Triangles in homogeneous 2-multi-tournaments: Case 1 (Notes)
Gregory Cherlin November, 2020

Contents

1. Statement of the problem

We conjecture that the triangle constraints in a homogeneous 2-multi-tournament determine an amalgamation class. This conjecture has been reduced to the elimination of four cases; these are patterns of forbidden triangles which do not define an amalgamation class, but for which every amalgamation problem of order 5 has a solution.

The elimination of this case by direct amalgamation methods is undoubtedly complicated, though there is a reasonable approach and in particular with some computer assistance it should be manageable. And there is the possibility that some new type of example would turn up, notably one involving some additional constraints of order 4, though the work done so far suggests this is unlikely, unless there is some underlying algebraic structure that can be exploited.

We prove a very fragmentary result here to illustrate that one can work effectively and systematically by focusing first on the consequences of 5-amalgamation.

It will be helpful to introduce some terminology.

Definition 1.1. A hereditary class of finite 2-multi-tournaments is *k-trivial* if every 2-multi-tournament of order at most k which contains no forbidden triangle belongs to the class.

Then we aim to prove something like the following, for each of the classes defined by a particular pattern of forbidden triangles which requires exclusion.

(*a*) The class is 4-trivial (relying mainly on amalgamations of order 5).
(*b*) If the class if 4-trivial, a contradiction results (using amalgamations of order 6).

More precisely, one should treat the second point first, seeing which configurations of order 4 actually play a role, and then prove the corresponding portion of the first point.

The basis for the second point would be a consideration of all the amalgamation diagrams of order 6 with no solution, in which the four auxiliary points are chosen so that each blocks one of the four possible 2-types by completing a forbidden triangle.

In each case one of the two factors of order 5 is forbidden, and if the class is 4-trivial then each of these factors can be represented in ten ways as the result of an amalgamation determining one 2-type; each such has, a priori, three possible solutions other than the forbidden factor, and in practice, fewer if forbidden triangles occur for some of the solutions.

We consider what this means in the specific case (which we call *Case 1*) in which monochromatic 3-cycles are forbidden and all other triangles are realized.

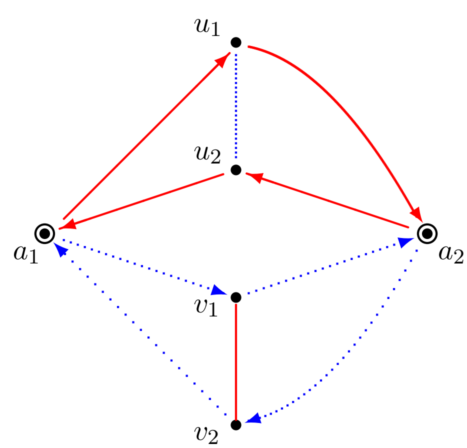
(Case 1) $C_3(111)$, $C_3(222)$ are forbidden.
All other triangles are realized.

The shorthand notation used here—C_3 for an oriented 3-cycle, and 111 for three 1-arcs, 222 for three 2-arcs—will be extended later to cover all tournaments of order 4 and used very heavily in the discussion of step (a).

2. Diagrams of order 6

The elimination of Case 1 is easy in the 5-trivial case. In the 4-trivial case (our step (b)) it is unclear, but we make a few comments.

The key amalgamation diagrams in the 5-trivial case have the following form.



This diagram has no solution, however it is completed, and it suffices to complete it without introducing any monochromatic triangles in (u_1, u_2, v_1, v_2).

At the cost of possibly interchanging a_1, a_2 we may take $a_1 \xrightarrow{2} a_2$. There are then 388 diagrams of this type, each imposing the condition that one of the two factors of order 5 must be forbidden. In the 4-trivial case, each of these factors can be viewed as an amalgamation diagram determining the type of one pair in ten ways, with factors present, so if a given factor is forbidden then it gives rise to ten positive conditions on variants which must be present.

In particular we have the extreme case of Figure 1 below.

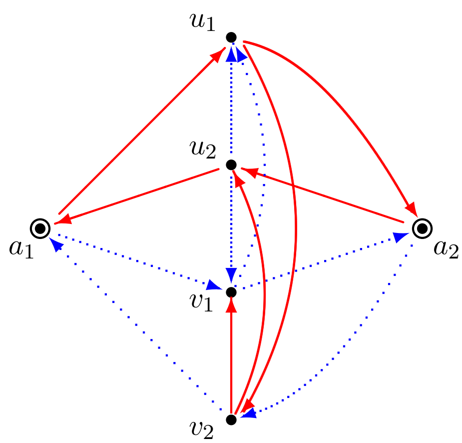


FIGURE 1. No solutions

Here, if the factor omitting a_2 is viewed as an amalgamation diagram determining the type of (u_1, u_2), then it has a unique solution. Thus assuming 4-triviality, the factor omitting a_1 must be forbidden. Viewing this factor also an amalgamation determining the type of (u_1, u_2), there is a unique solution other than the given factor, so this must be realized. We may also reach the same conclusions with 1-arcs or 2-arcs reversed, or colors interchanged.

It is not clear whether the desired contradiction can actually be reached staying within the realm of diagrams of order at most 6, but it is clear that one has a good deal to work with here.

3. Configurations of order 4

In the remainder of these notes we give some arguments using amalgamation diagrams of order at most 5 which are motivated by the problem of showing 4-triviality (or some approximation thereof). The main result is Lemma 3.4 which appears to be roughly half of the argument needed to show that a particular configuration is realized (along with all its variants under changes of language). As that already has a complicated argument—which would need to be checked in detail before continuing—we stop at that point, having laid out both a method (not very explicitly) and some useful notation.

On the methodological side, it will be clear that all of our arguments involve an alternating sequence of amalagamation arguments of just two types, and that one can lay out the tree of possible arguments of this type systematically. With relatively modest assumptions on the configurations of order 4 which are realized or forbidden this tends to lead to a reasonably efficient contradition in each case, and by subdividing cases according to whether certain key configurations are realized or forbidden, one can arrive at a useful result. As there are 132 2-multi-tournaments of order 4 one is not going to work through the 2^{132} possible specifications by a brute force computer search, but what one sees is the relevant exponent is not 132 but something closer to 6.

3.1. **Notation.** We introduce notation for 2-multi-tournaments of order 4.

Notation 3.1. The four tournaments of order 4 are denoted L_4 (transitive), C_4 (local order, not transitive), IC_3 (vertex dominating a 3-cycle), and C_3I (3-cycle dominating a vertex).

The vertices of a tournament of order 4 are labeled 1–4 according to the following scheme:

L_4	Natural order
C_4	4-cycle with $1 \to 3$, $2 \to 4$
IC_3	1 dominates the 3-cycle (2,3,4)
C_3I	the 3-cycle $(1, 2, 3)$ dominates 4

As we forbid monochromatic 3-cycles we may normalize further by taking the colors of the first two arcs in the 3-cycles in IC_3 or C_3I to agree.

A 2-multi-tournament of order 4 is denoted by the symbol for the underlying tournament and a list of the colors of the 6 edges in the following standardized order.

L_4	(1,2), (2,3), (3,4); (1,3),(2,4);(1,4)
C_4	(1,2), (2,3), (3,4), (4,1); (1,3), (2,4)
IC_3	(1,2), (1,3), (1,4); (2,3), (3,4), (4,2)
C_3I	(1,2), (2,3), (3,1); (1,4), (2,4); (3,4)

.

Thus our notations for 2-multi-tournaments of order 4 have the forms

$L_4(i_1i_2i_3; j_1j_2; k)$, $C_4(i_1i_2i_3i_4; j_1j_2)$, $IC_3(i_1i_2i_3; j_1j_2j_3)$, and $C_3I(j_1j_2j_3; i_1i_2i_3)$.

In what follows, we focus on the consideration of the configuration

$$L_4(112; 21; 1)$$

A first objective in Step (a) in the treatment of Case 1 would be the following.

Conjecture 1. *In Case 1, the configuration $L_4(112; 21; 1)$ is realized.*

Allowing for changes of language, this would give eight configurations of order four realized.

So far we have pursued the analysis far enough to conclude the following.

Proposition 3.2. *In Case 1, if the configuration $L_4(112; 21; 1)$ is forbidden, then its reversal $L_4(211; 12; 1)$ is realized.*

Allowing for changes of language, this has a total of 4 variants. An analysis similar to the proof of the proposition is likely to prove the conjecture.

We discuss this in detail.

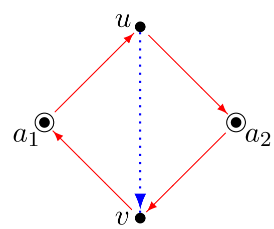
3.2. $C_4(1111;22)$ **and** $C_4(2222;11)$**.** For the proof of the proposition one can work exclusively with amalgamation diagrams of order at most 5. To begin with, amalgamation diagrams of order 4 yield the following simple but useful result.

Lemma 3.3. *In Case 1, the configurations*

$$C_4(1111;22) \text{ and } C_4(2222;11)$$

are realized.

Proof.



□

3.3. $C_4(1112;22)$**,** $L_4(112;11;2)$**,** $L_4(211;11;2)$**.** We will rely systematically on arguments based on amalgamation diagrams of order 4 or 5, which can be represented very compactly in a manner we now describe.

For expository purposes we deal with the following, which should really be treated as one claim in the proof of a lemma. The point is that this can be proved by a single application of a general method, and we wish mainly to establish our notation for the presentation of such arguments here.

Lemma 3.4. *With the forbidden and realized triangles as in Case 1, suppose the following.*

$$L_4(112;21;1),\ L_4(211;12;1),\ \text{and } L_4(122;11;1) \text{ are forbidden.}$$

Then $C_4(1112;22)$ *is realized.*

In the proof of this lemma, and again afterward, we use letter codes to refer to some key configurations of order 4. To begin with, we have the following.

$$\begin{array}{llll} A: & L_4(112;21;1) & C: & L_4(122;11;1) \\ B: & L_4(211;12;1) & D: & C_4(1112;22) \end{array}$$

In the proof of Lemma 3.4, we will assume that configurations A–D are forbidden, and arrive at a contradiction.

The proof is summarized in the following table. The notation will be explained in the proof.

Proof. We suppose toward a contradiction that the configuration $C_4(1112;22)$ is forbidden. We then arrive at a contradiction by a series of amalgamation

#	*Clause*	*Config.*	*Source*
1	Contradiction	$IC_3(111;221)_{1,2}$	#2, #3, #4
2	✗	$IC_3(111;221)$	#2.1, #2.2
3	✗	$IC_3(211;221)$	#3.1, #3.2
4	✗	$C_4(2121;21)$	#4.1, #4.2
2.1	✗ (#2, #2.2)	$B_{12}+C_{12}$	C ✗, A ✗, B ✗, #2.1.1
2.2	$D_{12}(2^-)$✓	$IC_3(221;221)$	D ✗, #3
3.1	✗ (#3, #3.2)	$C_{14}+A_{13}$	C ✗, A ✗, #4.1.1
3.2	$C_{34}(1^-)$✓	$L_4(111;12;1)$	C ✗, $3.2.1, #3.2.2
4.1	✗ (#4,#4.2)	$C_{14}+C_3(222)$	C ✗, D ✗, #4.1.1
4.2	✓	$C_4(2222;11)$	Lemma 3.3
2.1.1	✗	$L_4(212;21;1)$	#4.2, $2.1.1.1
3.2.1	✗	$L_4(121;11;1)$	#3.2.1.1, #3.2.1.2
3.2.2	✗	$L_4(112;12;1)$	#3.2.2.1, #3.2.2.2
4.1.1	✗	$C_4(1222;11)$	#3.1.1.1, #3.1.1.2
2.1.1.1	✗ (#2.1.1, #4.2)	$C_{14}+C_3(222)$	C ✗, D ✗
3.1.1.1	✗ (#3.1.1,#3.1.1.2)	$A_{14}+D_{24}$	A ✗, D ✗
3.1.1.2	✓	$C_4(1111;22)$	Lemma 3.3
3.2.1.1	✗ (3.2.1,#3.2.1.2)	$C_{14}+A_{13}$	See #3.1
3.2.1.2	$C_{12}(2)$✓	$L_4(222;11;1)$	C ✗, A ✗, #2.1.1
3.2.2.1	✗ (#3.2.2,#3.2.2.2)	$A_{13}+C_{12}$	C ✗, A ✗, #2.1.1
3.2.2.2	$C_{14}(2)$✓	$L_4(122;11;2)$	C ✗, #3.1.1

Table 1. Proof of Lemma 3.4

arguments summarized by Table 1. We discuss in detail how to read the table.

This table represents a sequence of numbered assertions, or more properly, a tree, with earlier assertions depending on later ones (with some apparent exceptions to this rule, which we will take note of).

The type of assertion made is indicated in the "Clause" column. In most cases the precise content of the assertion is further indicated in the "Configuration" column. The "Source" column refers to the supporting lines, or previously known facts. The references to supporting lines should give a tree structure, but to avoid duplication of branches some references cross over. For example, line #3.2.2.2 refers back to line #3.1.1 rather than duplicating the branch below that line.

The reading of a given line depends on which type of clause occurs. There are four types, two of which require amalgamation arguments, and one of which involve only tautologies, and one exceptional type which applies only to the first line. We discuss each of these types in detail, with a relevant example.

- The contradiction: Line 1.

Line 1 is exceptional: it is the statement that a contradiction has been reached. Typically we would expect the contradiction to be one of propositional logic (in which case no configuration would be given) but in the case at hand we actually refer to an amalgamation argument in a notation which is heavily used throughout. Namely, we see the following notation.

$$IC_3(111;221)_{1,2}$$

This means that the configuration $IC_3(111;221)$ is to be viewed as an amalgamation diagram of order 4 with the type of the pair (1,2) to be determined. There are four possible solutions, one of which must be realized. However the solution with $a_2 \xrightarrow{1} a_1$ involves a monochromatic 3-cycle and is discarded without comment in the table. The other three solutions, with $a_1 \xrightarrow{1} a_2$, $a_1 \xrightarrow{2} a_2$, $a_2 \xrightarrow{2} 1$ correspond respectively to the configurations shown in lines #2–4, and as we will see next, these lines assert that those configurations are not realized. Hence the "source" of the contradiction is lines #2, #3, #4, as listed in the "Source" column; and the configuration serves further to indicate the nature of the contradiction.

Note in particular that any configuration containing a monochromatic 3-cycle is dropped from explicit mention in the table.

- Forbidding a configuration. Example: Line #2.

The symbol " ✗" standing alone indicates that the configuration shown in the configuration column is forbidden. This is reduced to two other assertions, referenced in the source column, by propositional logic, as will be clear momentarily.

Now we come to the substantive assertions, relying on amalgamation arguments. These assertions may be either positive or negative.

- Realizing a configuration. Example: Line #2.2.

Here we have the clause entry "$D_{12}(2^-)$✓" and the configuration entry "$IC_3(221;221)$." The main content of this is the assertion

$$\checkmark IC_3(221;221)$$

meaning that the displayed configuration is realized. We have also the numerical code

$$D_{12}(2^-)$$

which provides a particular way of viewing the configuration $IC_3(221;221)$ which is relevant to the proof. This notation represents the configuration which is derived from the configuration D by replacing the type of $(1,2)$ by $1 \xleftarrow{2} 2$; here "2^-" is a code for "$\xleftarrow{2}$." The result is the desired configuration $IC_3(221;221)$.

The proof that the configuration $D_{12}(2^-)$ is realized goes as follows. View the configuration D as an amalgamation diagram of order 4 in which the type of the pair $(1,2)$ is to be determined. Eliminate all solutions other than $2 \xrightarrow{2} 1$ to conclude.

The source column gives the references for the elimination of the other solutions. One possible solution, $1 \xrightarrow{2} 2$, would produce a monochromatic 3-cycle and is therefore discarded without comment. The remaining solutions, corresponding to $1 \xrightarrow{1} 2$ or $2 \xrightarrow{1} 1$, give D and $IC_3(211;221)$, respectively. Consequently in the source column there is a reminder that D is forbidden, and a reference to line #3 where the configuration $IC_3(211;221)$ is discussed.

A simplified presentation occurs in the case of line #4.2, since the desired configuration is one which was treated earlier, and it suffices to cite the relevant lemma as a source.

- Forbidding one of a pair of configurations. Example: Line #2.1.

This is the most elaborate of the four types of assertion.

The notation " ✗ (#2,#2.2)" signifies that at least one of the configurations on display in lines #2, #2.2 is forbidden. Since line #2.2 states that the configuration shown there is realized, this justifies the assertion in line #2.

The method of proof in such cases involves an amalgamation diagram of order 5. This amalgamation diagram is one that can be shown to have no solution; hence one of its two factors must be forbidden, and these two factors will be the ones listed in the corresponding pair of lines. The source column supports the assertion that no solution to the amalgamation problem is realized; more specifically, for those solutions not giving rise to a monochromatic triangle, the source column will refer to a configuration of order 4 which embeds in the purported solution and can itself be shown to be forbidden.

To see this in detail, we first need to reconstruct the relevant amalgamation diagram. A rough description of that diagram will be found in the configuration column, using a subscript notation similar to the one we saw above.

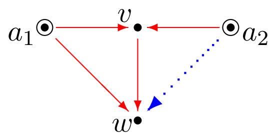
We will work through this in detail in the case of line #2, where the relevant description is as follows.

$$B_{12} + C_{12}$$

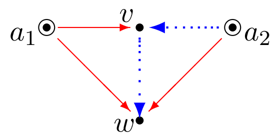
Here B_{12} and C_{12} refer to amalgamation diagrams of order 4 corresponding to B, C respectively, with the type of the pair (1,2) to be determined. These are as follows.

The two diagrams agree on the vertices a_1, a_2, w so we may form an amalgamation diagram of order 5 by combining the two as shown below.

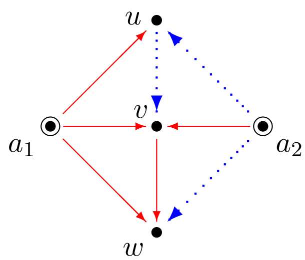
There is some ambiguity remaining, as we did not yet specify the type of (u,w) here. In fact, the argument that this diagram does not have a solution does not involve the type of (u,w), but of course the resulting factors do, and



B_{12}



C_{12}



$B_{12} + C_{12}$

the type must be chosen so as to produce the two factors already specified: $IC_3(111; 221)$ and $IC_3(221; 221)$. We take $w \xrightarrow{2} u$ to get those factors.

Now in the source column for line #2 we find four entries corresponding to the possible solutions of this diagram: $a_1 \xrightarrow{1} a_2$, $a_2 \xrightarrow{1} a_1$, $a_1 \xrightarrow{2} a_2$, or $a_2 \xrightarrow{2} a_1$. By the construction of the diagram, two of these solutions will involve either B or C, and a third solution turns out to be A. As these are all assumed to be forbidden we need only eliminate the last possibility, which is $L_4(212; 21; 1)$; this is dealt with in line #2.1.

This completes our discussion of line #2, and with this, we have covered all four types of line occurring in the table.

Now the whole table may be read in this manner and from this one can reconstruct an argument involving only two types of amalgamation diagram, those of order 4 giving rise to positive conditions, and those of order 5 giving rise to negative conditions. As this ends in a contradiction (working up the table to the first line), the lemma follows. □

Notice that the table given on page 6 contains all the information needed to reconstruct the full proof efficiently. We will continue to use this type of notation in subsequent arguments, to give a compact representation of proofs of this type.

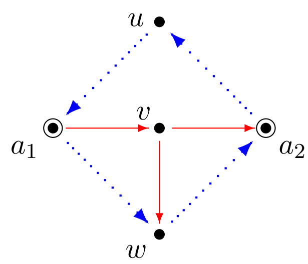
The previous lemma has some useful corollaries, based on the following.

Lemma 3.5. *Under the assumptions of Case 1, if $L_4(112; 21; 1)$ is forbidden and $C_4(1112; 22)$ is realized, then $L_4(122; 11; 2)$ is forbidden.*

Proof. Here the proof is very short but nonetheless we give it in tabular form.

As this is very short indeed, we elucidate once more. In line #1.1 the notation $A_{14} + C_3(222)$ references the following type of amalgamation diagram.

#	*Clause*	*Config.*	*Source*
1	✗	$L_4(122;11;2)$	#1.1, #1.2
1.1	✗ (#1, #1.2)	$A_{14}+C_3(222)$	A ✗
1.2	✓	$C_4(1112;22)$	Hyp.



$A_{14}+C_3(222)$

However this is completed, the only possible solution is $a_1 \xrightarrow{1} a_2$, and then A embeds. So this has no completion and one of the factors is forbidden.

Filling in the diagram with $w \xrightarrow{1} u$ and $v \xrightarrow{2} u$ yields the configurations of lines #1.2 and #1 as factors. □

As the hypotheses of Lemma 3.5 are preserved by reversal of arcs, the same applies to the conclusion. In particular, applying Lemma 3.4 we arrive at the following.

Corollary 3.6. *With the forbidden and realized triangles as in Case 1, suppose the following.*

$$L_4(112;21;1),\ L_4(211;12;1),\ \textit{and}\ L_4(122;11;1)\ \textit{are forbidden.}$$

Then the configurations

$$L_4(112;11;2), L_4(211;11;2)$$

are forbidden.

3.4. $L_4(122;11;1)$**.** As stated, our goal is to show, under the hypotheses of Case 1, that if $L_4(112;21;1)$ and $L_4(211;12;1)$ are both forbidden, then a contradiction results.

A significant step toward this is given by the following lemma.

Lemma 3.7. *Under the assumptions of Case 1, if $L_4(112;21;1)$ and $L_4(211;12;1)$ are forbidden then $L_4(122;11;1)$ is realized.*

We began working toward this in the previous section, where we also presented the notation we use to represent amalgamation diagrams of this type.

Proof. Suppose toward a contradiction that $L_4(112;211;1)$, $L_4(211;12;1)$, and $L_4(122;11;1)$ are all forbidden.

We recall some of our previous notation and introduce some additional notation for key configurations.[1]

A:	$L_4(112;21;1)$	E:	$L_4(122;11;2)$	I:	$C_4(1212;22)$
B:	$L_4(211;12;1)$	F:	$L_4(221;11;2)$	J:	$L_4(211;22;1)$
C:	$L_4(122;11;1)$	G:	$C_4(2111;22)$		
D:	$C_4(1112;22)$	H:	$C_4(1212;12)$		

So now we are assuming that configurations A, B, C are forbidden.

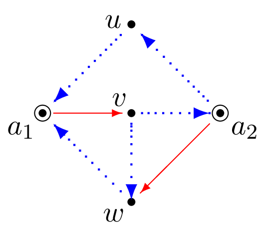
Claim 1. $L_4(221;22;1)$ is forbidden.

We suppose the contrary, and then argue as follows, where the notation is as in the previous case.

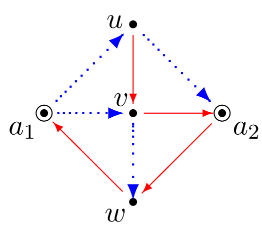
#	*Clause*	*Config.*	*Source*
1.	Contradiction		#2, #3, #4
2.	✗ (#3,#4)	$H_{13}+C_3(222)^*$ $(u \xrightarrow{2} v, u \xrightarrow{1} w)$	#2.1, #2.2, #2.3
3.	✓	$C_4(2222;11)$	Lemma 3.3
4.	✓	$L_4(221;22;1)$	Hyp.
2.1	✗	$H{=}C_4(1212;12)$	#2.1.1, #2.1.2
2.2	✗	$C_4(2121;21)$	#2.2.1, #2.2.2
2.3	✗	$C_4(1212;22)$	#2.3.1, #2.3.2
2.1.1	✗ (#2.1, #2.1.2)	$G_{13}+G_{14}$ $(w \xrightarrow{1} u)$	#4, #2.1.1.1
2.1.2	✓	$C_4(1111;22)$	Lemma 3.3
2.2.1	✗ (#2.2, #2.2.2)	$G_{24}+A_{14}$ $(w \xrightarrow{2} u)$	A ✗, #2.1.1.2
2.2.2	$G_{14}(2)$✓	$L_4(211;22;2)$	#4, #2.1.1.1
2.3.1	✗ (#2.3, #2.3.2)	$G_{13}+G_{14}$ $(u \xrightarrow{2} w)$	See 2.1.1
2.3.2	✓	$C_4(1112;22)$	Hyp.
2.1.1.1	✗	$G{=}C_4(2111;22)$	#2.1.1.1.1,#2.1.1.1.2
2.1.1.1.1	✗ (#2.1.1.1, #2.1.1.1.2)	$A_{14}+C_3(222)$; $(u \xrightarrow{2} v,\ w \xrightarrow{1} u)$	A ✗
2.1.1.1.2	$C_{14}(2^-)$✓	$C_4(1222;11)$	C ✗, #2.1.1.1.2.1
2.1.1.1.2.1	✗	$L_4(122;11;2)$	#2.1.1.1.2.1.1, #2.3.2
2.1.1.1.2.1.1	✗ (#2.1.1.1.2.1, #2.3.2)	$A_{14}+C_3(222)$ $(v \xrightarrow{2} u,\ w \xrightarrow{1} u)$	See #2.1.1.1.1

[1]Some of these may not be needed, in the end.

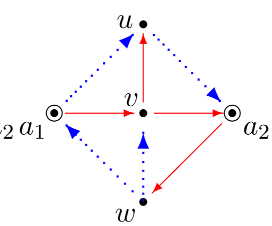
This time, the contradiction in line 1 is a matter of propositional logic. The amalgamation diagrams of order 5 are specified in detail and illustrated after the table.



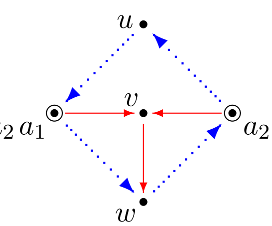
$H_{13} + C_3(222)$



$G_{13} + G_{14}$



$G_{24} + A_{14}$



$A_{14} + C_3(222)$

The proof of our next claim will be very similar but longer.

Claim 2. $C_3I(221;211)$ is realized.

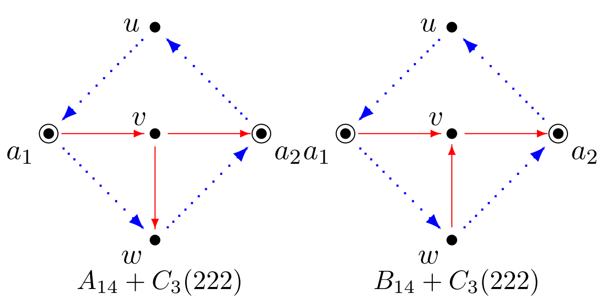
We suppose the contrary, and argue as shown in the following tables. The contradiction comes from forbidding all completions of the diagram E_{34}.

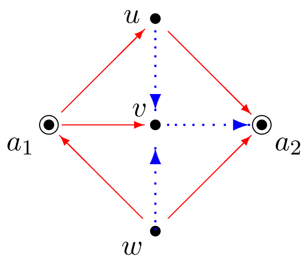
Part I

#	*Clause*	*Config.*	*Source*
1	Contradiction	E_{34}	#2, #3, #4, #5
2	✗	$E_{34}(1) = L_4(121;11;2)$	#2.1, #2.2
3	✗	$E_{34}(1^-) = L_4(111;22;1)$	#3.1, #3.2
4	✗	$E_{34}(2) = L_4(122;11;2)$	#4.1, #4.2
5	✗	$E_{34}(2^-) = L_4(112;22;1)$	#5.1, #5.2
2.1	✗ (#2, #2.2)	$F_{14} + F_{24}$	#2.1.1, #2.1.2
2.2	✓	$N_{13}(2) = L_4(222;11;1)$	#2.2.1, #2.2.2
3.1	✗ (#3, #3.2)	$G_{13} + G_{14}$	#3.1.1
3.2	✓	$K_{12}(2^-) = C_4(2121;21)$	#3.2.1, #3.2.2
4.1	✗ (#4, #4.2)	$A_{14} + C_3(222)$	A ✗
4.2	✓	$C_4(1112;22)$	Hyp.
5.1	✗ (#5, #5.2)	$J_{13} + A^*_{12}$	C ✗, A ✗, J ✗
5.2	✓	$I_{34}(1^-) = C_3I(221;212)$	#5.2.1, #5.2.2
2.1.1	✗	$L_4(221;11;2)$	#2.1.1.1, #4.2
2.1.2	✗	$IC_3(122;112)$	#2.1.2.1, #2.1.2.2
2.2.1	✗	$C_3I(221;112)$	#2.2.1.1, #2.2.1.2
2.2.2	✗	$L_4(222;21;1)$	#2.2.2.1, #2.2.2.2
3.1.1	✗	$C_4(2111;22)$	#3.1.1.1, #3.1.1.2
3.2.1	✗	$IC_3(111;221)$	#3.2.1.1, #3.2.1.2
3.2.2	✗	$IC_3(211;221)$	#3.2.2.1, #3.2.2.2
5.2.1	✗	$C_4(1212;22)$	#5.2.1.1, #4.2
5.2.2	✗	$C_3I(221;222)$	#5.2.2.1, #5.2.2.2

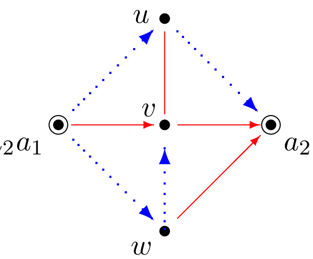
Tracing contradiction: Part II

#	*Clause*	*Config.*	*Source*
2.1.1.1	✗ (#2.1.1, #4.2)	$B_{14} + C_3(222)$	B ✗
2.1.2.1	✗ (#2.1.2, #2.2)	$J_{13} + B_{14}$	B ✗, J ✗
2.2.1.1	✗ (#2.2.1, #2.2.1.2)	$F_{14} + F_{24}$	See #2.1
2.2.1.2	✓	$C_4(2222; 11)$	Lemma 3.3
2.2.2.1	✗ (#2.2.2, #2.2.2.2)	$A_{24} + C_{14}$	A ✗, #4
2.2.2.2	✓	$A_{14}(2) = L_4(112; 21; 2)$	A ✗
3.1.1.1	✗ (#3.1.1, #3.1.1.2)	$A_{14} + C_3(222)$	See #4.1
3.1.1.2	✓	$C_{14}(2^-) = C_4(1222; 11)$	C ✗, #4
3.2.1.1	✗ (#3.2.1, #5.2)	$H_{14} + C_{14}$ (11)	C ✗, #4, #3.2.1.1.1
3.2.2.1	✗ (#3.2.2, #3.2.2.2)	$B_{14} + C_3(222)$	See #2.1.1.1
3.2.2.2	✓	$C_3I(221; 211)$	Hyp.
5.2.1.1	✗ (#5.2.1, #4.2)	$G_{13} + G_{14}$	See #3.1
5.2.2.1	✗ (#5.2.2, #5.2.2.2)	$H_{14} + C_{14}$ (12)	C ✗, #4, #3.2.1.1.1
5.2.2.2	✓	$L_{24}(2) = L_4(222; 22; 1)$	#2.2.2, #5.2.2.2.1
3.2.1.1.1	✗	$C_4(1212; 12)$	#3.2.1.1.1.1, #3.2.1.1.1.2
5.2.2.2.1	✗	$IC_3(221; 221)$	#5.2.2.2.2.1, #3.2.2.2
3.2.1.1.1.1	✗ (#3.2.1.1.1, #3.2.1.1.1.2)	$G_{13} + G_{14}$	See #3.1
3.2.1.1.1.2	✓	$C_4(1111; 22)$	Lemma 3.3
5.2.2.2.1.1	✗ (#5.2.2.2.2, #2.1.2.2)	$A_{14} + F_{14}$	A ✗, #2.1.1

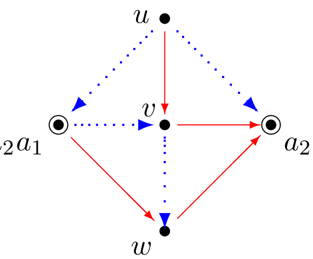




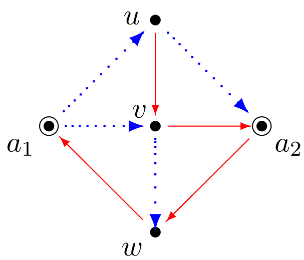
$A_{24} + C_{14}$



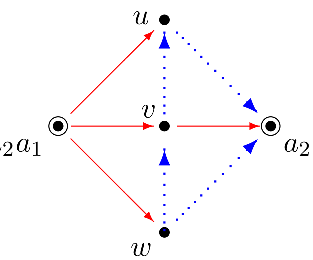
$F_{14} + (A_{14}, B_{14})$ (11)



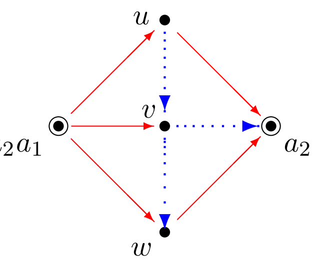
$F_{14} + F_{24}$



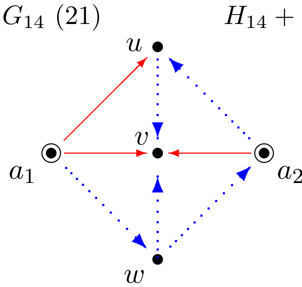
$G_{13} + G_{14}$ (21)



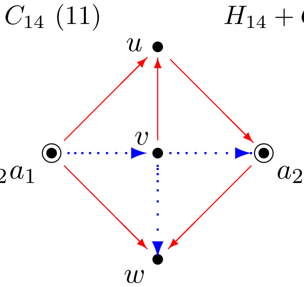
$H_{14} + C_{14}$ (11)



$H_{14} + C_{14}$ (12)



$J_{13} + A^*_{12}$ $(11')$



$J_{13} + B_{14}$

Now to conclude we make another elaborate argument in the same vein. We give the relevant illustrations in considerable detail.

See the illustrations following for more detail on diagrams of order 5.

Part I

#	*Clause*	*Config.*	*Source*
1	Contradiction		#2, #3, #4
2	✗ (#3, #4)	$L_{24} + C_3(111)$	#2.1, #2.2
3	✓	$L_{34}(2^-) = L_4(222;12;2)$	#3.1, #3.2, #2.2
4	✓	$N_{12}(2^-) = C_4(2221;21)$	#4.1, #4.2
2.1	✗	$IC_3(221;221)$	#2.1.1, #2.1.2
2.2	✗	$L_4(222;22;1)$	#2.2.1, #2.2.2
3.1	✗	$L_4(221;22;1)$	Hyp.
3.2	✗	$L_4(221;12;2)$	#3.2.1, #3.2.2
4.1	✗	$IC_3(122;221)$	#4.1.1, #4.1.2
4.2	✗	$IC_3(222;221)$	#4.2.1, #4.2.2
2.1.1	✗ (#2.1,# 2.1.2)	$E_{34} + I^*_{23}$	#2.1.1.1, #2.1.1.2, #2.1.1.3, #2.1.1.4
2.1.2	✓	$N_{13}(1) = IC_3(211;221)$	#2.1.1.4, #4.2
2.2.1	✗ (#2.2,# 2.2.2)	$J_{13} + F_{14}$	#2.2.1.1, #2.2.1.2, #2.2.1.3
2.2.2	✓	$F_{13}(2) = L_4(221;21;2)$	#2.2.1.3, K ✗
3.2.1	✗ (#3.2,# 3.2.2)	$K_{23} + C^*_{14}$	C ✗, #2.1.1.3, #3.2.1.1
3.2.2	✓	$C_{24}(2) = L_4(122;12;1)$	C ✗, #3.2.2.1
4.1.1	✗ (#4.1,# 4.1.2)	$F_{13} + I_{23}$	#2.2.1.3, K ✗, #2.1.1.4
4.1.2	✓	$G_{34}(2) = C_4(2121;22)$	#4.1.2.1, K ✗, $4.1.2.2
4.2.1	✗ (#4.2,# 4.2.2)	$J_{34} + I^*_{34}$	#4.2.1.1, #2.1.1.4, #3.1
4.2.2	✓	$L_{23}(1) = L_4(212;22;1)$	#2.2
2.1.1.1	✗	$C_4(2122;11)$	#2.1.1.1.1, #2.1.1.1.2
2.1.1.2	✗	$L_4(111;22;1)$	#2.1.1.2.1, #2.1.1.1.2
2.1.1.3	✗	$L_4(122;11;2)$	#2.1.1.3.1, #2.1.1.3.2
2.1.1.4	✗	$C_4(1212;22)$	#2.1.1.4.1, #2.1.1.3.2
2.2.1.1	✗	$L_4(211;11;2)$	#2.2.1.1.1, #2.1.1.1.2
2.2.1.2	✗	$C_3I(221;121)$	#2.2.1.2.1, #2.2.1.2.2
2.2.1.3	✗	$L_4(221;11;2)$	#2.2.1.3.1, #2.1.1.3.2
3.2.1.1	✗	$L_4(121;22;1)$	#3.2.1.1.1, #4.2.2
3.2.2.1	✗	$IC_3(111;221)$	#3.2.2.1.1, #2.1.2
4.1.2.1	✗	$C_4(2111;22)$	#4.1.2.1.1, #4.1.2.1.2
4.2.1.1	✗	$C_3I(221;212)$	#4.2.1.1.1, #4.2.1.1.2

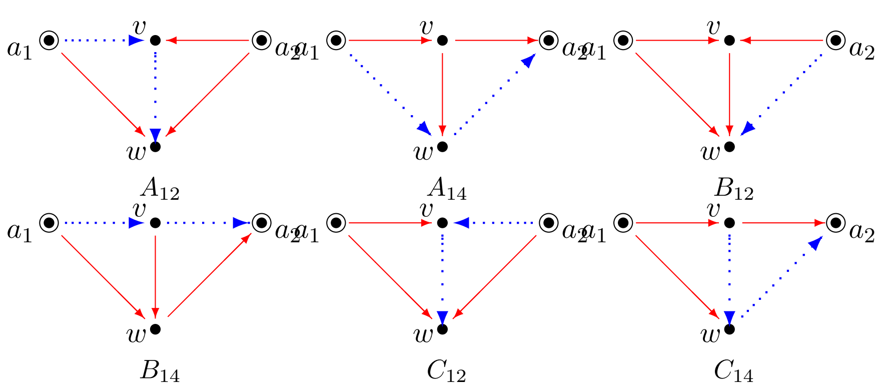
Part II

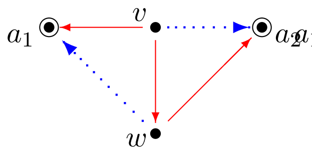
#	*Clause*	*Config.*	*Source*
2.1.1.1.1	✗ (#2.1.1.1, #2.1.1.1.2)	$A_{14} + C_3(222)$ $(b1)$	A ✗
2.1.1.1.2	✓	$G_{14}(1) = L_4(211; 22; 1)$	#4.1.2.1, 2.1.1.1.2.1
2.1.1.2.1	✗ (#2.1.1.2, #2.1.1.1.2)	$F_{14} + B_{14}$	B ✗, #2.2.1.3
2.1.1.3.1	✗ (#2.1.3.1, #2.1.3.1.2)	$A_{14} + C_3(222)$ (a)	See 2.1.1.1.1
2.1.1.3.2	✓	$C_4(1112; 22)$	Hyp.
2.1.1.4.1	✗ (#2.1.1.4, #2.1.1.3.2)	$G_{13} + G_{14}$	#4.1.2.1
2.2.1.1.1	✗ (#2.2.1.1, #2.1.1.1.2)	$F_{13} + J_{13}$	#2.2.1.3, K ✗, J ✗
2.2.1.2.1	✗ (#2.2.1.2, #2.2.1.2.2)	$H_{13} + H^*_{13}$	#2.2.1.2.2.1,, #2.1.1.4
2.2.1.2.2	✓	$I_{12}(1') = IC_3(212; 221)$	#2.1.1.4, #4.2
2.2.1.3.1	✗ (#2.2.1.3, #2.1.1.3.2)	$B_{14} + C_3(222)$	B ✗
3.2.1.1.1	✗ (#3.2.1.1, #4.2.2)	$C_{14} + C_3(222)^*$	C ✗, #2.1.1.3
3.2.2.1.1	✗ (#3.2.2.1, #2.1.2)	$B_{12} + A^*_{12}$	A ✗, C ✗, B ✗, #2.2.1.1
4.1.2.1.1	✗ (#4.1.2.1, #4.1.2.1.2)	$A_{14} + C_3(222)$ $(b2)$	See 2.1.1.1.1
4.1.2.1.2	✓	$C_{14}(2') = C_4(1222; 11)$	C ✗, #2.1.1.3
4.2.1.1.1	✗ (#4.2.1.1, #2.1.2)	$C_{14} + C_3(222)^*$	See 3.2.1.1.1
2.1.1.1.2.1	✗	$L_4(211; 22; 2)$	#2.1.1.1.2.1.1, #2.1.1.1.2.1.2
2.1.1.1.2.1.1	✗ (#2.1.1.1.2.1, #2.1.1.1.2.1.2)	$J_{13} + C_{12}$	C ✗, #2.2.1.2, J ✗
2.1.1.1.2.1.2	✓	$J_{13}(1) = L_4(221; 12; 1)$	#2.2.1.2, J ✗

This last contradiction completes the proof. (Illustrations follow.)

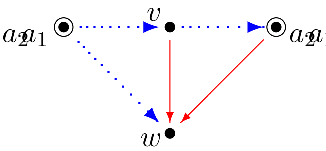
□

ILLUSTRATIONS

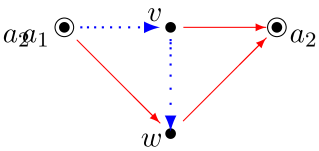




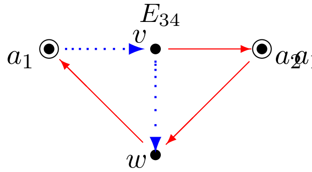
E_{34}



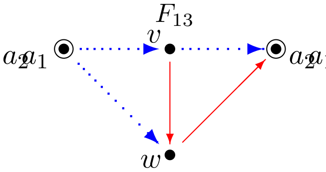
F_{13}



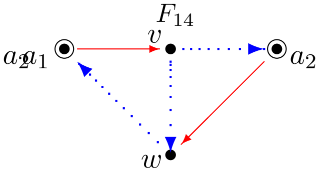
F_{14}



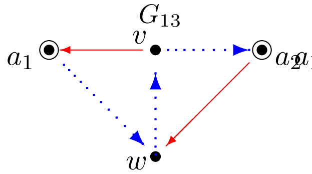
G_{13}



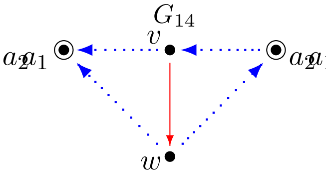
G_{14}



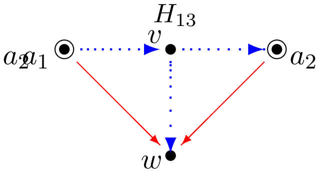
H_{13}



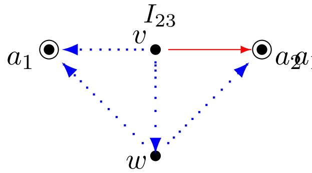
I_{23}



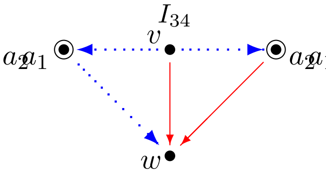
I_{34}



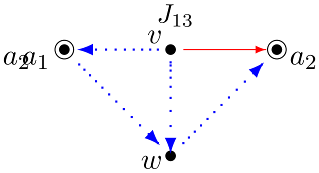
J_{13}



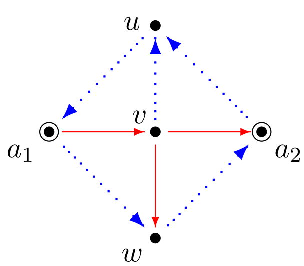
J_{34}



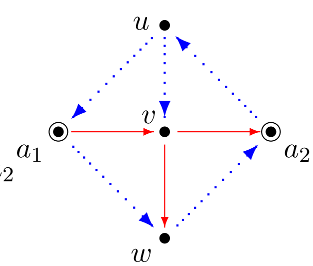
K_{23}



L_{24}

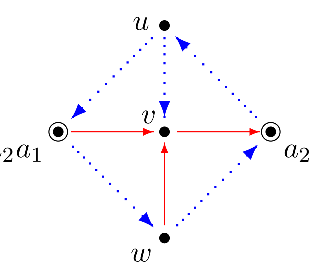


$A_{14} + C_3(222)$ (a)



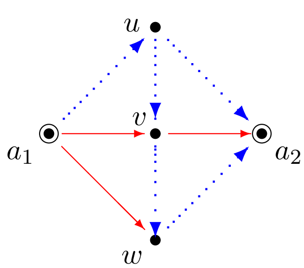
$A_{14} + C_3(222)$ $(b1, b2)$

$u \xrightarrow{1} w$ or $w \xrightarrow{1} u$



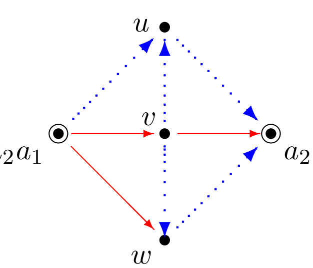
$B_{14} + C_3(222)$

$u \xrightarrow{1} w$



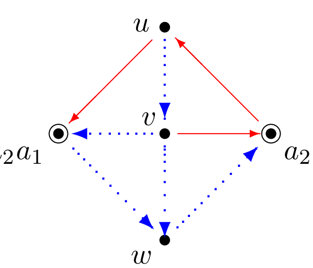
$C_{14} + C_3(222)^*$ (a)

$w \xrightarrow{1} u$



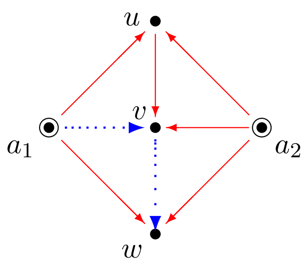
$C_{14} + C_3(222)^*$ (b)

$u \xrightarrow{1} w$

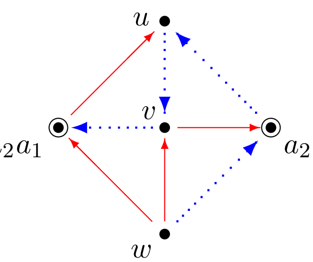


$L_{24} + C_3(111)$

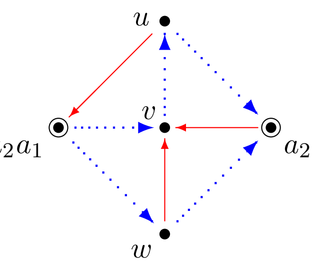
$u \xrightarrow{2} w$



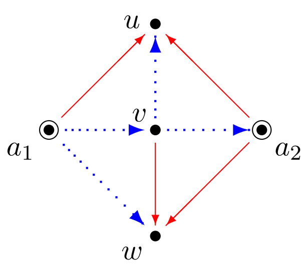
$A_{12} + B_{12}^{*}$ $(21')$
$u \xrightarrow{2} w$



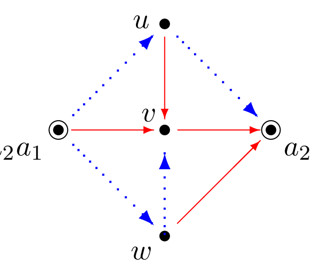
$E_{34} + I_{23}^{*}$ $(2'1)$
$w \xrightarrow{2} u$



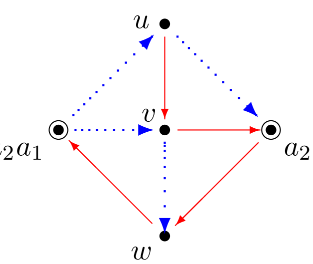
$F_{13} + I_{23}$ $(21')$
$w \xrightarrow{2} u$



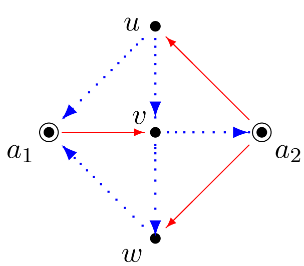
$F_{13} + J_{13}$ (22)
$w \xrightarrow{1} u$



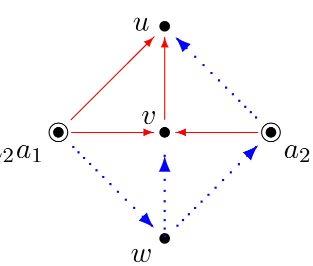
$F_{14} + B_{14}$ (11
$u \xrightarrow{1}$ 2)



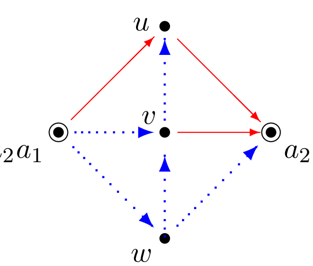
$G_{13} + G_{14}$ (21)
$w \xrightarrow{2} u$



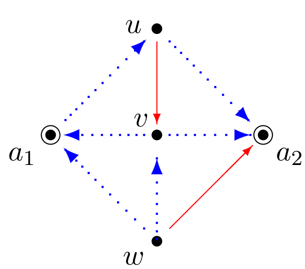
$H_{13} + H_{13}^{*}$ (12)
$u \xrightarrow{1} w$



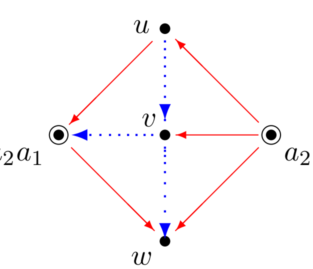
$J_{13} + C_{12}$ $(11')$
$w \xrightarrow{2} u$



$J_{14} + F_{14}$ (21)
$w \xrightarrow{2} u$



$J_{34} + I_{34}^{*}$ $(2'2)$
$w \xrightarrow{2} u$



$K_{23} + C_{14}^{*}$ $(2'1')$
$u \xrightarrow{2} w$