13** [4, Proposition 2.43]. *Let $G = H \rtimes Q$ be a group of finite Morley rank where $H$, $Q$, and the action of $Q$ on $H$ are definable. Let $H_1 \lhd H$ be a solvable $Q$-invariant definable $\pi$-subgroup of bounded exponent in $G$. Assume that $Q$ is a solvable $\pi^{\perp}$-subgroup. Then $C_H(Q)H_1/H_1 = C_{H/H_1}(Q)$.*

**Fact 4.14** [11, Lemma 6.1]. *If $G$ is a connected group of finite Morley rank and $Z(G)$ is finite, then $Z(G/Z(G)) = 1$.*

**Definition 4.15.** For $G$ a group of finite Morley rank, $\sigma(G)$ denotes its *solvable radical*, the subgroup generated by all its normal solvable subgroups. It was proven in [22] that this group is definable and solvable.

**Definition 4.16.** For $G$ a group of finite Morley rank, $O(G)$ is the largest definable connected normal *solvable* subgroup without involutions.

In published work on groups of finite Morley rank, $O$ has been defined without the solvability assumption. In the $K^*$ context, where $O$ is frequently used, all proper definable connected subgroups of degenerate type are solvable and hence without involutions by Fact 4.12. Since in the $L^*$ context, nondegeneracy does not necessarily imply solvability for proper definable connected subgroups, we have added this assumption.

### *4.6. $E(G)$*

Let $G$ be a group of finite Morley rank. Then $E(G)$ is the subgroup of $G$ generated by the subnormal quasisimple definable subgroups of $G$. It is definable and is a central product of quasisimple subgroups [11]; if $G$ is connected, then $E(G)$ and its factors are connected.

### *4.7. $K$-group structure*

For $G$ a group of finite Morley rank, the *socle* $\mathrm{soc}(G)$ is the subgroup of $G$ generated by its minimal normal nontrivial subgroups. In general, $\mathrm{soc}(G)$ may be trivial (if there are no such) or may not be definable. However:

**Fact 4.17** ([22], [11, Theorem 7.8]). *Let $G$ be a connected group of finite Morley rank such that $\sigma(G) = 1$ and $G \neq 1$. Then* $\mathrm{soc}(G)$ *is the direct sum of a finite number of infinite definable simple groups, and every nontrivial normal subgroup of $G$ meets* $\mathrm{soc}(G)$ *nontrivially.*

**Fact 4.18** [8]. *Let $G$ be a perfect group of finite Morley rank such that $G/Z(G)$ is a simple algebraic group. Then $G$ is an algebraic group. In particular, $Z(G)$ is finite.*

This is a good place to repair a possible defect in our definition of $K$-group. We will need the following, a slight variant of a result of Poizat in [24].

**Fact 4.19.** *If $K$ is a field of finite Morley rank of characteristic $p \neq 0$, then every simple definable section $G$ of* $\mathrm{GL}_n(K)$ *is definably isomorphic to an algebraic group over $K$.*

This result is of some importance, as otherwise we would not be entitled to consider algebraic groups, in enriched languages, as $K$-groups! Poizat states the result only for simple subgroups of $\mathrm{GL}_n(K)$. We give the general lines of an argument that reduces the general result to the case treated by Poizat. We will freely use facts about linear algebraic groups which can be found in [16]. We believe that Erulan Mustafin has also given a proof of this result in an unpublished note.

Let $H/N$ be the simple section in question, and let $R$ be the Zariski closure of $\sigma(H)$. $R$ is solvable and normalized by $H$. Since $\sigma(H) \leqslant R$, it follows that $H \cap R = \sigma(H)$. Since $R$ is closed so is $N(R)$. Therefore $N(R)/R$ is algebraic. We have $HR/R \leqslant N(R)/R$ and $HR/R \cong H/\sigma(H)$, thus $H/\sigma(H)$ is definably isomorphic to a subgroup of an algebraic group in characteristic $p$, namely $N(R)/R$. By Fact 4.17, $\mathrm{soc}(H/\sigma(H)) = Y_1/\sigma(H) \oplus \cdots \oplus Y_k/\sigma(H)$ where the $Y_i/\sigma(H)$ are simple groups. Since $H/\sigma(H)$ is definably isomorphic to a subgroup of an algebraic group in characteristic $p$, Poizat's result implies that the $Y_i/\sigma(H)$ are definably isomorphic to algebraic groups. Using this and Fact 4.22 in an argument similar to that of Proposition 3.5, we conclude that $H/\sigma(H) = \mathrm{soc}(H/\sigma(H))$. Since $H/N$ is simple, $\sigma(H) \leqslant N$. Hence $N/\sigma(H)$ is a normal subgroup of $H/\sigma(H)$. By properties of completely reducible groups, we conclude that $H/N$ is definably isomorphic to one of the $Y_i/\sigma(H)$.

**Corollary 4.20.** *Let $G$ be a group of finite Morley rank, and suppose that $G$ has a definable composition series for which all simple quotients are isomorphic to algebraic groups over algebraically closed fields of positive characteristic. Then $G$ is a $K$-group.*

As mentioned in the introduction, Fact 4.19 and its corollary depend on the classification of the finite simple groups. For our purposes the following, which requires only the Feit–Thompson Theorem, would be sufficient.

**Fact 4.21.** *If $K$ is a field of finite Morley rank of characteristic $p \neq 0$ then every nonsolvable definable section $G$ of* $\mathrm{GL}_n(K)$ *contains an involution.*

However this does not imply Corollary 4.20, and consequently where we invoke $K$-group theory in the present paper, if one wishes to get by using only the Feit–Thompson Theorem, then one would have to check that the $K$-group arguments go through under the hypothesis of that corollary, with occasional uses of Fact 4.21.

*4.8. Automorphisms*

**Fact 4.22** [11, Theorem 8.4]. *Let $\mathcal{G} = G \rtimes H$ be a group of finite Morley rank where $G$ and $H$ are definable, $G$ is an infinite simple algebraic group over an algebraically closed field, and $C_H(G) = 1$. Then, viewing $H$ as a subgroup of $\mathrm{Aut}(G)$, we have $H \leqslant \mathrm{Inn}(G)\Gamma$ where $\mathrm{Inn}(G)$ is the group of inner automorphisms of $G$ and $\Gamma$ are the graph automorphisms.*

**Fact 4.23** [11, Exercise 14, p. 73]. *Let $G$ be a group of finite Morley rank without involutions. If $\alpha$ is a definable involutive automorphism of $G$ then $G = C_G(\alpha)G^-$, where $G^- = \{g \in G\colon\ g^\alpha = g^{-1}\}$. Moreover, if $c \in C_G(\alpha)$ and $g \in G^-$, then $(c, g) \mapsto cg$ is a definable bijection. In particular, $G$ is connected if and only if $C_G(\alpha)$ is connected, and $G^-$ is of Morley degree 1.*

**Fact 4.24** ([17, Lemme 4.7], [4, Proposition 9.4]). *Let $Q \rtimes X$ be a group of finite Morley rank where $Q$, $X$, and the action of $X$ on $Q$ are definable. If $Q$ is an abelian 2-group of bounded exponent and $X$ is a $2^\perp$-group which centralizes the involutions of $Q$, then $X$ centralizes $Q$.*

*4.9. Torsion*

**Fact 4.25** [11, Exercise 11, p. 93]. *Let $G$ be a group of finite Morley rank, $H$ a definable subgroup of $G$, and $x \in G$. If for some prime number $p$ we have $x^p \in H$, then the coset of $x$ modulo $H$ contains a $p$-element.*

*4.10. Fusion*

**Fact 4.26** [11, Proposition 10.2]. *Let $G$ be a group of finite Morley rank and $i, j \in G$ two involutions. Then either $i$ and $j$ are $d(\langle ij\rangle)$-conjugate, or they commute with an involution in $d(\langle ij\rangle)$.*

**Fact 4.27** [11, Lemma 10.22]. *Let $S$ be a Sylow$^\circ$ 2-subgroup of a group $G$ of finite Morley rank. If $T$ is the maximal torus in $S$ then $N_G(T)$ controls fusion in $S$, in the sense that any two subsets of $S$ which are conjugate in $G$, are conjugate in $N_G(T)$.*

**Corollary 4.28** ([18, Fait 2.18], [1, Fact 2.48]). *Let $G$ be a group of finite Morley rank of mixed type. Suppose $S$ is Sylow$^\circ$ 2-subgroup, and $S = U * T$ where $U$ is the maximal unipotent 2-subgroup of $S$ and $T$ its maximal 2-torus. If $i \in I(T)$ then $i^G \cap U \subseteq T$. In particular, $U$ has only finitely many involutions that are conjugate to involutions in $T$.*

*4.11. Strong embedding*

**Definition 4.29.** Let $G$ be a group of finite Morley rank. A proper definable subgroup $M$ of $G$ is said to be *strongly embedded* in $G$ if

(i) $I(M) \neq \emptyset$;
(ii) for every $g \in G \setminus M$, $I(M \cap M^g) = \emptyset$.

**Fact 4.30** [15, Theorem 9.2.1]. *Let $G$ be a group of finite Morley rank with a proper definable subgroup $M$. Then the following are equivalent*:

(1) *$M$ is a strongly embedded subgroup.*
(2) *$I(M) \neq \emptyset$, $C_G(i) \leqslant M$ for every $i \in I(M)$, and $N_G(S) \leqslant M$ for every Sylow 2-subgroup of $M$.*
(3) *$I(M) \neq \emptyset$ and $N_G(S) \leqslant M$ for every nontrivial 2-subgroup $S$ of $M$.*

**Definition 4.31.** Let $G$ be a group. For $x \in G$, $C_G^*(x) = \{g \in G\colon x^g = x \text{ or } x^{-1}\}$. An element of $G$ is said to be *strongly real* if it is the product two involutions.

**Fact 4.32** ([15, Theorem 9.2.1], [11, Theorem 10.19]). *Let $G$ be a group of finite Morley rank with a strongly embedded subgroup $M$. Then the following hold*:

(1) $\mathrm{Syl}_2(M) \subseteq \mathrm{Syl}_2(G)$.
(2) *$I(G)$ is a single conjugacy class.*
(3) *The involutions in $M$ are conjugate in $M$.*
(4) *If $i \in I(M)$ and $x$ is a nontrivial strongly real element in $C_G(i)$, then $C_G^*(x) \leqslant M$.*

**Fact 4.33.** *Let $G$ be a group of finite Morley rank with a strongly embedded subgroup $M$, and $X$ a normal subgroup of $M$ with an infinite Sylow 2-subgroup. Then $I(M) \subseteq X^\circ$.*

**Proof.** Otherwise, in view of point (3) above, $I(M) \cap X^\circ = \emptyset$, so $X$ has finite Sylow 2-subgroups. □

**Corollary 4.34.** *If $G$ is a group of finite Morley rank of even type with a strongly embedded subgroup $M$, then $I(M) = I(M^\circ)$.*

**Fact 4.35** [1, Proposition 3.4]. *Let $G$ be a group of finite Morley rank with a strongly embedded subgroup $M$. If $N$ is a proper definable subgroup of $G$ which contains $M$, then $N$ is strongly embedded in $G$.*

*4.12. Weak embedding*

**Definition 4.36.** Let $G$ be a group of finite Morley rank. A proper definable subgroup $M$ of $G$ is said to be *weakly embedded* in $G$ if

(i) $M$ has infinite Sylow 2-subgroups;
(ii) for every $g \in G \setminus M$, $M \cap M^g$ has finite Sylow 2-subgroups.

For groups with infinite Sylow 2-subgroups, this is a substantial generalization of the notion of strong embedding, and experience shows that in the case of groups of even type it is the notion one needs to work with ultimately. The following characterization is straightforward.

**Fact 4.37** [3]. *Let $G$ be a group of finite Morley rank, $M$ a proper definable subgroup of $G$. $M$ is weakly embedded if and only if the following hold*:

(i) *$M$ has infinite Sylow 2-subgroups.*
(ii) *For any nontrivial unipotent 2-subgroup $U$ and nontrivial 2-torus $T$ in $M$, $N_G(U) \leqslant M$ and $N_G(T) \leqslant M$.*

*4.13. The graph $\mathcal{U}(G)$*

**Definition 4.38.** Let $G$ be a group of finite Morley rank. Then the graph $\mathcal{U}(G)$ is defined as follows. The vertices of $G$ are the nontrivial 2-unipotent subgroups of $G$. The edges of $G$ are the pairs of distinct nontrivial 2-unipotent subgroups of $G$ which commute. $G$ acts naturally on $\mathcal{U}(G)$ by conjugation.

Our main interest is in the connected components of this graph (which is, however, usually connected). There is a variant of this graph, which we may call $\widehat{\mathcal{U}}(G)$, with the same vertices, but with edges consisting of pairs of nontrivial 2-unipotent subgroups which *normalize* rather than centralize each other. This graph has more edges than $\mathcal{U}$, but the same connected components, since if $U_1$ and $U_2$ normalize each other then they are at distance at most 2 in $\mathcal{U}(G)$, with $Z^\circ(U_1U_2)$ as an intermediate vertex.

**Fact 4.39** [3, Proposition 5.18, Corollary 5.19]. *Let $G$ be a group of finite Morley rank and $\mathcal{C}$ a connected component of the graph $\mathcal{U}(G)$. Let $\langle\mathcal{C}\rangle$ be the group generated by the vertices of $\mathcal{C}$, and let $M$ be the set-wise stabilizer of $\mathcal{C}$ in $G$. Then*

(1) *$M = N_G(\langle\mathcal{C}\rangle)$, so $M$ is definable.*
(2) *$\mathcal{U}(M) = \mathcal{C}$.*

**Fact 4.40** [3, Proposition 5.21]. *Let $H$ be a $B$-type $K$-group. If $\mathcal{U}(H)$ is not connected, then $H \simeq \mathrm{PSL}_2(K)$ for some algebraically closed field $K$ of characteristic* 2.

## 5. Theorem 1 (strongly embedded case)

The starting point for the classification of simple $K^*$-groups of even type is the following result of Jaligot.

**Fact 5.1** [19]. *A simple $K^*$-group of even type with a weakly embedded subgroup is isomorphic to* $\mathrm{PSL}_2(K)$, *where $K$ is an algebraically closed field of characteristic* 2.

The analysis is subtle, and departs considerably from the lines that would be taken in the corresponding case in finite group theory. As such, it provides an excellent test of our resources in the $L^*$-context. As we have mentioned, the first objective in the $K^*$-case was to show that the weakly embedded subgroup $M$ has $M^\circ$ solvable. In the $L^*$-context, we do not expect to achieve so much by a direct argument. Instead we aim at:

**Theorem 1.** *Let $G$ be a simple $L^*$-group of even type with a weakly embedded subgroup $M$. Then $M^\circ/O_2^\circ(M^\circ)$ is of degenerate type.*

Note that in a $K^*$-context, if the quotient $M^\circ/O_2^\circ(M)$ is of degenerate type, then it is solvable, and hence $M^\circ$ is solvable.

**Lemma 5.2.** *Let $M$ be an $L$-group of even type. Then the following are equivalent.*

(1) *$B(M)$ is solvable.*
(2) *$B(M) = O_2^\circ(M)$.*
(3) *$M/O_2^\circ(M)$ is of degenerate type.*

**Proof.** If $B(M)$ is solvable then, by Lemma 3.9, $B(M)/O_2^\circ(M)$ is trivial. Thus $B(M) = O_2^\circ(M)$. The converse is clear, so the first two conditions are equivalent.

Since $B(M)$ is the smallest normal definable subgroup $N$ of $M$ such that $M/N$ is of degenerate type, the equivalence of the last two conditions is clear. □

Turning to the proof of Theorem 1, in this section we will treat the more delicate case in which $M$ is strongly embedded, and in the following section we turn to the case of weakly embedded subgroups which are not strongly embedded, which follows more closely on earlier lines [4, Section 5].

There are two sources for the following line of argument: [1,4]. We begin along the lines of [1], then after Corollary 5.11 we go more in the direction of [4], with some deviations, notably in the treatment of Proposition 5.14. We will consider the group $M_1 = B(M)^{(\infty)}$, and aim eventually at $M_1 = 1$.

The following lemmas and propositions were proven in [1], based on analogous finite group theory results from [15, Chapter 9]; in order to make this line of argument work in the context of groups of finite Morley rank, one must work with the connected components of the groups involved. As the next lemmas provide all the basic building blocks, as well as the notation, for our argument, we run through this material in some detail.

**Fact 5.3** ([15, Theorem 9.2.1(iii)], [1, Lemma 3.8]). *Let $G$ be a group of finite Morley with a strongly embedded subgroup $M$. Then there is an involution $w \in G \setminus M$ such that* $\mathrm{rk}(I(wM)) \geqslant \mathrm{rk}(I(M))$.

**Proof.** This is a rank computation: one shows that "almost all" involutions have this property, by considering the rank of the set of involutions. □

**Notation 5.4.**

(1) Fix an involution $w$ with $\mathrm{rk}(I(wM)) \geqslant \mathrm{rk}(I(M))$ for the remainder of the argument.
(2) $Y = \{uw\colon u \in I(wM)\}$. (By choice of $w$, $\mathrm{rk}(Y) \geqslant \mathrm{rk}\, I(M)$.)
(3) $K = d(Y)$, the smallest definable group containing $Y$.
(4) $Y_0 = \{y \in K^\circ\colon y^w = y^{-1}\}$.
(5) $K_1 = d(Y_0) \leqslant K^\circ$.

**Fact 5.5** [1]. *Let $G$ be a group of finite Morley rank with a strongly embedded subgroup. Then the group $K = d(Y)$ as defined above contains no involutions.*

**Proof.** This is found in [15, Theorem 9.2.1] in the finite case and in [1, Proposition 3.9] adapted to the case at hand. □

**Fact 5.6** [1, Proposition 3.10]. *Let $G$ be a group of finite Morley rank with a strongly embedded subgroup $M$. Then for $i \in I(M)$, $M^\circ = C_G^\circ(i)K^\circ$.*

**Fact 5.7** [1]. *Let $G$ be a group of finite Morley rank with a strongly embedded subgroup $M$ containing infinitely many involutions. Then $K^\circ = C_{K^\circ}(w)Y_0$ and* $\mathrm{rk}(Y_0) = \mathrm{rk}(Y)$.

**Proof.** As $I(M)$ is infinite, by the choice of the involution $w$, $Y$ and hence $K$ are also infinite, and thus $K^\circ$ is nontrivial. Applying Fact 4.23 to $K^\circ$, with respect to the automorphism $\alpha$ induced by the action of $w$, we find $K^\circ = C_{K^\circ}(w)Y_0$ and $\mathrm{rk}(K^\circ) = \mathrm{rk}(C_{K^\circ}(w)) + \mathrm{rk}(Y_0)$. Applying the same fact to $K$ and the action of $w$ on $K$, we find similarly $\mathrm{rk}(K) = \mathrm{rk}(C_K(w)) + \mathrm{rk}(Y)$. Since $\mathrm{rk}(K^\circ) = \mathrm{rk}(K)$ and $\mathrm{rk}(C_{K^\circ}(w)) = \mathrm{rk}(C_K(w))$, we find that $\mathrm{rk}(Y_0) = \mathrm{rk}(Y)$. □

**Fact 5.8** [1]. *Let $G$ be a group of finite Morley rank with a strongly embedded subgroup $M$ containing infinitely many involutions. Then* $\mathrm{rk}(I(M)) = \mathrm{rk}(Y) = \mathrm{rk}(Y_0) = \mathrm{rk}(i^{Y_0})$, *where $i \in I(M)$.*

**Proof.** We deal first with $\mathrm{rk}(Y_0)$. We already have $\mathrm{rk}(Y_0) = \mathrm{rk}(Y) \geqslant \mathrm{rk}(I(M))$, by choice of $w$. For the reverse inequality, we claim that the involutions $i^y$ ($y \in Y_0$) are all distinct.

If $i^{y_1} = i^{y_2}$, with $y_1, y_2 \in Y_0$, let $x_1 = wy_1$, $x_2 = wy_2$; these are involutions in $wM$. We have by hypothesis $(x_2x_1)^w \in C(i)$. Assuming $x_1 \neq x_2$, we have $C_G^*((x_2x_1)^w) \leqslant M$ by Fact 4.32(4). Now $x_1^w$ inverts $(x_2x_1)^w$, so this forces $x_1^w \in M$, and hence

$$wM = x_1M = x_1x_1^wM = x_1wx_1wM = (y_1)^{-2}M = M,$$

a contradiction. So the involutions $i^y$ ($y \in Y_0$) are distinct, and $\mathrm{rk}(Y_0) = \mathrm{rk}(i^{Y_0}) \leqslant \mathrm{rk}(I(M))$. □

After these preparations we can begin to analyze the structure of $M^\circ$ when $G$ is a simple $L^*$-group of even type. Note that $I(M) = I(M^\circ)$ by Corollary 4.34.

**Definition 5.9.** $A = \langle I(M)\rangle$.

**Proposition 5.10** [4, Lemma 5.4]. *Let $G$ be a simple $L^*$-group of even type with a strongly embedded subgroup $M$ and $A = \langle I(M)\rangle$. Then $A \leqslant Z(B(M))$ is a definable connected elementary abelian 2-group such that $A = I(M) \cup \{1\}$.*

**Proof.** By Lemma 3.11 applied with $X = B(M)$ and $Y = K^\circ$, there is a Sylow° 2-subgroup $U$ of $M$ normalized by $K^\circ$. Let $i \in I(Z(U))$, $C = C(i)$, so that $U \leqslant C^\circ$. As $M^\circ = C^\circ K^\circ$ by Fact 5.6, we find

$$B(M) = \langle U^{M^\circ}\rangle = \langle U^{K^\circ C^\circ}\rangle = \langle U^{C^\circ}\rangle \leqslant C^\circ$$

which shows that $i \in Z(B(M))$. As $Z(B(M)) \lhd M$, Fact 4.32(3) implies that $I(M) \subseteq Z(B(M))$. It follows that $A \leqslant Z(B(M))$ and $A$ is an infinite elementary abelian 2-subgroup such that $A = I(M) \cup \{1\}$. Clearly $A^\circ \neq 1$ and $A \lhd M$. Therefore, another application of Fact 4.32(3) shows that $A = A^\circ$. □

**Corollary 5.11** [1, Corollary 4.6]. *Let $G$ be a simple $L^*$-group of even type with a strongly embedded subgroup $M$. If $a, i, j \in G^\times$ and $i$ and $j$ are involutions, with $i$ commuting with $a$ and $j$ inverting $a$, then $a$ is also an involution.*

**Proof.** We may assume that $i \in M$. Then $a$ in $M$ is strongly real and $j \in C_G^*(a)$ so, by Fact 4.32, also $j \in M$. Therefore $j, ja \in A$ and $a \in A$. □

**Corollary 5.12.** *Let $G$ be a simple $L^*$-group of even type with a strongly embedded subgroup $M$. Let $K_1$ and $Y_0$ be as in Notation* 5.4. *Then any subgroup of $M^\circ$ containing $Y_0$, in particular $K_1$, acts transitively on $I(M)$.*

**Proof.** Let $i, j \in I(M)$. By Proposition 5.10, $I(M)$ is a set of Morley degree 1. Thus by Fact 5.8, $i^{Y_0} \cap j^{Y_0} \neq \emptyset$. Let $x, y \in Y_0$ such that $i^x = j^y$, equivalently $i^{xy^{-1}} = j$. If $H$ is a subgroup containing $Y_0$ then we have $xy^{-1} \in H$. □

**Definition 5.13.** $M_1 = B(M)^{(\infty)}$; $\overline{M}_1 = M_1/O_2(M_1)$.

We will use this notation as well as the hypotheses of the last corollary in the rest of the argument.

**Proposition 5.14.** *If $M_1 \neq 1$ then $K^\circ$ is abelian.*

**Proof.** Let $K_0 = C_{K^\circ}(\overline{M}_1)$. Using the argument at the beginning of the proof of Lemma 3.10, we conclude that $\overline{M}_1$ is the central product of a finite number of quasisimple algebraic groups over algebraically closed fields of characteristic 2.

Fact 4.22 shows that $K^\circ$ acts on $\overline{M}_1$ by inner automorphisms. Since $K^\circ/K_0$ has no involutions (Fact 4.25), Fact 4.21 and Proposition 3.5 imply that $K^\circ/K_0$ has no simple sections, and is therefore isomorphic to a connected solvable subgroup of $\overline{M}_1$ without involutions, hence acts as a subgroup of a torus on $\overline{M}_1$. In particular, $K^{\circ\prime} \leqslant K_0$.

We will show next that

$$C_{K_0}(w) = 1. \tag{$*$}$$

For $x \in K_0$, Fact 4.13 applied with $Q = d(\langle x\rangle)$ shows that $C_{M_1}(x)$ covers $\overline{M}_1$. In particular, $C_M(x)$ has an infinite Sylow 2-subgroup $S$.

Suppose, in particular, that $x \in C_{K_0}(w)$ is nontrivial. Then $S, S^w \leqslant C(x)$ and hence $C^\circ(x) \cap M$ is strongly embedded in $C^\circ(x)$. By Lemma 3.12, $C^\circ(x) = L \times D$ with $L = B(C^\circ(x))$ and $D = C_{C^\circ(x)}(L)$, where $L \simeq \mathrm{PSL}_2(K)$ with $K$ algebraically closed of characteristic 2, and $L \cap M^\circ$ is a Borel subgroup of $L$. But then no subgroup of $C^\circ(x)$ can cover $\overline{M}_1$, a contradiction. So $(*)$ follows.

In particular, $C_{K^{\circ\prime}}(w) = 1$ and as $w$ acts on $K^{\circ\prime}$, it follows from Fact 4.23 that $w$ inverts $K^{\circ\prime}$. If $x \in K^{\circ\prime}$ is nontrivial then we have the following: $x$ centralizes an involution in $M$; $w$ inverts $x$. Then by Corollary 5.11 $x$ is an involution, a contradiction. Thus $K^{\circ\prime} = 1$. □

**Proposition 5.15.** *If $M_1 \neq 1$ then $O_2^\circ(M)$ is abelian.*

**Proof.** By the preceding lemma, $K_1$ is abelian, and since $K_1 = d(Y_0)$, $w$ inverts $K_1$. Hence, by Corollary 5.11, no nontrivial element of $K_1$ centralizes an involution in $M$.

As $K_1$ acts transitively on $I(M)$ and $A \leqslant O_2^\circ(M)$, the structure $(O_2^\circ(M), K_1)$ is a Suzuki 2-group of finite Morley rank. By our opening remarks, this is in fact a free Suzuki 2-group and therefore, by Fact 4.6, $O_2^\circ(M)$ is abelian. □

**Lemma 5.16** [4, Lemma 5.5]. *If $O_2^\circ(M)$ is abelian then* $[O_2^\circ(M), M_1] = 1$.

**Proof.** Let $X$ be any definable subgroup of $M_1$ without involutions. By Proposition 5.10, $X$ centralizes $A$. Then, since $A = \Omega_1(O_2^\circ(M))$, and $O_2^\circ(M)$ is of bounded exponent and is assumed to be abelian, Fact 4.24 implies that $X$ centralizes $O_2^\circ(M)$. Now the conclusion follows, since such subgroups generate $M_1$ by Lemma 3.10. □

Our final argument follows the lines of an argument given for the weakly embedded case in [4, Proof of Theorem 5.1].

**Proof of Theorem 1.** We claim that $M_1 = 1$. Assuming the contrary, then, by Proposition 5.15, $O_2^\circ(M)$ is abelian. Then Lemma 5.16 implies that $O_2^\circ(M_1) \leqslant Z(M_1)$. Using Facts 4.10–4.12 and Corollary 3.4, we conclude that $\sigma^\circ(M_1) = O(M_1) \times O_2^\circ(M_1)$. By Proposition 3.2, $O(M_1) \leqslant Z(M_1)$. Fact 4.18 then implies easily that $\sigma^\circ(M_1) = 1$. It follows that $A \cap M_1 = 1$. But $M_1$ has nontrivial Sylow 2-subgroups and hence meets $A$, a contradiction. □

## 6. Theorem 1 (weakly embedded case)

We now take up the case of Theorem 1 in which the weakly embedded subgroup $M$ is *not* strongly embedded. Since the proof in this case follows the solvability proof in [4, Section 5] very closely, we will simply outline the argument.

For the remainder of this section, $G$ will denote a simple $L^*$-group with a weakly embedded subgroup $M$ which is not strongly embedded. In this situation Fact 4.37 shows the existence of an *offending involution* $\alpha \in M$ whose centralizer is not contained in $M$. The following lemma is the starting point in the analysis.

**Lemma 6.1** [4, Lemma 3.1, Proposition 3.2]. *Let $G$ be a simple $L^*$-group of even type with a weakly embedded subgroup $M$ that is not strongly embedded. Then there exists an involution $\alpha \in M$ with the following properties*:

(1) $C_\alpha^\circ \not\leqslant M$.
(2) $C_\alpha^\circ \cap M$ *is a weakly embedded subgroup of* $C_\alpha^\circ$.
(3) $C_\alpha^\circ = L \times D$ *where* $L = B(C_\alpha)$, $D = C_{C_\alpha^\circ}(L)$, *and* $L \simeq \mathrm{PSL}_2(K)$ *with $K$ algebraically closed of characteristic* 2, *and $D$ is a definable connected subgroup of degenerate type.*
(4) $C_M^\circ(\alpha) = (L \cap M) \times D$ *and* $L \cap M$ *is a Borel subgroup of* $L$.
(5) $\alpha \notin \sigma^\circ(C_\alpha^\circ)$.

This is proved as in [4], taking into account Lemma 3.12. (See also Lemma 7.18.) Let $A = \Omega_1(L \cap M)$ and $T$ be a complement to $A$ in $L \cap M$. Next, corresponding to Proposition 3.5 and Corollary 3.6 in [4], one shows using Lemma 3.11 that $M$ has an $(\langle\alpha\rangle \times T)$-invariant Sylow° 2-subgroup $S$ that contains $A$, and that $C_S^\circ(\alpha) = A$. In this situation, the following classification result of Landrock–Solomon type applies.

**Fact 6.2** ([4, Theorem 4.1], [20]). *Let $H = S \rtimes T$ be a group of finite Morley rank, where $S$ is a unipotent 2-group and $T$ is also definable. Assume that $S$ has a definable subgroup $A$ such that $A \rtimes T \cong K_+ \rtimes K^\times$ for some algebraically closed field $K$ of characteristic 2, with the multiplicative group acting naturally on the additive group. Assume also that $\alpha$ is a definable involutory automorphism of $H$ such that $C_H^\circ(\alpha) = A \rtimes T$. Under these assumptions $S$ is isomorphic to one of the following groups*:

(i) *If $S$ is abelian then either $S$ is homocyclic with $I(S) = A^\times$, or $S = E \oplus E^\alpha$, where $E$ is an elementary abelian group isomorphic to $K_+$. In the latter case, $A = \{xx^\alpha\colon x \in E\}$.*
(ii) *If $S$ is non-abelian then $S$ is an algebraic group over $K$ whose underlying set is $K \times K \times K$ and the group multiplication is as follows*:

*for* $a_1, b_1, c_1, a_2, b_2, c_2 \in K$,

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = \left(a_1 + a_2, b_1 + b_2, c_1 + c_2 + \epsilon\sqrt{a_1 a_2} + \sqrt{b_1 b_2} + \sqrt{b_1 a_2}\right),$$

*where $\epsilon$ is either* 0 *or* 1. *In this case $\alpha$ acts by* $(a, b, c)^\alpha = (a, a + b, a + b + c + \sqrt{ab})$ *and* $[\alpha, S] = \{(0, b, c)\colon b, c \in K\}$.

*In particular, if $S$ is non-abelian then $S$ has exponent* 4.

One can then prove the following by straightforward adaptation of the corresponding arguments in [4]. As in the previous section, $M_1$ denotes $B(M)^{(\infty)}$, and we assume $M_1 \neq 1$.

**Lemma 6.3** [4, Lemma 5.3]. $O_2^\circ(M) \neq 1$.

**Lemma 6.4** [4, Lemma 5.4]. $A \leqslant Z(B(M))$.

**Lemma 6.5** [4, Lemma 5.5]. $[O_2^\circ(M), M_1] = 1$.

At this point one may conclude with a contradiction as at the end of the previous section. (This corresponds to the proof of Theorem 5.1 of [4].)

## 7. Groups of mixed type

This section is devoted to the analysis of simple groups of mixed type and the proof of Theorem 2 and its corollary.

**Theorem 2.** *Let $G$ be a simple group of finite Morley rank, all of whose proper definable infinite simple sections of even type are algebraic. Then $G$ cannot be of mixed type.*

**Corollary 7.1.** *Assume that all simple groups of finite Morley rank of even type are algebraic. Then there is no simple group of finite Morley rank and of mixed type.*

In other words, if one can handle the simple groups of finite Morley rank of even type, then the mixed type problem goes away. This line of argument is somewhat roundabout. The advantage of focussing on groups of even type is the good control one has of groups of $B$-type. Subgroups of $B$-type will also play a crucial role in this section.

The proof of Theorem 2 follows the proof of Fact 3.1 as given in [18]. There are two major steps. In the first step, a counterexample to Fact 3.1 is shown to have a weakly embedded subgroup. In the second step one shows that the group in question is strongly embedded: that is, a failure of strong embedding leads to a contradiction by considering fusion of involutions. On the other hand, once the group in question is strongly embedded, an easy fusion argument produces an immediate contradiction. The underlying idea is the following: in groups of mixed type there are two particular “kinds” of involutions, those associated with unipotent subgroups and those associated with 2-tori. Morally speaking, involutions of these two kinds “ought to” commute with each other; but since each kind is a union of conjugacy classes, this would contradict the simplicity of $G$.

In our proof, the first step, aiming at weak embedding, requires the more substantial alterations. The second step remains almost intact, except at one point (Lemma 7.21). Wherever deviations are necessary, they are due to the possible presence of nonsolvable degenerate sections; among other things, the possible existence of involutions in such sections must be borne in mind. The action of such a section on a minimal elementary abelian 2-group can be "linearized" and hence understood when the section is solvable, but this is not necessarily the case for sections of degenerate type in general.

In the proof of Theorem 2 we consider a counterexample $G$ of minimal Morley rank. Thus the infinite proper definable simple sections of this counterexample are not of mixed type; but some of them may be of degenerate type. One should also note that we do not assume that the sections of odd type are algebraic. However, $G$ is an $L^*$-group, and we will find sufficient scope for the application of $L$-group theory by considering subgroups of $B$-type.

### 7.1. *D-type groups*

The analysis of groups of mixed type, first in the "tame" $K^*$ setting and later in the general $K^*$ setting, depended on the study of the interaction of subgroups of $B$-type with a dual class, called $D$-type. The same analysis goes over in the more general context of $L$-groups, given Proposition 3.5.

**Definition 7.2.** Let $H$ be a group of finite Morley rank. $D(H)$ is the subgroup of $H$ generated by the definable closures in $H$ of its 2-tori; this is definable and connected by Zilber's Indecomposability Theorem.

$H$ is said to be of *D-type* if $H = D(H)$.

We first recall the essential $K$-group facts.

**Fact 7.3** [3, Corollary 5.8]. *Let $H$ be a $K$-group. Then $B(H)$ and $D(H)$ commute.*

**Fact 7.4** [3, Lemma 5.9]. *Let $H$ be a $K$-group of $B$-type. Then $D(H) = 1$.*

**Lemma 7.5.** *Let $H$ be a group of finite Morley rank which has no definable simple sections of mixed type, and such that all of its definable sections of even type are algebraic. Then $B(H)$ and $D(H)$ commute.*

**Proof.** It suffices to show that if $T$ is the definable closure of a 2-torus in $H$, then $T$ centralizes $B(H)$. Now $B(H)$ is a $K$-group by Corollary 3.6, and $T$ is abelian, so $B(H)T$ is a $K$-group. Applying Fact 7.3 to $B(H)T$, we find that $T$ centralizes $B(H)$, as required. □

### 7.2. *The critical configuration*

Our analysis will drive us toward a particular configuration involving $\mathrm{PSL}_2(K)$ in characteristic 2, which must be eliminated by a special close analysis. So we dispose of this in advance. The result in question is:

**Theorem 3.** *Let $G$ be a simple group of mixed type none of whose proper simple definable sections are of mixed type. Let $R$ be a maximal 2-torus of $G$. Then $B(C_G(R))$ is not of the form $\mathrm{PSL}_2(K)$, with $K$ algebraically closed of characteristic 2.*

This is an analog of [18, Théorème 4.2]. The proof requires extensive preparation. Ultimately, a contradiction is reached via the following quite technical fact.

**Fact 7.6** [18, Lemme 4.1]. *Let $G$ be a group of finite Morley rank with involutions $i, j, k, k'$, $L = B(C_G(k))$, satisfying the following properties.*

(1) *$i$ and $j$ are not conjugate*;
(2) *$L \simeq \mathrm{PSL}_2(K)$ with $K$ an algebraically closed field of characteristic* 2;
(3) *$k'$ is the unique involution in $d(\langle ij \rangle)$*;
(4) *$i \in L$, $j \in C_G(k)$, and $k' \notin L$.*

*Then $jk' \in L$.*

For the rest of the present subsection we assume that $G$ satisfies the hypotheses of Theorem 3. In addition, we fix the following notation.

**Notation 7.7.**

(1) $S$ is a Sylow° 2-subgroup of $G$.
(2) $S = U * R$ with $U$ 2-unipotent and $R$ a 2-torus.
(3) $L = B(C_G(R))$. Note that $U$ is then a Sylow° 2-subgroup of $L$.

We will assume, toward a contradiction, that

$$L \simeq \mathrm{PSL}_2(K) \quad \text{with } K \text{ an algebraically closed field of characteristic 2.}$$

In particular, $U$ is abelian.

An involution which lies in a 2-torus will be said to have *odd type*. An involution which lies in a 2-unipotent subgroup of $G$, will be said to have *even type*. As $G$ is of mixed type there are involutions of both types. Furthermore, by Corollary 4.28, there are involutions of even type which are not of odd type; we will say that such involutions are *properly* of even type. Note that because of the transitive action on $I(U)$ of a torus in $L$ normalizing $U$, all even type involutions are properly so. Eventually we will show that involutions of odd type commute with involutions which are properly of even type. This produces nontrivial commuting normal subgroups of $G$, and a contradiction.

**Lemma 7.8.** *If $i$ is an involution of even type, then $B(C(i))$ is conjugate to $U$ and $i$ belongs to a unique maximal unipotent* 2*-subgroup of $G$.*

**Proof.** After conjugating we may suppose that $i \in U$ and hence $R \leqslant D(C(i))$. By Lemma 7.5, we have $B(C(i))$ centralizing $D(C(i))$ and hence $B(C(i)) \leqslant B(C(R)) = L$. Hence $B(C(i)) = U$. □

**Notation 7.9.** If $i$ is an involution of even type, $U_i$ will denote the unique maximal unipotent 2-subgroup of $G$ containing $i$.

**Lemma 7.10.** *Let $i, j$ be involutions with $i$ of even type and $j$ of odd type. If $j$ normalizes $U_i$ then $j$ centralizes $U_i$.*

**Proof.** We may suppose that $j \in R$. Since $j$ normalizes $U_i$, it follows from Fact 4.9 that $C^\circ_{U_i}(j) \neq 1$. Let $A = C^\circ_{U_i}(j)$. By Lemma 7.5 applied to $C(j)$, $R$ centralizes $A$. By Lemma 7.5 applied to $C(A)$, $R$ centralizes $B(C(A))$, which contains $U_i$. As $j \in R$, $j$ centralizes $U_i$. □

**Lemma 7.11** [18, Corollaire 4.4]. *Let $i, i'$ be two commuting involutions of even type. Then $U_i = U_{i'}$.*

**Proof.** $i'$ normalizes $U_i$ and hence centralizes a nontrivial connected subgroup of $U_i$ (Fact 4.9). So $U_{i'} = U_i$. □

Let $I_\mathrm{e}$ be the set of involutions which are properly of even type, and $I_\mathrm{o}$ the set of involutions of odd type. As $G$ is simple, there must be involutions $i \in I_\mathrm{e}$, $j \in I_\mathrm{o}$ which do not commute. Fix such for the remainder of this subsection. We move in the direction of Fact 7.6, with $k = k'$ to be chosen as follows.

**Lemma 7.12** [18, Lemme 4.5]. *$d(\langle ij\rangle)$ contains a unique involution.*

**Proof.** Since $i$ and $j$ are not conjugate, by Fact 4.26 $d(\langle ij\rangle)$ contains at least one involution. For the uniqueness, in view of Fact 4.8, it suffices to show that $d(\langle ij\rangle)^\circ$ contains no nontrivial 2-tori.

Suppose on the contrary that $R_0$ is a nontrivial 2-torus in $d(\langle ij\rangle)^\circ$. Since $i$ inverts $ij$ and $d(\langle ij\rangle)$ is abelian, $i$ inverts $d(\langle ij\rangle)$. In particular, $i$ inverts $R_0$. But if $t \in I(R_0)$ then $[t, i] = 1$ and, by Lemma 7.10, $t$ centralizes $U_i$. So $i \in B(C(t))$ and, by Corollary 7.5, $i$ centralizes $R_0$, a contradiction. □

The unique involution of $d(\langle ij\rangle)$ will be denoted $k$.

**Lemma 7.13** [18, Lemme 4.6]. *$B(C_G(k)) \simeq \mathrm{PSL}_2(K)$ for some algebraically closed field $K$ of characteristic 2.*

**Proof.** Note that as $i, j$ do not commute, $j$ cannot normalize $U_i$ by Lemma 7.10.

Since $k$ commutes with $i$, it normalizes $U_i$ by Lemma 7.8, and centralizes a nontrivial connected subgroup $A_i \leqslant U_i$. As $k$ commutes with $j$, $k$ also centralizes $A_i^j \leqslant U_i^j$. Thus $L_k = B(C(k))$ contains two distinct Sylow° 2-subgroups and, as the maximal unipotent subgroups of $G$ have pairwise trivial intersections (Lemma 7.8), $\mathcal{U}(L_k)$ is disconnected. By Fact 4.40, $L_k \simeq \mathrm{PSL}_2(K)$ for some algebraically closed field $K$ of characteristic 2. □

**Proof of Theorem 3.** We will now arrive at a contradiction to the choice of $i$ and $j$. We may suppose $j \in R$. Take $i'$ an involution in $C^\circ_{U_i}(k)$ with $i'$ not conjugate to $j$ (since $I(C^\circ_{U_i}(k))$ is infinite, this can be done).

We apply Fact 7.6 to $i'$, $j, k$ (with $k' = k$ as well). There are four conditions to be verified. By our choice of $i'$ and $j$, they are not conjugate. The previous two lemmas verify conditions (2) and (3) of Fact 7.6. The last condition is

$$i' \in B\bigl(C_G(k)\bigr), \qquad j \in C_G(k), \quad \text{and} \quad k \notin B\bigl(C_G(k)\bigr).$$

Now $i' \in B(C_G(k))$ and $j \in C_G(k)$ by our choice of $i'$ and $k$, and $k \notin B(C_G(k))$, as otherwise we would have $k$ of even type, and $B(C_G(k)) = U_k$.

So Fact 7.6 applies and yields $jk \in B(C(k))$. In particular, $jk$ is of even type. Since $j$ centralizes $jk$, $j$ centralizes $U_{jk}$ by Lemma 7.10. By Corollary 7.5, $D(C(j))$ centralizes $B(C(j))$ and, in particular, $R$ centralizes $U_{jk}$, and hence $R$ centralizes $k$. Then by Lemma 7.5, $R$ centralizes $B(C(k))$, and, in particular, $j$ centralizes $B(C(k))$. Then $jk \in Z(B(C(k)))$, a contradiction to the simplicity of $B(C(k))$.

This contradiction shows that the properly even type involutions and the odd type involutions commute, producing commuting normal subgroups of the simple group $G$, a contradiction. □

### *7.3. The proof*

Now we proceed to the proof of Theorem 2.

**Notation 7.14.**

(1) $G$ is a simple group of finite Morley rank of mixed type, and of minimal rank. All the simple proper definable infinite sections of even type of $G$ are algebraic.
(2) $\mathcal{U} = \mathcal{U}(G)$ is the associated graph, with vertices nontrivial 2-unipotent subgroups of $G$ and edges consisting of pairs of vertices which commute.
(3) $S$ is a Sylow° 2-subgroup of $G$.
(4) $S = U * R$ with $U$ 2-unipotent and $R$ a 2-torus.

If $G$ were actually a $K$-group of mixed type, one would expect both $D(G)$ and $B(G)$ to be proper and normal in $G$, and one would expect $\mathcal{U}$ to be connected unless $G$ has $\mathrm{PSL}_2(K)$ as a normal subgroup with $K$ of characteristic 2. Now as it turns out that, if $\mathcal{U}$ is connected, one easily finds a nontrivial proper normal subgroup of $G$ of $D$-type, giving a contradiction. The "exceptional case" in which $\mathcal{U}$ is disconnected is in fact the only one that requires prolonged analysis. In this case one obtains a weakly embedded subgroup of $G$ by considering the stabilizer in $G$ of a connected component of $\mathcal{U}$.

**Lemma 7.15.** *The graph $\mathcal{U}$ is not connected.*

**Proof.** For $U$ a nontrivial 2-unipotent subgroup of $G$, let $D_U = D(C_G(U))$. Observe that, as $G$ is of mixed type, $D_U \neq 1$ for such $U$.

Evidently for $g \in G$ we have $D_{U^g} = D_U^g$. We claim that, for $(U_1, U_2)$ an edge of $\mathcal{U}$, we have $D_{U_1} = D_{U_2}$. If $\mathcal{U}$ is connected, this implies that $D_U$ is independent of the choice of $U$ and hence normal in $G$, contradicting the simplicity of $G$.

So fix $U_1, U_2$ commuting nontrivial unipotent 2-subgroups of $G$. Then $U_2$ normalizes $D_{U_1}$ and centralizes $D_{U_1}$ by Lemma 7.5. So $D_{U_1} \leqslant C_G(U_2)$, implying $D_{U_1} \leqslant D_{U_2}$. By symmetry $D_{U_1} = D_{U_2}$, as claimed. □

The construction of a weakly embedded subgroup follows the same line as in [18].

**Notation 7.16.** Let $M$ be the set-wise stabilizer, under the natural action of $G$, of the connected component $\mathcal{C}$ of the graph $\mathcal{U}$ which contains $U$.

Recall that $M$ is definable, and $\mathcal{U}(M) = \mathcal{C}$ (Fact 4.39).

**Theorem 4.** *$M$ is a weakly embedded subgroup of $G$.*

**Proof.** Observe that every connected component of $\mathcal{U}$ contains a maximal 2-unipotent subgroup of $G$, hence $G$ operates transitively on the set of connected components of $\mathcal{U}$, and in particular $M < G$. Furthermore, if $U$ is a nontrivial 2-unipotent subgroup of $M$, then $U \in \mathcal{C}$ by Fact 4.39, and hence $N_G(U) \leqslant M$. By the criterion for weak embedding given in Fact 4.37, it suffices to check the following:

$$\text{For any nontrivial 2-torus } T \text{ of } M, \quad N_G(T) \leqslant M.$$

Suppose on the contrary that $T$ is a 2-torus in $M$ such that $N_G(T) \nleqslant M$; that is, $N_G(T)$ does not stabilize $\mathcal{C}$ set-wise. Let $Q = B(N_G(T)) = B(C_G(T))$. Then $\mathcal{U}(Q) = \mathcal{U}(N_G(T))$ is disconnected. Thus Corollary 3.6 and Fact 4.40 imply that $Q \simeq \mathrm{PSL}_2(K)$ for some algebraically closed field $K$ of characteristic 2.

If $R$ is a maximal torus containing $T$, then $R \leqslant D(C(T))$ and Lemma 7.5 implies that $Q \leqslant C(R)$, so $N(R) \nleqslant M$ and thus also $B(N(R)) \simeq \mathrm{PSL}_2(K)$ for some algebraically closed field $K$ of characteristic 2. Now we can apply Theorem 3 to reach a contradiction and conclude that $N_G(R) \leqslant M$. □

This completes the first step of the proof of Theorem 2. In the second step we must show that the weakly embedded subgroup $M$ is in fact strongly embedded. This provides an immediate contradiction, since $G$ has at least two conjugacy classes of involutions, in view of Fact 4.32(2).

For this second part, we may proceed very much as in [18], with some deviation in Lemma 7.21. We will give the details, suitably adjusted. In what follows, $M$ may be taken to be any weakly embedded subgroup of $G$; its construction is no longer important.

**Theorem 5.** *If $M$ is a weakly embedded subgroup of $G$ then $M$ is strongly embedded.*

Assuming the contrary, as in Section 6, one has an *offending involution* $\alpha$ in $M$, whose centralizer is not contained in $M$. As in case of groups of even type, one can pin down the structure of this centralizer, as in the slightly more specialized situation of Lemma 6.1.

**Notation 7.17.** $I_M^-$ will denote the set of offending involutions in $M$.

**Lemma 7.18.** *Suppose $\alpha \in I_M^-$. Then the following hold*:

(1) $C(\alpha)^\circ = L \times X$ *with* $L = B(C(\alpha))$, $X = C_{C^\circ(\alpha)}(L)$, $L \simeq \mathrm{PSL}_2(K)$ *with $K$ algebraically closed of characteristic* 2, *and $X$ a group of degenerate type.*
(2) $M^\circ \cap L = A \rtimes T$ *is a Borel subgroup of $L$ and* $X \leqslant M$.
(3) $D(C(\alpha)) = 1$.
(4) *If* $i \in I(A)$ *then* $C(i) \leqslant M$.

**Proof.** Set $H = C(\alpha)^\circ$, $L = B(H)$, and $X = C_H(L)$.

(1) and (2). If $L \leqslant M$, then a Frattini argument shows that $C(\alpha) \leqslant M$, contradicting our hypothesis. So $L \nleqslant M$. So $L \cap M$ is a weakly embedded subgroup of $L$ and, as $L$ is a $K$-group, Lemma 3.12 gives the structure of $L$ and proves (2).

As $M$ is weakly embedded, $C_H(L) \leqslant M$ and, in particular, $X \leqslant M$. If $X$ has an infinite 2-Sylow subgroup then $C(X) \leqslant M$ by weak embedding, hence $L \leqslant M$, a contradiction. This proves (1) and (2).

(3) This follows from (1).

(4) If $i \in I(A)$ then $i$ is conjugate to an involution in $U$, so $C(i)$ contains a nontrivial 2-torus. By (3) $i \notin I_M^-$, so $C(i) \leqslant M$. □

**Notation 7.19.** For $\alpha \in I_M^-$, let $A_\alpha$ be $O_2(B(C(\alpha)) \cap M)$, which is the unique Sylow$^\circ$ 2-subgroup of $C(\alpha)$ contained in $M$.

Our goal now is to show that all involutions of odd type belong to $M$. Since this implies that $M$ contains a nontrivial normal subgroup of $G$, this will provide a contradiction completing the proof of Theorem 5 (and hence the proof of Theorem 2).

**Lemma 7.20** [18, Lemme 5.3]. *Suppose $j$ is an involution of odd type in $G \setminus M$, $\alpha \in I_M^-$, and $i \in A_\alpha$. Then $d(\langle ij\rangle)$ contains a unique involution $\beta$, and $\beta \in I_M^-$.*

**Proof.** All the involutions in $A_\alpha$ are conjugate (in $B(C(\alpha)) \cap M$). Thus by Corollary 4.28, $i$ and $j$ are not conjugate and, by Fact 4.26, $d(\langle ij\rangle)$ contains at least one involution $\beta$.

To prove the uniqueness of $\beta$, by Fact 4.8, it suffices to show that $d(\langle ij\rangle)$ does not contain a nontrivial 2-torus. Suppose $T$ is one such. Let $t \in I(T)$. Then $t \in C(i) \leqslant M$. By Lemma 7.18(3), $t \notin I_M^-$, so $C(t) \leqslant M$. Then $j \in M$, a contradiction. So $\beta$ is unique.

Now the involution $\beta$ of $d(\langle ij\rangle)$ commutes with both $i$ and $j$, and $C(i) \leqslant M$, $j \notin M$. So $\beta \in M$, but $C(\beta) \nleqslant M$, and thus $\beta \in I_M^-$. □

**Lemma 7.21** [18, Lemme 5.4]. *Every involution of odd type belongs to $M$.*

**Proof.** We suppose the contrary: $j \in G \setminus M$ is an involution of odd type.

Let $\alpha_0 \in I_M^-$ and $i_0 \in A_{\alpha_0}^\times$. By Lemma 7.20, $d(\langle i_0 j\rangle)$ contains a unique involution $\alpha_1$ and $\alpha_1$ is an offending involution. Now take $i_1 \in A_{\alpha_1}^\times$. A second application of Lemma 7.20 to $j$ and $i_1$ yields that $\alpha_2$ in $d(\langle i_1 j\rangle)$. Let $L_1 = B(C(\alpha_1))$.

We intend to apply Fact 7.6 with $i_1, j, \alpha_1, \alpha_2$ in the roles of $i, j, k, k'$. The first three hypotheses of that lemma hold by the choice of the involutions. The last hypothesis is

$$i_1 \in L_1, \qquad j \in C_G(\alpha_1), \quad \text{and} \quad \alpha_2 \notin L_1.$$

The only point that needs to be checked is the last.

If $\alpha_2 \in L_1$, then $\alpha_2$ is in a unipotent 2-subgroup of $G$. Hence $D(C(\alpha_2)) \neq 1$. This contradicts Lemma 7.18(3).

So Fact 7.6 applies and gives $j\alpha_2 \in B(C(\alpha_1))$. As $j \in C(\alpha_1)$, $j$ normalizes $B(C(\alpha_1))$. The action is by inner automorphisms, and $j$ centralizes $j\alpha_2$, so $j$ acts like an element of $A_{j\alpha_2}$. In particular, $j$ centralizes $A_{j\alpha_2}$, so $j\alpha_2 \in B(C(j))$. Let $R_j$ be a maximal 2-torus containing $j$. By Lemma 7.5, $D(C(j))$ and $B(C(j))$ commute, so $R_j$ centralizes $j\alpha_2$. Hence $R_j$ centralizes $\alpha_2$; but $D(C(\alpha_2)) = 1$ by Lemma 7.18(3), a contradiction. □

**Proof of Theorem 5.** Lemma 7.21 contradicts the simplicity of $G$. □

**Proof of Theorem 2.** Theorems 4 and 5, Fact 4.32, and Corollary 4.28. □

## Added in proof

Fact 4.19 is also noted and a proof is sketched in [24, Remarque 3].

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## References

[1] T. Altınel, Groups of finite Morley rank with strongly embedded subgroups, J. Algebra 180 (1996) 778–807.
[2] T. Altınel, Groupes de rang de Morley fini de type pair avec des sous-groupes faiblement inclus. Thèse d'habilitation, Université Claude Bernard, Lyon, France, 2001.
[3] T. Altınel, A. Borovik, G. Cherlin, Groups of mixed type, J. Algebra 192 (1997) 524–571.
[4] T. Altınel, A. Borovik, G. Cherlin, On groups of finite Morley rank with weakly embedded subgroups, J. Algebra 211 (1999) 409–456.
[5] T. Altınel, A. Borovik, G. Cherlin, Groups of finite Morley rank and even type with strongly closed abelian subgroups, J. Algebra 232 (2000) 420–461.
[6] T. Altınel, A. Borovik, G. Cherlin, Pushing up and $C(G,T)$ in groups of finite Morley rank of even type, J. Algebra 247 (2002) 541–576.
[7] T. Altınel, G. Cherlin, Simple $L^*$-groups of even type with strongly embedded subgroups, in preparation.
[8] T. Altınel, G. Cherlin, On central extensions of algebraic groups, J. Symbolic Logic 67 (1999) 68–74.
[9] T. Altınel, G. Cherlin, L.-J. Corredor, A. Nesin, A Hall theorem for $\omega$-stable groups, J. London Math. Soc. (2) 57 (1998) 385–397.
[10] A. Borovik, A. Nesin, Schur–Zassenhaus theorem revisited, J. Symbolic Logic 59 (1994) 283–291.

[11] A.V. Borovik, A. Nesin, Groups of Finite Morley Rank, Oxford University Press, 1994.
[12] A.V. Borovik, B. Poizat, Tores et $p$-groupes, J. Symbolic Logic 55 (1990) 478–490.
[13] M. Davis, A. Nesin, Suzuki 2-groups of finite Morley rank (or over quadratically closed fields), J. Group Theory, to appear.
[14] O. Frécon, Sous-groupes anormaux dans les groupes de rang de Morley fini résolubles, J. Algebra 229 (2000) 118–152.
[15] D. Gorenstein, Finite Groups, Chelsea, New York, 1980.
[16] J.E. Humphreys, Linear Algebraic Groups, second edition, Springer-Verlag, Berlin, 1981.
[17] E. Jaligot, Contributions à la classification des groupes simples de rang de Morley fini, PhD thesis, Université Claude Bernard–Lyon I, 1999.
[18] E. Jaligot, Groupes de type mixte, J. Algebra 212 (1999) 753–768.
[19] E. Jaligot, Groupes de type pair avec un sous-groupe faiblement inclus, J. Algebra 240 (2001) 413–444.
[20] P. Landrock, R. Solomon, A characterization of the Sylow 2-subgroups of $PSU(3,2^n)$ and $PSL(3,2^n)$, Technical Report 13, Aarhus Universitet, Matematisk Institut, January 1975.
[21] A. Nesin, On solvable groups of finite Morley rank, Trans. Amer. Math. Soc. 321 (1990) 659–690.
[22] A. Nesin, Generalized Fitting subgroup of a group of finite Morley rank, J. Symbolic Logic 56 (1991) 1391–1399.
[23] A. Nesin, Poly-separated and $\omega$-stable nilpotent groups, J. Symbolic Logic 56 (1991) 694–699.
[24] B. Poizat, Quelques modestes remarques à propos d'une conséquence inattendue d'un résultat surprenant de Monsieur Frank Olaf Wagner, J. Symbolic Logic 66 (2001) 1637–1646.
[25] F. Wagner, Stable Groups, in: London Math. Soc. Lecture Note Ser., Vol. 240, Cambridge University Press, Cambridge, UK, 1997.