L^*-GROUPS OF ODD TYPE WITH RESTRICTED 2-TORAL ACTIONS IV. TOWARD THE IDENTIFICATION OF G_2

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Abstract. We have given an identification theorem for PSp_4 as part of an investigation of simple K^*-groups of finite Morley rank of odd type having Prüfer 2-rank 2 and 2-rank at least 3, and more generally in a suitable restricted odd type L^*-setting. Here we pursue the alternative line which should lead to G_2. One branch of the analysis arrives at the desired identification theorem. A second branch, at an early stage, leads to a configuration in characteristic three known also in finite group theory, but eliminated there through the use of character theory, in which a maximal unipotent subgroup is pathologically small. The latter configuration deserves further attention.
Note. Some material in §3 repeats portions of [BC22c], mainly because the division of the text has been adjusted at various times and involves some redundancy.

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Contents

1. Introduction 3
2. Preliminaries 5
2.1. L^*-groups and the condition NTA_2 5
2.2. Unipotence theory 5
2.3. Component Analysis 8
2.4. Auxiliary results and notation 10
2.5. Some subgroups 14
3. The Weyl group and unipotent subgroups 15
3.1. The Weyl group 15
3.2. The group N 15
3.3. Unipotent subgroups 17
3.4. Tori 18

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4. G_2 type: SL_3-type subgroups and root groups 20
4.1. Characteristic other than three (or 2) 20
4.2. Root subgroups 21
5. The pathological case: $N_G^\circ(\mathbb{X}) \leq C_G(i)$ 25
5.1. The base fields 25
5.2. Local analysis: a black hole principle 28
5.3. Involutions and rank computations 32
5.4. Invisible elements and dark matter 35
6. Identification of G_2 when $U \not\leq C_G(i)$. 39
6.1. Unipotent subgroups 39
6.2. The BN-pair 45
7. Appendix 49
Acknowledgments, and a note on the text 49
References 49

1. Introduction

We consider the Algebraicity Conjecture for groups of finite Morley rank in odd type with Prüfer 2-rank two and 2-rank at least 3. We work in the setting of odd type L^*-groups with restricted actions of 2-tori (the condition NTA_2 given in Definition 2.1). This includes the case of K^*-groups—groups whose proper definable simple sections are algebraic—which is the usual inductive setting for such problems. In that setting the aim is to identify the minimal possible counterexamples to the Algebraicity Conjecture as a whole.

By taking a somewhat wider setting we aim to identify the minimal configurations of odd type which would be counterexamples to the Algebraicity Conjecture in their own right—that is, we permit some degenerate type definable simple sections, but constrain their automorphism groups, and then proceed much as in the K^* setting. One of the obstacles to this program is a satisfactory treatment of the case of a "strongly embedded" subgroup, but in the case that concerns us here, this does not arise.

Prior work [BC22a] showed that in this wider setting all simple groups of finite Morley rank of odd type and Prüfer 2-rank at least three are algebraic, or else possess a strongly embedded subgroup, and a Sylow 2-subgroup which reduces to a 2-torus. That work generalizes, and follows closely in the footsteps of, an earlier treatment of the K^*-case.

In [BC22b, BC22c] we continued on to the case of Prüfer 2-rank two and 2-rank at least three. The lengthy paper [BC22b] analyzes centralizers of involutions in detail, and arrives at two possible configurations, of the sort encountered in PSp_4 and G_2. One useful distinction between the two cases is the following: the number of conjugacy classes of involutions is either one or two. The details are given in Fact 2.10. In higher Prüfer rank the corresponding analysis can be carried out more efficiently, more uniformly, and more simply.

It is shown in [BC22c] that the case of two conjugacy classes of involutions leads to the identification of the group as PSp_4 over an algebraically closed field. So the present paper takes up the case of one conjugacy class—with less success. The target in this case is the group G_2.

The initial configuration to be studied is given by Fact 2.10, part (2). This is supplemented by the determination of the Weyl group in [BC22c].

So our analysis here begins at a reasonably advanced stage. The aim then is to construct a split BN-pair and apply a result given in fullest generality in [Ten04] (or the earlier form of [TVM03], which covers the case in question), together with the classification of Moufang polygons of finite Morley rank.

As in the identification of PSp_4 in [BC22c], the idea is to define a "torus" $\mathbb{T}$, and a "Borel" subgroup $\mathbb{B}$ explicitly, taking $\mathbb{B} = \mathbb{T}U$ where U is a "maximal unipotent" subgroup in an abstract sense, and then work with the groups $\mathbb{N} = N_G(\mathbb{T})$ and $\mathbb{B}$. Again as in the case of PSp_4, one of the first question that arises is whether the group U is contained in the centralizer of an involution. This possibility is quickly eliminated in the setting corresponding to PSp_4 but remains alive here as a potential pathological configuration, very similar to a configuration encountered in the finite case as well, and which is eliminated there using character theory.

On what we consider the main branch of the analysis, where the group U as constructed does not fall into the centralizer of an involution, we identify the group G_2. The other branch appears to be recalcitrant but fairly tightly structured. In particular, the characteristic is three in that case. We would be quite happy at this point to invoke a K^*-hypothesis to dispose of this branch, if we knew how to do that, but we do not in fact see any major simplifications in that case. Still it is not out of the question that the additional techniques available in that setting could be brought to bear. There is a somewhat similar situation in the treatment of strongly embedded subgroups for which the extension to our context of L^*-groups of odd type with NTA_2 is still not known, but where the configuration can be eliminated in the K^* context by taking advantage of the rich theory of solvable groups of finite Morley rank.

A noteworthy earlier result under stronger hypotheses is found in [Alt98] (unpublished), which leaves aside the case of characteristic three.

2. Preliminaries

2.1. **L^*-groups and the condition** NTA_2**.** We work throughout in the context of groups of finite Morley rank of odd type.

Definition 2.1. Let G be a group of finite Morley rank and odd type.

Then G is an L^*-group if every proper definable simple section of odd type is algebraic.

G satisfies the condition NTA_2 ("no 2-toral actions") if any connected definable section which acts definably and faithfully on a degenerate type simple section of G is itself of degenerate type.

The 2-rank of a group is the maximal rank of an elementary abelian 2-subgroup (we work only with finite 2-rank).

A 2-torus is an abelian divisible 2-subgroup. The Prüfer 2-rank of group of finite Morley rank is the 2-rank of a maximal 2-torus.

The notion of L^*-group used in odd type differs from the notion used in even type. Earlier articles in this series have more comprehensive reviews of useful terminology and prior work on other aspects of the Algebraicity Conjecture, and in particular discuss the reasons for focusing on odd type and working with Prüfer 2-rank. We refer the reader to [BC22a] for a discussion of the merits of treating L^*-groups with NTA_2 rather than K^*-groups, and for the prior results that make this possible. Of course, the prior results which are still needed at this point in the analysis will be reviewed in this section.

The following captures the extent to which our hypotheses approximate the K^*-hypothesis.

Fact 2.2 ([BC22b])**.** *Let H be a connected L-group of finite Morley rank and odd type satisfying* NTA_2*. Let $\bar{H} = H/OF(H)$.*

Then

$$\bar{H} = E_{alg}(\bar{H}) * \bar{K} \text{ where } \bar{K} \text{ is connected and } \bar{K}/Z^\circ(\bar{K}) \text{ has degenerate type.}$$

In the K^* setting $\bar{K}$ would be solvable. In our setting, the extensive theory of solvable groups is generally not helpful, but a great deal can be done without it.

2.2. **Unipotence theory.** We review some aspects of the theory of unipotence in groups of finite Morley rank.

Definition 2.3. A *unipotence parameter* is either a prime p or a pair $(0, r)$ with $r \geq 0$.

For a prime p, a p-unipotent group is a definable connected nilpotent p-group, and the p-unipotent radical of a group H of finite Morley rank is the largest definable normal p-unipotent subgroup. One of the useful points is that in a solvable group H of finite Morley rank, every p-unipotent subgroup lies in the p-unipotent radical (hence in $F(H)$).

The "characteristic zero" unipotence theory uses the full range of parameters $(0, r)$, with the intuition being that larger values of r correspond to "more unipotent" subgroups, smaller values to "more semisimple" ones. One has notions of $(0, r)$-unipotence and the corresponding $(0, r)$-unipotent radical. It is not the case in general that all $(0, r)$-unipotent subgroups belong to the Fitting subgroup of a solvable group of finite Morley rank, but this does hold for the most unipotent subgroups: that is, for the $(0, r)$-unipotent subgroups corresponding to the largest value of r for which non-trivial $(0, r)$-unipotent subgroups exist. One therefore focuses attention on the parameter $\bar{\mathrm{r}}_0(H)$, defined as the largest such value of r.

Generally speaking the theory of $(0, r)$-unipotence runs parallel to the more straightforward p-unipotence theory. One recurring point of the $(0, r)$-unipotence theory is that by definition $(0, r)$-unipotent groups are generated by connected abelian $(0, r)$-unipotent groups, so that when entering into details one frequently returns to the abelian case.

We write $U_\pi(H)$ for the subgroup of H generated by its π-unipotent subgroups. We use this notation mainly in the case when this group coincides with the π-unipotent radical. In general (as in the case of algebraic groups) $U_\pi(H)$ need not itself be π-unipotent.

In the nilpotent case the theory becomes particularly transparent, and the U_π-notation becomes particularly useful.

Fact 2.4 ([Bur04, Thm. 2.31]; [Bur06, Cor. 3.6])**.** *Let G be a nilpotent group of finite Morley rank. Then $G = D * B$ is a central product of definable characteristic subgroups $D, B \leq G$ where D is divisible and B is connected of bounded exponent. Let T be the torsion part of D. Then we have decompositions of D and B into central products as follows.*

$$\begin{aligned} D &= d(T) * U_{0,1}(G) * U_{0,2}(G) * \cdots \\ B &= U_2(G) \times U_3(G) \times U_5(G) \times \cdots \end{aligned}$$

In particular, when k is a field of finite Morley rank, one has the following points.

(1) If the characteristic is non-zero then $\bar{\mathrm{r}}_0(k^\times) = \bar{\mathrm{r}}_0(k_+) = 0$.

(2) If the characteristic is zero then $\bar{\mathrm{r}}_0(k_+) = \mathrm{rk}(k_+) > \bar{\mathrm{r}}_0(k^\times)$. Thus, as one might hope, the additive group is more unipotent than the multiplicative group.

Definition 2.5. Let k be a field of finite Morley rank. The *Morley characteristic* $\chi_M(k)$ is defined as follows.

(1) If the characteristic is $p > 0$, the Morley characteristic is also p.

(2) If the characteristic is zero, the Morley characteristic is the pair $(0, \mathrm{rk}(k))$.

In other words, the Morley characteristic associates an abstract notion of unipotence to the field (or really, to its additive group).

We also require a partial order on Morley characteristics. The notation "$\pi' \geq_M \pi$" means the following.

(1) If $\pi = p > 0$: then $\pi' = \pi$.

(2) If $\pi = (0, r)$: then either π' is a prime p, or $\pi' = (0, r')$ and $r' \geq r$.

In this connection we have also the following, slightly rephrasing the foregoing.

Fact 2.6. *Let k be a field of finite Morley rank, π its Morley characteristic, and $\pi' \geq \pi$. Then $U_{\pi'}(k^\times) = 1$.*

Here (and throughout) one treats the classical case $\pi = p > 0$ and the case $\pi = (0, r)$ separately, but in parallel.

In the next lemma one is interested mainly in the special case of simple algebraic groups, where the definability hypotheses are automatically satisfied.

Lemma 2.7. *Let L be an affine algebraic group over an algebraically closed field k of characteristic zero, equipped with its structure as an* *algebraic group (with the field as an additional sort), and possibly additional structure. Let $r \geq \mathrm{rk}(k)$. Then a $(0,r)$-unipotent subgroup of L is a unipotent subgroup (non-trivial only if $r = \mathrm{rk}(k)$).*

Proof. Certainly k_+ is a $(0,r)$-unipotent group and therefore any unipotent subgroup of L is as well.

For the converse it suffices to consider an abelian $(0,r)$-unipotent subgroup of L. This is then a product of a unipotent group and a torus and as we have a $(0,r)$-unipotent group, the torus is trivial. □

A similar result, in a less transparent notation, is the following.

Fact 2.8 ([BC22b]). *Let G be a connected simple group of finite Morley rank which is an L^*-group of odd type satisfying the condition* NTA_2*, of Prüfer 2-rank at least two, and 2-rank at least three.*

Let t be an involution of G.

Then

$$\Delta_\rho(C_G(t)) = E_{\mathcal{E}}(C_G(t)).$$

Here ρ is the maximum "reduced rank" of the multiplicative group of the base field of a component of an involution and Δ_ρ is the subgroup generated by all p-unipotent subgroups and all $(0,r)$-unipotent subgroups for which $r > \rho$.

The fact implies that for r at least the maximum rank of any such base field of characteristic zero, any $(0,r)$-unipotent subgroup of $C_G(t)$ will be contained in $EC_G(t)$ (hence unipotent).

One important ingredient of the proof, of major importance in its own right, is the following. This comes from the signalizer functor theory.

Fact 2.9. *Under the hypotheses of Fact 2.8, for any involution i we have*

$$\bar{\mathrm{r}}_0(OFC_G(i)) \leq \rho.$$

In addition, $OFC_G(i)$ is torsion-free.

2.3. **Component Analysis.** We discuss the main results of [BC22b], passing quickly over those which lead to identification as PSp_4.

We use the notation $E_{alg}(H)$ for the product of the algebraic components of $E(H)$; there may also be degenerate type quasisimple sections, but we tend to ignore them.

From [BC22b, Thm. 1.2] we have the following two possibilities.

Fact 2.10. *Let G be a connected simple L^* group of finite Morley rank of odd type satisfying the condition* NTA_2*, with Prüfer 2-rank 2 and*

$$m_2(G) \geq 3.$$

Then there are at most two conjugacy classes of involutions, and one of the following applies.

(1) *There are two conjugacy classes of involutions.*

Then the 2-rank of G is 4; and the Sylow 2-subgroup is as in PSp_4.

One conjugacy class of involutions satisfies

$$C_G^\circ(i) \simeq \mathrm{PSL}_2(k) \times k^\times,$$

and the other class satisfies

$$C_G(i) \simeq \mathrm{SL}_2(k) *_2 \mathrm{SL}_2(k),$$

with the two components of $\mathrm{SL}_2(k) *_2 \mathrm{SL}_2(k)$ *conjugate.*

(2) *There is one conjugacy class of involutions, and these satisfy*

$$C_G(i) = \mathrm{SL}_2(k_1) *_2 \mathrm{SL}_2(k_2)$$

where the base fields k_1, k_2 have the same characteristic. Furthermore, in characteristic zero, we have

$$\bar{\mathrm{r}}_0(k_1^\times) = \bar{\mathrm{r}}_0(k_2^\times)$$

in the sense of characteristic zero unipotence theory (§3.3).

In particular $C_G(i)$ is connected. Furthermore, $C_G(i)$ contains a Sylow 2-subgroup of G, isomorphic to that of $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ *(in characteristic other than 2).*

The notation $*_2$ denotes a central product with central involutions identified.

The first case was treated in [BC22c].

Fact 2.11 ([BC22c, Theorem 1.8])**.** *Let G be a group of finite Morley rank and odd type. Suppose that G is a simple L^*-group satisfying* NTA_2*, with Prüfer 2-rank two and 2-rank at least three, and having precisely two conjugacy classes of involutions. Then G has the form* $\mathrm{PSp}_4(k)$ *for some algebraically closed field k.*

So we take up the second case. That is, we will work with the following assumptions.

Hypothesis 2.12. G is a simple group of finite Morley rank, an L^*-group of odd type satisfying NTA_2, with one conjugacy class of involutions. For any involution i we have

$$C_G(i) = L_{i,1} *_2 L_{i,2}$$

with $L_{i,\ell} \simeq \mathrm{SL}_2(k_\ell)$ where the base fields have the same characteristic and their multiplicative groups have the same reduced rank. We suppose $\mathrm{rk}(k_1) \geq \mathrm{rk}(k_2)$, and that for fixed ℓ the components $L_{i,\ell}$ (with i varying) are conjugate. We let $\pi = \chi_M(k_1)$.

For L a component of the centralizer of an involution in G, with base field k, we define the Morley characteristic $\chi_M(L)$ as the Morley characteristic of the base field k in the sense of Definition 2.5.

We will sometimes quote results from prior work for groups satisfying Hypothesis 2.12 which were in fact proved previously in greater generality. One should consult the prior works directly if one wants to know the appropriate level of generality for a given result. In this paper we have no reason to consider other cases (and in the case that leads to the identification of PSp_4, that result tends to make such questions moot).

2.4. **Auxiliary results and notation.** We record some useful principles.

2.4.1. *Torsion subgroups.*

Definition 2.13. For P a set of primes, a *P-torus* is a divisible abelian P-group. We write Π for the set of all primes, so that a Π-torus is a maximal divisible abelian torsion group.

The definable hull of a Π-torus is called a *decent torus*. If all definable subgroups of a definable group are decent tori, it is called a *good torus*

The following is part of the deep background and is for the most part taken advantage of without particular comment, but it is worth noting explicitly.

Fact 2.14 ([Che05]). *Let G be a group of finite Morley rank. Then any two maximal Π-tori of G are conjugate; equivalently, any two maximal decent tori are conjugate.*

Fact 2.15 ([Fré06, Lemma 3.1]). *Let G be a group of finite Morley rank, N a definable normal subgroup of G. Let T be a maximal decent torus in G. Then TN/N is a maximal decent torus of G/N, and every maximal decent torus of G/N has this form.*

Fact 2.16 ([ABC08, Prop. II 11.7]). *Let P be a set of primes. Let H be a connected solvable $P^\perp$-group of finite Morley rank acting faithfully on a nilpotent P-group U of bounded exponent. Then H is a good torus.*

The proof is by reduction to the case of the multiplicative group of a field of nonzero characteristic (Wagner).

Fact 2.17 ([AB08, Theorem 1]). *If G is a connected group of finite Morley rank and T is a p-torus of G, then $C_G(T)$ is connected.*

Fact 2.18 ([ABC99, Prop. 2.43], [ABC08, Prop. I.9.12]). *Let $G = H \rtimes T$ be a group of finite Morley rank, $Q \lhd H$, and π a set of primes, such that Q, H, T are definable and*

- *Q and T are solvable;*
- *T is a π-group of bounded exponent;*
- *Q is a T-invariant $\pi^\perp$-subgroup.*

Then

$$C_{H/Q}(T) = C_H(T)Q/Q$$

Fact 2.19 ([ABCC03], [Bur09, Lemma 3.5]). *Let G be a connected solvable $p^\perp$-group of finite Morley rank, and let P be a finite p-group of definable automorphisms of G. Then $C_G(P)$ is connected.*

If in addition G is a nilpotent $(0, r)$-unipotent group then $C_G(P)$ is a $(0, r)$-unipotent group.

Remark 2.20. In the foregoing result it suffices to have P a locally finite p-group as the centralizer of P will then be the centralizer of a finite subgroup. The case of particular interest has $p = 2$ and P a 2-torus.

Fact 2.21 ([BC09, Theorem 3*])**.** *Let G be a connected group of finite Morley rank and odd type. Then any 2-element in G lies in some 2-torus of G.*

Fact 2.22 ([ABC08, I.10.4])**.** *Let G be a group of finite Morley rank without involutions, and α a definable involutory automorphism of G. Then*

$$G = C_G(\alpha) \times G^-$$

(i.e., the multiplication map going from right to left is bijective).

Here G^- is the subset inverted by α.

The following is a variation on [BN94, Prop. 13.4] (a simple bad group has no definable involutive automorphism).

Lemma 2.23. *Let G be a connected group of finite Morley rank of degenerate type, and let α be an involutive automorphism of G. Then $\bigcup C_G(\alpha)^G$ is disjoint from $G^- \setminus \{1\}$, where $G^- = \{x \in G \mid x^\alpha = x^{-1}\}$.*

Proof. The group G has no involutions. By Fact 2.22 we have $G = C_G(\alpha)G^-$.

Suppose $g \in G$ and $x \in G^- \cap C_G(\alpha)^g$. We may take $g \in G^-$. So

$$\begin{aligned} x^{g^{-1}} &= (x^{g^{-1}})^\alpha = (x^\alpha)^{g^{-\alpha}} = (x^{-1})^g; \\ x^{g^{-2}} &= x^{-1}. \end{aligned}$$

Thus $g^4 \in C(x)$, but $g \in d(\langle g^4 \rangle)$, so $g \in C(x)$ and $x^{-1} = x$, $x = 1$. $\square$

We give the next result in its original form, which is considerably more general than is needed here.

Fact 2.24 ([Wag97, Theorem 2.4.3])**.** *If G is a stable group of generic exponent 3, then G is nilpotent of exponent 3.*

2.4.2. *Solvable groups, Carter subgroups, genericity.*

Fact 2.25 ([ABC08, Lemma 8.3]). *Let G be a connected solvable group of finite Morley rank. Then $G/F^\circ(G)$ is divisible abelian.*

Definition 2.26. Let G be a group of finite Morley rank. A *Carter subgroup* of G is a definable connected nilpotent subgroup of finite index in its normalizer.

Fact 2.27 ([Fré00, Cor. 5.20]). *If C is a Carter subgroup of a connected solvable group G of finite Morley rank and N is a normal subgroup of G (not necessarily definable), then the image of C in G/N is a Carter subgroup of G/N, and all Carter subgroups of G/N have this form. In particular, the Carter subgroups of G/N are conjugate.*

We are only interested in the case in which N is definable but we retain the generality of the original statement. The conjugacy statement is less subtle in that case but in any case this is not the aspect that will concern us here.

There is one point here which we will find convenient to put in another form. It is likely that this formulation also occurs in the literature, but we have not noticed a suitable reference.

Lemma 2.28. *If C is a Carter subgroup of a connected group G of finite Morley rank and N is a definable normal connected solvable subgroup of G, then the image of C in G/N is again a Carter subgroup of the quotient.*

Proof. Otherwise $N_G(CN)/N$ is infinite. Therefore there is a connected subgroup H of $N_G(CN)$ with H/CN infinite abelian. This violates Fact 2.27 applied to H. □

Fact 2.29 ([BBC07, Lemma 4.1]). *Let G be a group of finite Morley rank, H a definable subgroup of G, and X a definable subset of G. Suppose that*

$$\mathrm{rk}(X \setminus \bigcup_{g \notin H} X^g) \geq \mathrm{rk}(H)$$

Then $\mathrm{rk}(\bigcup X^G) = \mathrm{rk}(G)$.

Fact 2.30 ([BC09, Thm. 1 (1,2)]). *Let G be a connected group of finite Morley rank, p a prime, and let a be a generic element of G. Then*

- *the element a commutes with a unique maximal p-torus T_a of G, and*
- *the definable hull $d(\langle a\rangle)$ contains T_a.*

2.5. **Some subgroups.**

Fact 2.31 ([Poi01, Théorème 1]). *Si K est un corps de rang de Morley fini et de caractéristique p non nulle, tout sous-groupe definissable simple G de $\mathrm{GL}_n(K)$ est définissablement isomorphe à un groupe algébrique sur K.*

Fact 2.32 ([BC08, Theorem 2.1]). *Let G be a connected L-group of finite Morley rank and odd or degeneate type. Let V be an elementary abelian 2-group acting definably on G.*

Then $\Gamma_V = G$.

The following emerges from component analysis.

Fact 2.33 ([BC22b, Lemma 7.8]). *Let G be a group of finite Morley rank satisfying Hypothesis 2.12 and having one conjugacy class of involutions. Let i be an involution and let L be an $\mathcal{E}$-component of $C_G(i)$.*

Let H be a definable proper subgroup of G containing L and having 2-rank at least 2. Let $\hat{L}$ be the normal closure of L in H and $Q = OF(\hat{L})$.

Then one of the following applies.

(1) *$Q = 1$ and $\hat{L} \leq E_{alg}(H)$.*
   (a) *$H \leq C_G(i)$, or*
   (b) *$E_{alg}(H)$ is of type SL_3.*

   or

(2) *$Q > 1$ and $\hat{L} = Q \cdot L$.*
   (a) *i inverts Q, or*
   (b) *$C_G(i) = E_{alg}(C_G(i)) = E_{\mathcal{E}}(C_G(i))$ has two components of type SL_2. Their base fields have the same characteristic and, in the case of characertistic zero, the same rank.*

3. The Weyl group and unipotent subgroups

We now move towards the construction of a BN-pair and the resulting identification of G_2, with a "bad branch" still to come.

3.1. **The Weyl group.** The first part of the analysis concerns the definition and structure of the Weyl group, and this runs parallel to the analysis for PSp_4. This topic was treated with sufficient generality in [BC22c]. We recapitulate here for the case of present interest.

Notation 3.1. Let G be a group of finite Morley rank of odd type.

For T a maximal 2-torus of G, and for i an involution in T, set

$$W_T = N_G(T)/C_G(T); \qquad W_i = N_{C(i)}(T)/C_{C(i)}(T).$$

We call these Weyl group of G, and of $C(j)$, respectively (with respect to T). As one may replace T by $d(T)$, these groups are quotients of definable groups.

By conjugacy of maximal 2-tori the Weyl group $W = W_T$ of G is well-defined up to conjugacy.

W_i may be identified with a subgroup of W_T, namely the image of $N_{C(i)}(T)$ in W_T.

In particular, any element of W_i will be considered also as an element of W_T.

Fact 3.2 ([BC22c, Lemma 3.6])**.** *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Let T be a maximal 2-torus of G and i an involution of T.*

Then the Weyl group of G is a dihedral group of order 12, with generators $\bar{w}_1, \bar{w}_2, \bar{\sigma}$ where w_ℓ is an element of $N_{L_{i,\ell}}(T)$ of order 4, and $\sigma \in N(T)$ is an element of order three acting as the 3-cycle (i, j, ij) on $\Omega_1(T)$.

3.2. **The group N.** Here we continue to overlap with [BC22c] but nonetheless we give the details relevant to the case of G_2.

Definition 3.3. Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Let T be a maximal 2-torus of G. We define a subgroup N of $N(T)$ as follows.

For an involution i of G, let x_i be a representative for the Weyl group $W_{L_{i,1}}$ in $L_{i,1}$. That is, $x_i \in L_{i,1}$ has order 4 and inverts $T \cap L_{i,1}$ while centralizing $L_{i,2}$. Let w_i be an involution of $C_G(i)$ inverting T.

Take two involutions $i, j \in T$ and set

$$N = d(T)\,\langle x_i, x_j, w_i \rangle\,.$$

Lemma 3.4. *With hypotheses and notation as in Definition 3.3, the element $x_j x_i$ represents an element of order three in the Weyl group, operating on $\Omega_1(T)$ as the 3-cycle*

$$(i, j, ij).$$

Proof. The involution i acts as the transposition (j, ij) and j acts as the transposition (i, ij) on $\Omega_1(T)$ so the action of $x_j x_i$ is clear.

Consider the action of $(x_j x_i)^3$ on $T \cap L_{i,1}$.

For $a \in L_{i,1}$ we have $i^{x_j x_i} = j$ and thus $a^{x_j x_i} \in L_{j,1}$, so for $a \in T \cap L_{i,1}$ we have

$$\begin{aligned} a^{(x_j x_i x_j)} &= (a^{-1})^{x_j x_i}; \\ a^{(x_j x_i)^3} &= (a^{-1})^{(x_j x_i)(x_i x_j x_i)} = (a^{-1})^{x_i} = a. \end{aligned}$$

Thus $(x_j x_i)^3$ centralizes $T \cap L_{i,1}$. As $(x_j x_i)^3$ also centralizes $(x_j x_i)$, it centralizes $T \cap L_{j,1}$ and $T \cap L_{ij,1}$. As any two of these 2-tori generate T, $(x_j x_i)^3$ centralizes T and $(x_j x_i)$ acts as a Weyl group element of order three. □

Lemma 3.5. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Let N be defined as above.*

Then

$$C_N(T) = N^\circ = d(T); \qquad N(T) = C(T) \cdot N; \qquad N/d(T) \simeq W_T.$$

Proof. By construction $d(T) \le N$. Since $N/d(T)$ is finite, in view of Fact 2.17 we have $C_N(T) = C_N^\circ(T) = d(T)$. So $C_N(T) = N^\circ = d(T)$.

The statements $N(T) = C(T) \cdot N$ and $N/d(T) \simeq W_T$ are equivalent. We have seen that the group N induces Sym_3 on $\Omega_1(T)$, and the element inverting T has been included. □

Lemma 3.6. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Let N be defined as above.*

Then the following hold.

(1) Any proper definable connected simple algebraic section L of G is of type PSL_2 *or* PSL_3*.*

(2) There is no proper connected definable subgroup of G containing N.

Proof.

Ad 1. Otherwise, the section L is of type G_2. Let $L = H/K$ with K normal and definable in H.

By Fact 2.2 $E_{alg}(H/OF(H)) \simeq L$. So we may suppose $L = H/OF(H)$. By Fact 2.18 the centralizer of an involution in H covers the centralizer in L.

We consider an involution i of H. Then $C_H(i)$ has a quotient of type $\mathrm{SL}_2 *_2 \mathrm{SL}_2$ and it follows that $C_G(i)$ is contained in H. Then by Fact 2.33 we find $H \le C_G(i)$, for a contradiction.

This proves the first point.

Ad 2. Suppose H is a proper connected definable subgroup of G containing N. Then the Sylow 2-subgroup and Weyl group of H agree with that of G_2, so H has a definable section of type G_2, for a contradiction. □

3.3. **Unipotent subgroups.** This section reviews a number of facts from [BC22c, §3.3]. concerning a "maximal unipotent" subgroup U in the sense of Fact 3.11. This is a problematic subject in our context and is associated with a possible pathological configuration.

Fact 3.7. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Let T be a maximal 2-torus of G, and let U be a maximal definable T-invariant connected nilpotent group.*

Then U is maximal among definable connected nilpotent subgroups of G, and $N^\circ(U)/\sigma^\circ(U)$ is of degenerate type.

Fact 3.8. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Let T be a maximal 2-torus of G, and let U be a maximal definable connected T-invariant nilpotent subgroup of G.*

Let π be either a prime different from the characteristic of any base field of a component of the centralizer of an involution, or a symbol

$(0, r)$ with r greater than the Morley rank of the base field of any component of the centralizer of an involution. Then

$$U_\pi(U) = 1$$

Notation 3.9. Let G be a group of finite Morley rank satisfying Hypothesis 2.12. let T be a maximal 2-torus of G (fixed, and not generally referred to explicitly in the associated notation). Let i be an involution of T.

For $\ell = 1$ or 2, $\mathbb{B}_\ell$ denotes some Borel subgroup of $L_{i,\ell}$ normalized by (thus, containing) T for $\ell = 1, 2$, and $\mathbb{B} = \mathbb{B}_1\mathbb{B}_2$. We write $\mathbb{X}_\ell$ for the unipotent radical of $\mathbb{B}_\ell$ and we set $\mathbb{X} = \mathbb{X}_1\mathbb{X}_2$.

Let U be a fixed maximal definable connected nilpotent subgroup of G which contains $\mathbb{X}$ and is T-invariant (hence U is also maximal among definable connected nilpotent subgroups of G).

Fact 3.10. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. With notation as in 3.9, we have*

$$U \cap d(T) = 1.$$

Fact 3.11. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Let i be an involution.*

Then with notation as in 3.9, we may choose U to contain a maximal definable nilpotent U_π-subgroup of G.

The main question—which confronts us immediately—is whether or not $U = \mathbb{X}$. When $U > \mathbb{X}$ our analysis proceeds to an identification of G_2. When $U = \mathbb{X}$ our analysis will be inconclusive. We shall return to this topic shortly.

3.4. **Tori.** We continue with our review of [BC22c, §3], specialized (as always) to our current context.

Notation 3.12. Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Fix a maximal 2-torus T of G.

Then we let (correspondingly) $\mathbb{T} = C_G(T)$.

Remark 3.13. N normalizes $\mathbb{T}$.

This is clear by the construction of N.

We think of $\mathbb{T}$ as the "algebraic" torus containing T. This is somewhat justified by the following.

Fact 3.14.

(1) $\mathbb{T} = \mathbb{T}_1\mathbb{T}_2$ *with* $\mathbb{T}_\ell$ *the algebraic torus of* $L_{i,\ell}$ *containing* $T \cap L_{i,\ell}$*.*

(2) For $t \in I(T)$*, and* K *a component of* $E_{alg}(C_G(t))$*, the group* $\mathbb{T} \cap K$ *is a maximal torus of* K*.*

(3) $U \cap \mathbb{T} = 1$

Fact 3.15. $\mathbb{T}$ *normalizes the group* U*.*

Furthermore, U *is generated by unipotent subgroups of algebraic components of centralizers of involutions* $t \in I(T)$*.*

This gives us the point of departure for our analysis.

4. G_2 type: SL_3-type subgroups and root groups

As in the case of PSp_4, the key to building a BN-pair is to show that our subgroup U is not contained in $C_G(i)$; cf. [BC22c]. But this is more difficult in the case of groups of type G_2 than in the case of PSp_4. In the finite case, one uses character theory to settle this point when the characteristic of the base field is three.

In characteristic not three, we will find a group of type SL_3 embedded as the centralizer of an element of order three, whose maximal unipotent subgroup links the long root groups in $C_G(j)$ for $j \in \Omega_1(T)$. In characteristic three, we arrive at an alternate pathological configuration where in fact $U \leq C_G(i)$. We will not eliminate this configuration, and it deserves further attention. A more "intrinsic" description of this situation is: $N^\circ_G(\mathbb{X}) \leq C_G(i)$.

We operate under Hypothesis 2.12 and in the previous section we have discussed various useful subgroups of the group G. We will be keeping a maximal 2-torus T fixed and use the associated notation (which depends on T but does not mention the dependence).

Notation 4.1. $V = \Omega_1(T) = \langle i, j \rangle$.

4.1. Characteristic other than three (or 2).

The odd type hypothesis removes characteristic 2 from consideration. Characteristic three presents its own particular issues.

Lemma 4.2. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Then the following conditions are equivalent.*

(1) G contains a non-trivial 3-torus .

(2) The base fields of the components of $C_G(i)$ have characteristic not equal to three.

If these conditions hold, then there is an element $c \in C^\circ_G(\Omega_1(T))$ of order three centralized by a Weyl group element σ of order three, and which lies in one of the components of $C_G(i)$. For any such element c, we have

$$E_{alg}(C_G(c)/OF(C_G(c))) \simeq SL_3 \,.$$

Proof. We first check the equivalence of the two conditions.

$(2 \implies 1)$

We recall that the two base fields have the same characteristic. If this characteristic is not three then $\mathbb{T}$ contains a 3-torus.of Prüfer rank two.

$(1 \implies 2)$

For the converse, if there is a non-trivial 3-torus, then by Fact 2.14 there is a non-trivial 3-torus T_3 centralizing the 2-torus T. In particular $T_3 \leq \mathbb{T}$, and the characteristic is not equal to three.

So the two conditions are equivalent. Now we assume these conditions hold, and let T_3 be a maximal 3-torus in $\mathbb{T}$.

Then T_3 has Prüfer 3-rank two, and the Weyl group of G with respect to T also acts on T_3. So there is some $c \in \Omega_1(T_3)$ which is fixed by an element σ of order three in W.

Now the Weyl group element w_1 inverts σ and hence acts on $C_{T_3}(\sigma)$. It follows that $C_{T_3}(\sigma)$ contains an element c of order three which is either fixed or inverted by w_1, hence lies in $L_{i,1}$ or $L_{i,2}$. Let $K_{i,1}$ be the component of $C_G(i)$ containing c and $K_{i,2}$ the other component. Then $K_{i,2} \leq C_G(i, c)$. Similarly there are components $K_{t',2}$ of $C_G(t'))$ in $C_G(c)$ for $t' \in I(V)$.

Let $H = C_G^\circ(c)$, $\bar{H} = H/OF(H) = \bar{E} * \bar{K}$ the usual decomposition, and correspondingly, $H = E \cdot K$. We claim that E is itself quasi-simple, that is, $[E, OF(H)] = 1$. As E is generated by $C_E^\circ(t')$ for t' an involution of V, and $E(C_E(i)) = K_{i,2}$, this follows. As the group E has Prüfer 2-rank two, it must be a quasi-simple cover of PSL_3, with the base field as in $K_{i,2}$.

Thus at this point we have $H = E * K$ with E of type SL_3 or type PSL_3, and with K of degenerate type.

Claim 1. *$K \leq \mathbb{T}$.*

As usual it suffices to consider $C_K^\circ(i) \leq C_{C_G(i)}(K_{i,2}) = K_{i,1}$. So $C_K^\circ(i) \leq C_{K_{i,1}}(c)) \leq \mathbb{T}$. The claim follows.

Now a maximal 3-torus of E has Prüfer rank two and thus contains the 3-torsion of $\mathbb{T}$. So the factor K contains no 3-torsion. Hence $c \in Z(E)$ and $E \simeq \mathrm{SL}_3$. □

4.2. **Root subgroups.** Recall that $\mathbb{X}_\ell$ is a maximal unipotent subgroup of $L_{i,\ell}$ (a *root group*).

Lemma 4.3. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Then* $(\bigcup \mathbb{X}_1^G) \cap (\bigcup \mathbb{X}_2^G) = 1$.

Proof. Supposing the contrary, there is some $g \in G$ for which

$$\mathbb{X}_1 \cap \mathbb{X}_2^g > 1.$$

Take a non-trivial element $h \in \mathbb{X}_1 \cap \mathbb{X}_2^g$. Then

$$L_{i,2}, L_{i,1}^g \leq C_G(h).$$

In particular $i, i^g \in C_G^\circ(h)$. and so after further conjugation in $C_G(h)$ By Facts 2.21 and Fact 2.14 we may suppose that i and i^g lie in a common 2-torus T_H of H.

Now i normalizes $L_{i,1}^g$ and centralizes h, so centralizes $L_{i,1}^g$. It follows that $i = i^g$ and $L_{i,2}^g$ is $L_{i,2}$. But then $\mathbb{X}_1$ meets $L_{i,2}$, a contradiction. □

Lemma 4.4. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Then $\mathbb{X}_1^z, \mathbb{X}_2^z \not\leq C_G(i)$ for any $z \in N \setminus C_G(i)$.*

Proof. Suppose e.g. that $\mathbb{X}_1^z \leq C_G(i)$. Then i, i^z centralize $\mathbb{X}_1^z$ so after conjugating in $C(\mathbb{X}_1^z)$ we may suppose that i^z commutes with i. Then i normalizes $L_{i,1}^z$ and acts trivially on $\mathbb{X}_1^z$, so $L_{i,1}^z \leq C_G(i)$ and $L_{i,1}^z = L_{i,1}$, and z centralizes i. □

Lemma 4.5. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Then w_2 acts non-trivially on $L_{t,\ell}$ for $t \in I(V)$, $\ell = 1, 2$, except when $t = i$ and $\ell = 1$.*

Proof. If w_2 centralizes $L_{t,\ell}$ then $i = w_2^2$ centralizes $L_{t,\ell}$ and so $t = i$, in which case clearly $\ell = 1$. □

Notation 4.6. Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Take $\sigma \in N$ of order three. Set

$$\langle \mathbb{X}_1 \rangle_\sigma = \left\langle \mathbb{X}_1^\gamma | \gamma \in \langle \sigma \rangle \right\rangle,$$

with $\langle \mathbb{X}_2 \rangle_\sigma$ defined similarly.

Lemma 4.7. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Take $\sigma \in N$ of order three. Then $\langle \mathbb{X}_1 \rangle_\sigma$ is not nilpotent.*

Proof. We may suppose σ acts on $\Omega_1(V)$ as the 3-cycle (i, j, ij). Set $H = \langle \mathbb{X}_1 \rangle_\sigma$.

Claim 1. *H is nonabelian.*

Suppose toward a contradiction that H is abelian and let $K = C_G^\circ(\mathbb{X}_1)$. Then

$$H, L_{i,2}, L_{i,2}^\sigma \leq K.$$

By Fact 2.33 we find that either $K \leq C_G(i)$ or $E_{alg}(K)$ is of type SL_3.

If $K \leq C_G(i)$ then in particular $\mathbb{X}_1^\sigma \leq C_G(i)$. From the point of view of $C_G(j)$ this is visibly not the case.

If $E_{alg}(K)$ is of type SL_3 then there is an element of order three in K centralizing $L_{i,2}$, and hence lying in $C_K(i)$. This element must centralizes $\mathbb{X}_1$ and $L_{i,2}$, giving a contradiction.

The claim follows.

Now suppose toward a contradiction that H is nilpotent. Let π be $\chi_M(L_{i,1})$.

Let $A = U_\pi(Z(H))$. Then A is nontrivial by the theory of unipotence (notably, but not exclusively, Fact 2.4).

Now by Fact 2.32

$$A = \langle C_A^\circ(t) : t \in I(V) \rangle .$$

As σ normalizes A, it follows that all the groups $A_t = C_A^\circ(t)$ for $t \in I(V)$ are non-trivial. By Fact 2.19, the groups A_t are U_π-groups.

Now A_i is T-invariant and hence has the form

$$A_{i,1} \times A_{i,2}$$

where $A_{i.\ell} = A_i \cap L_{i,\ell}$.

The group $A_{i,1}$ is a unipotent subgroup of $L_{i,1}$, hence is $\mathbb{X}_1$ or trivial. If $A_{i,1} = \mathbb{X}_1$ then $\mathbb{X}_1 \leq Z(H)$. Then H is abelian, a contradiction. So

$$A_i \leq L_{i,2}.$$

If A_i is a unipotent subgroup of $L_{i,2}$ then since since it is nontrivial and $\mathbb{T}$-invariant it is a root subgroup. But then $\left\langle A_i^g : g \in \langle\sigma\rangle \right\rangle$ is also contained in A, and hence $\langle A_i \rangle_\sigma$ is abelian, giving much the same contradiction as above.

There remains the case in which

$$A_i \leq \mathbb{T} \cap L_{i,2}.$$

It then follows that all $A_t \leq \mathbb{T}$ for $t \in I(V)$, and hence $A \leq \mathbb{T}$.

As A centralizes $\mathbb{X}_1$, A centralizes $L_{i,1}$. Similarly A centralizes $L_{j,1}$ and $L_{ij,1}$.

Let $H_A = C_G^\circ(A)$, $\bar{H}_A = H_A/OF(H_A) = \bar{E}_A * \bar{K}_A$ with $\bar{E}_A = E_{alg}(\bar{H}_A)$. Then $\bar{E}_A$ contains $L_{v,1}$ for $v \in I(V)$. By Fact 2.33, $E_{alg}(H_A)$ is of type SL_3 and is generated by $L_{t,1}$ with $t \in V$. But then $\mathbb{X}_1^\sigma$ is another root group of $E_{alg}(H_A)$ and we arrive at a contradiction.

This proves the result. □

Lemma 4.8. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Take $\sigma \in N$ of order three. Set*

$$\langle \mathbb{X}_1 \rangle_\sigma = \left\langle \mathbb{X}_1^\gamma | \gamma \in \langle \sigma \rangle \right\rangle$$

Then we have one of the following.

(1) $\langle \mathbb{X}_1 \rangle_\sigma = G$; *or*

(2) $E_{alg}(\langle \mathbb{X}_1 \rangle_\sigma) \simeq (\mathrm{P})\mathrm{SL}_3$.

The same applies to $\mathbb{X}_2$.

Proof. Set $H_0 = \langle \mathbb{X}_1 \rangle_\sigma$ and $H = H_0\mathbb{T}$. Suppose $H_0 < G$. Then $H < G$. Set $\bar{H} = H/OF(H) = \bar{E} * \bar{K}$ where $\bar{E} = E_{alg}(\bar{H})$ and $\bar{K}/Z^\circ(\bar{K})$ has degenerate type.

If $\bar{\mathbb{X}}_1 = 1$ then $\bar{H}_0 = 1$ and H_0 is nilpotent, a contradiction to Lemma 4.7. It follows that $\mathbb{X}_1 \cap OF(H) = 1$. On the other hand $\mathbb{X}_1$ is inverted by a 2-element of $\mathbb{T}$ and thus $\bar{\mathbb{X}}_1 \leq \bar{E}$. Thus $\bar{H}_0 \leq \bar{E}$. It follows easily that $\bar{E} \simeq \mathrm{PSL}_3$ or SL_3 and that $\bar{H}_0 = \bar{E}$. That is

$$H_0/OF(H_0) \simeq (\mathrm{P})\mathrm{SL}_3 .$$

In particular the Weyl group $\mathrm{Sym}(3)$ acts on H_0.

Again the base field of $\bar{H}_0$ is the base field of $L_{i,1}$ and the torus of $\bar{H}_0$ is $\bar{\mathbb{T}}$. We have

$$OF(H_0) = \left\langle C^\circ_{OF(H_0)}(t) \,:\, t \in I(V) \right\rangle .$$

Consider the group $Y = C^\circ_{OF(H_0)}(i)$. Since $[Y, \mathbb{X}_1\mathbb{T}]$ is nilpotent and Y normalizes $L_{i,1}$, the group Y centralizes $L_{i,1}$. Y is also normalized by $\mathbb{T}$ and $Y \cap \mathbb{T} = 1$, so $Y \leq L_{i,2}$ is unipotent.

If Y is non-trivial then Y is a maximal unipotent subgroup of $L_{i,2}$. But $\langle Y \rangle_\sigma \leq OF(H_0)$ is nilpotent so we have a contradiction. Therefore $Y = 1$, and $OF(H_0) = 1$. Thus H_0 is PSL_3 or SL_3. □

5. The pathological case: $N^\circ_G(\mathbb{X}) \leq C_G(i)$

We first examine the problematic case—though the reader is welcome to pass over this and continue with the more favorable case of the next section.

Hypothesis 5.1 (G_2 with pathology)**.** Let G be a group of finite Morley rank satisfying Hypothesis 2.12. We assume in addition that

$$N^\circ_G(\mathbb{X}) \leq C_G(i). \tag{2}$$

Remark 5.2. Under Hypothesis 5.1, $\mathbb{X}$ is a maximal definable connected nilpotent subgroup of G.

5.1. **The base fields.** Among other things, we show in the first stage of analysis that the base fields are of characteristic three and definably isomorphic.

Lemma 5.3. *Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Suppose $A = L_{i,2} \cap L_{j,2} > 1$. Then $C^\circ_G(A) \simeq \mathrm{(P)SL}_3 * A^\circ$.*

Proof. Notice that A contains no involution and is centralized by $\langle i, j \rangle$. As j acts on $L_{i,2}$ like a square root z of i in $\mathbb{T}$, it follows that A commutes with z. So A must lie in the unique torus of $L_{i,2}$ containing z, namely $\mathbb{T} \cap L_{i,2}$.

Let $H = C^\circ_G(A)$

Claim 1. *$OF(H) \leq \mathbb{T}$.*

Set $U = C^\circ_{OF(H)}(i)$. Then $L_{i,1}$ normalizes $U \leq C_G(i)$ with U nilpotent, so $U \leq L_{i,2}$. Furthermore A centralizes U so $U \leq \mathbb{T}$. Similarly $C^\circ_{OF(H)}(t) \leq \mathbb{T}$ for $t \in I(V)$. The claim follows.

In particular $\mathbb{T}$ acts trivially on $OF(H)$ and hence $L_{i,1}, L_{j,1}$ act trivially on $OF(H)$. In particular $\langle L_{i,1}, L_{j,1}\rangle \le C_H(OF(H))$ and it follows that $\langle L_{i,1}, L_{j,1}\rangle = E_{alg}(H) \simeq \mathrm{(P)SL}_3$, and

$$H = E * C_H(E)$$

with $E = E_{alg}(H)$.

Now $C_H(E) \le C_H(i) \le C_G(i)$ so $C_H(E) = H \cap C_{C_G(i)}(E) \le H \cap L_{i,2}$. Similarly $C_H(E) \le L_{j,2}$, so $C_H(E) = A^\circ$ and $H = E * A^\circ$. □

Lemma 5.4. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1 and σ a Weyl group element of order three. Then $\langle \mathbb{X}_1\rangle_\sigma = G$.*

Proof. Supposing the contrary, by Lemma 4.8 $E_{alg}(\langle \mathbb{X}_1\rangle_\sigma) \simeq \mathrm{(P)SL}_3$. Let $K = E_{alg}(\langle \mathbb{X}_1\rangle_\sigma)$.

Take a maximal unipotent subgroup U_K of K centralizing $\mathbb{X}_1$. Consider $H = C^\circ_G(\mathbb{X}_1)$, which contains U_K and $L_{i,2}$. H has Prüfer 2-rank 1, and $C_H(i) = \mathbb{X}_1 L_{i,2}$ covers $\bar{H} = H/OF(H) = \bar{L}_{i,2} \times C_{\bar{H}}(L_{i,2})$, so $H = OF(H)L_{i,2}$. Now with $\mathbb{B}_2$ the Borel subgroup of $L_{i,2}$ containing $\mathbb{X}_2$, we have $\mathbb{X}_2 \le (\mathbb{B}_2 \cdot OF(H))' \le F(\mathbb{B}_2 \cdot OF(H))$, so $\mathbb{X}_2 \cdot OF(H)$ is nilpotent.

By hypothesis $\mathbb{X}$ is a maximal connected nilpotent subgroup of G, so $OF(H) = \mathbb{X}_1$. But then $U_K \le H = \mathbb{X}_1 L_{i,2}$. By rank considerations $U_K \cap L_{i,2}$ must be a Borel subgroup. But U_K is either p-unipotent with p the base characteristic, or torsion-free. □

Lemma 5.5. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1. Then $L_{i,2} \cap L_{j.2} = 1$.*

Proof. Supposing the contrary, let $A = L_{i,2} \cap L_{j,2}$ and

$$K = E_{alg}(C^\circ_G(A)) \simeq \mathrm{(P)SL}_3 \,.$$

Then $\langle \mathbb{X}_1\rangle_\sigma \le K$, contradicting Lemma 5.4. □

Lemma 5.6. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1. Then the multiplicative groups of the fields k_1, k_2 are definably isomorphic. In particular the fields have the same characteristic and the same rank.*

Proof. Let

$$A_1 = \mathbb{T} \cap L_{i,1} \simeq k_1^\times; \qquad A_2 = \mathbb{T} \cap L_{i,2} \simeq k_2^\times;$$
$$B = \mathbb{T} \cap L_{j,2}.$$

Then $B \cap A_1 = 1$, since j commutes with B and inverts A_1, and B does not contain i. By Lemma 5.3, $B \cap A_2 = 1$.

On the other hand $B \leq \mathbb{T} = A_1 \times A_2$ and by rank considerations $\mathbb{T} = A_1 \times B$. So B is the graph of a definable isomorphism between the multiplicative groups of k_1 and k_2. Thus the characteristics are equal and the ranks are the same. □

Note the asymmetry in our notation between index 1 and 2, as set up initially in the statement of Hypothesis 2.12, is now abolished. In particular Lemma 5.4 applies to $\mathbb{X}_2$ as well.

Definition 5.7. Let G be a group of finite Morley rank satisfying Hypothesis 2.12.

The *base characteristic* is the characteristic of the base fields k_1, k_2.

Both parts of the next lemma seem significant, though the second is a corollary of the first.

Lemma 5.8. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

Then

(1) The base characteristic is three.

(2) $C_G(i)$ contains no definable torsion-free subgroup.

Proof.

Ad 1.

Apply Lemma 4.2 and the symmetric form of 5.4.

Ad 2.

Suppose $C_G(i)$ contains a non-trivial definable torsion-free subgroup. Then the projection to $L_{i,1}$ is torsion-free, and we may suppose it is non-trivial. It must then lie in an algebraic torus,

But by Fact 2.16 an algebraic torus of $L_{i,1}$ is a good torus. □

5.2. **Local analysis: a black hole principle.** We aim at the "black hole" principle stated in Proposition 5.14 below.

Lemma 5.9. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

Then any element t which is a representative for a Weyl group element σ of order three is conjugate to an element of $\mathbb{X}$.

Proof. Take $w \in C_G(i)$ an involution inverting $\mathbb{T}$. At the level of the Weyl group, w and t commute. So $a = [t, w] \in \mathbb{T}$.

By Lemma 5.5 we have $C_{\mathbb{T}}(t) = 1$. Thus $t^3 = 1$ and $[t, \mathbb{T}] = \mathbb{T}$. Take $b \in \mathbb{T}$ with $[t, b] = a^{-1}$. Then $[t, wb] = 1$. As wb is an involution, t is conjugate to an element t' of $C_G(i)$. As $t^3 = 1$, t' is conjugate to an element of $\mathbb{X}$ in $C_G(i)$. □

Lemma 5.10. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

For $g \in \mathbb{T}$ non-trivial, there is a $t \in I(V)$ such that $C_G^\circ(g) \le C_G(t)$.

Proof.

$$C_G^\circ(g) = \langle C_G^\circ(g, t) \,:\, t \in V \rangle \,.$$

Now $C_G^\circ(g, i)$ is one of $\mathbb{T}$, $\mathbb{T}L_{i,2}$, or $L_{i,1}\mathbb{T}$.

If at most one centralizer $C_G^\circ(g, t)$ is greater than $\mathbb{T}$ for $t \in I(V)$, the claim follows. So suppose

$$C_G^\circ(g, i), C_G^\circ(g, j) > \mathbb{T},$$

and specifically $C_G^\circ(g, i) = L_{i,1}\mathbb{T}$. Then $C_G^\circ(g, j) = L_{j,1}\mathbb{T}$. But then $\langle \mathbb{X}_1 \rangle_\sigma < G$, contradicting Lemma 5.4. □

Definition 5.11. Let G be a group of finite Morley rank satisfying Hypothesis 5.1. Then $\mathcal{U}_{\mathbb{X}}$ denotes the family of all definable connected nilpotent subgroups of G which meet $\mathbb{X}$ nontrivially and are not contained in $C_G(i)$.

A group $U \in \mathcal{U}$ will be called *$\mathcal{U}$-maximal* if it maximizes $U \cap \mathbb{X}$ across all groups in $\mathcal{U}$, and in addition is itself a maximal definable connected nilpotent subgroup of G.

Lemma 5.12. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1. Let $V = \Omega_1(T)$.*

Suppose that $U \in \mathcal{U}$ satisfies the following conditions.

(a) $(U \cap \mathbb{X})^\circ$ is maximized across $\mathcal{U}_\mathbb{X}$.

(b) U is a maximal definable connected nilpotent subgroup of G.

Then the following hold.

(1) $U \cap \mathbb{X}$ is connected.

(2) U is $\mathcal{U}$-maximal.

(3) $U \cap \mathbb{X} < \mathbb{X} < C_G^\circ(U \cap \mathbb{X})$.

In particular, if $\mathcal{U}$ is nonempty then $\mathcal{U}$-maximal groups exist.

Proof.

Ad 1,2.

By Lemma 2.19 $C_U(i)$ is connected and it follows that $U \cap \mathbb{X}$ is connected. This gives points (1,2).

Ad 3.

By assumption $\mathbb{X}$ is maximal definable nilpotent, hence $U \cap \mathbb{X} < \mathbb{X}$.

Clearly $\mathbb{X} \leq C_G^\circ(U \cap \mathbb{X})$.

Let $U_1 = N_U^\circ(U \cap \mathbb{X})$. Then $U_1 > U \cap \mathbb{X}$ and U_1 normalizes $C_G^\circ(U \cap \mathbb{X})$. If we suppose $\mathbb{X} = C_G^\circ(U \cap \mathbb{X})$ then U_1 normalizes $\mathbb{X}$ and consideration of $\mathbb{X} \cdot U_1$ violates the $\mathcal{U}$-maximality of U. So (3) follows. □

Lemma 5.13. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1. Let $V = \Omega_1(T)$.*

Suppose $U \in \mathcal{U}$ is $\mathcal{U}$-maximal and set $H = C_G^\circ(U \cap \mathbb{X})$. Then the following hold.

(1) $F^\circ(H) \leq \mathbb{X}$.

(2) $\mathbb{X}$ is not contained in $L_{i,1}$ or $L_{i,2}$.

(3) For $h \in H \setminus N_H(\mathbb{X})$ we have $\mathbb{X} \cap \mathbb{X}^h = U \cap \mathbb{X}$.

(4) The union $\bigcup_{h \in H} \mathbb{X}^h$ is generic in H.

Proof.

Ad 1.

Since the group $B = \mathbb{X}F^\circ(H)$ is solvable, Fact 2.25 implies that $\mathbb{X} \leq F(B)$ and thus $\mathbb{X} \cdot F^\circ(H)$ is nilpotent. By the maximality of $\mathbb{X}$ we find $F^\circ(H) \leq \mathbb{X}$.

Ad 2.

Suppose for example that $U \cap \mathbb{X} \leq L_{i,1}$, so that $L_{i,2} \leq H$. By Claim 1 the group $\mathbb{X}$ commutes with $OF(H)$, so $L_{i,2}$ acts trivially on $OF(H)$. It follows that $L_{i,2} \leq E_{alg}(H)$, and as the Prüfer 2-rank of H is at most one, that

$$L_{i,2} = E_{alg}(H).$$

Let $U_0 = N_U^\circ(U \cap \mathbb{X})$. Then U_0 normalizes $L_{i,2}$ and acts via inner automorphisms. Conjugating under the action of $L_{i,2}$, we may suppose that U_0 acts like a subgroup of the Borel subgroup $N(\mathbb{X}_2)$. But then the group

$$U^* = U_0 \cdot \mathbb{X}_2$$

is nilpotent and $U \cap \mathbb{X} < (U \cap \mathbb{X})\mathbb{X}_2 \leq U^* \cap \mathbb{X}$, contradicting the maximality of $U \cap \mathbb{X}$.

Ad 3.

Set

$$\bar{H} = H/(U \cap \mathbb{X}).$$

Claim 1. *$\bar{\mathbb{X}}$ is a Carter subgroup of $\bar{H}$ (Definition 2.26).*

We need to show that $N_{\bar{H}}^\circ(\bar{\mathbb{X}}) = \bar{\mathbb{X}}$.

Suppose toward a contradiction that $N_{\bar{H}}(\bar{\mathbb{X}})/\bar{\mathbb{X}}$ is infinite. Let $B \leq N_H^\circ(\mathbb{X})$ be the preimage of a Borel subgroup of $N_{\bar{H}}^\circ(\bar{\mathbb{X}})$. By the maximality of $\mathbb{X}$, $F^\circ(B) = \mathbb{X}$. By Fact 2.25, $B/F^\circ(B)$ is divisible abelian. The maximality of $\mathbb{X}$ implies that the action of this group on $\mathbb{X}$ is faithful. By Fact 2.16 $B/F^\circ(B)$ contains torsion. By Fact 2.15 B contains a non-trivial Π-torus T_0. Conjugating by an element of $N(\mathbb{X})$, we may suppose that $T_0 \leq \mathbb{T}$. Since $U \cap \mathbb{X}$ and T_0 commute, we may suppose $U \cap \mathbb{X} \leq L_{i,1}$ and $T_0 \leq L_{i,2}$. This contradicts (2) and proves the claim.

Now suppose $\mathbb{X} \cap \mathbb{X}^h > U \cap \mathbb{X}$, that is, $\bar{\mathbb{X}}^h \cap \bar{\mathbb{X}} > 1$.

Then $\mathbb{X}^h \cap \mathbb{X} > U \cap \mathbb{X}$. Then by the maximality of $U \cap \mathbb{X}$ we have $\mathbb{X}^h \leq C_G(i)$. But $\mathbb{X}^h$ contains $U \cap \mathbb{X}$ and is maximal unipotent, while $U \cap \mathbb{X}$ is not contained in $L_{i,1}$ or $L_{i,2}$, so this forces $\mathbb{X}^h = \mathbb{X}$. This proves (3).

Ad 4. From (3), by Fact 2.29, we have $\bigcup \bar{\mathbb{X}}^{\bar{H}}$ generic in $\bar{H}$, and (4) follows. ☐

Proposition 5.14. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

If U is a connected nilpotent subgroup of G that contain a non-trivial unipotent element of $C_G(i)$ then U lies inside $C_G(i)$.

Proof. Assuming the contrary, we can take a counterexample U which is $\mathcal{U}$-maximal and consider $H = C_G^\circ(U \cap \mathbb{X})$.

By Lemma 5.13, H is generically of exponent 3. By Fact 2.24, H is nilpotent. As H contains $\mathbb{X}$, this is a contradiction. ☐

We derive some consequences.

Lemma 5.15. *If $u \in \mathbb{X}$, $u \neq 1$, then i is the only involution centralizing u.*

Proof. Let H_u be the group generated by definable connected nilpotent subgroups of G containing u. Then $H_u \leq C_G(i)$ and H_u is either $\mathbb{X}$ or a product $\mathbb{X}_0 L$ with $\mathbb{X}_0$ contained in $\mathbb{X}$ and L a component of $C_G(i)$. Let H_u^* be the group generated by definable connected nilpotent subgroups of G containing a unipotent element of H_u. Then H_u^* is $C_G(i)$. Therefore any involution t centralizing u centralizes i. But if the involution t centralizes i and u then it is i. ☐

Lemma 5.16. *For i, j distinct involutions, the intersection*

$$C_G(i) \cap C_G(j)$$

lies inside a conjugate of $\mathbb{T}$.

Proof. This is a definable subgroup of $C_G(i)$ containing no unipotent element, by Lemma 5.15. The result follows by Fact 2.31. □

5.3. **Involutions and rank computations.** One can attempt to compute $\mathrm{rk}(G)$ by a study of involutions in the manner of Thompson by studying the generic product of involutions. In some pathological settings this leads to a fairly direct contradiction. Here however one encounters the problem of "dark matter," to be defined and discussed in the next subsection. Still, we make the computations.

Notation 5.17. Let f denote the rank of the base fields (see Lemma 6.2).

Then to begin with, we have $\mathrm{rk}(C_G(i)) = 6f$.

Now we consider the set Σ of *strongly real* elements of G: these are the elements inverted by some involution. This includes all elements of $C_G(i)$, since a torus is inverted by an involution, a unipotent element is inverted by an involution, and a mixed type element ua with u, a in different components of $C_G(i)$ and with u unipotent, a toral is inverted by elements t_1 and w_2 in the respective components with square i, so that au is inverted by the product $t_1 w_2$.

We consider the multiplication map

$$\mu : I(G) \times I(G) \to \Sigma.$$

In the context of groups of finite Morley rank, if one has a definable surjection

$$f : A \twoheadrightarrow B$$

then

$$\mathrm{rk}(A) = \max(\mathrm{rk}(B_r) + r)$$

with B_r the subset of B for which the rank of the corresponding fiber is r.

Specializing to the case at hand we have, first of all, the following.

Lemma 5.18. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

Then the inverse image of the set of toral elements $\bigcup \mathbb{T}^G \subseteq \Sigma$ under μ has rank $\mathrm{rk}(G) + 2f$.

Proof. We consider the rank of the fiber at $a \in \mathbb{T}$. This consists of the pairs

$$(t, ta)$$

where t is an involution inverting a.

If $a = 1$ then the rank of the fiber at a is

$$\mathrm{rk}(I(G)) = \mathrm{rk}(G) - \mathrm{rk}(C_G(i)). \tag{1}$$

and as there is only one such point a the rank of the inverse image is the same.

If $a = i$ is an involution of $\mathbb{T}$ then the rank of the fiber at a is

$$\mathrm{rk}(I(C_G(i))) = 4f.$$

As the rank of the set of involutions is $\mathrm{rk}(G) - 6f$ we find that the rank of the inverse image of the involutions under μ is

$$\mathrm{rk}(G) - 2f. \tag{2}$$

If $a \in \mathbb{T}$ is nontrivial, and not an involution, we apply Lemma 5.10. We may suppose that $C_G^\circ(a) \le C_G(i)$. We take an involution $w \in C_G(i)$ inverting $\mathbb{T}$.

If t is an involution inverting a we write

$$t = wt'$$

where w inverts t' and t' centralizes a. In particular, $t' \in C_G(i)$. More precisely, $a = a_1 a_2$ where $a_\ell \in L_{i,\ell} \cap \mathbb{T}$ and $t' = t_1' t_2'$ similarly, with components well-defined up to multiplication by i.

These cases are very similar, with some variation in detail according as a is in a component $L_{i,\ell}$ or not. We claim that the fiber rank is $2f$. On the one hand w inverts $\mathbb{T}$ and $\mathbb{T}$ centralizes a, so the fiber rank is at least $\mathrm{rk}(\mathbb{T}) = 2f$. On the other hand, as w must invert t' the fiber rank is at most $2f$. Now a varies over a generic subset of $\bigcup_G \mathbb{T}^G$, which is itself generic in G, so the inverse image for this subset has rank

$$\mathrm{rk}(G) + 2f. \tag{3}$$

The largest of these three numbers is the last.

$$\mathrm{rk}\left(\mu^{-1}\left(\bigcup \mathbb{T}^G\right)\right) = \mathrm{rk}(G) + 2f. \qquad \square$$

Lemma 5.19. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

Then $\mathrm{rk}(G) \geq 14f$.

Proof. The rank of $I(G)\times I(G)$ is $2\,\mathrm{rk}(G)-2\,\mathrm{rk}(C_G(i)) = 2\,\mathrm{rk}(G)-12f$ and the rank of the inverse image of the toral elements is $\mathrm{rk}(G)+2f$. The lemma follows. □

So the question becomes whether the rank of G is exactly $14f$, or greater. This comes down to computing, or estimating, the ranks of the inverse image of the remaining strongly real elements. It is not clear how to do that effectively.

But we will examine the other elements of G, beginning with those in $\bigcup_G (C_G(i))^G$.

Lemma 5.20. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

Then the inverse image of the ot the nontoral elements of $\bigcup C_G(i)^G \subseteq \Sigma$ *under μ has rank at most* $\mathrm{rk}(G)+f$.

Proof. We consider elements of the form ua with $u \in L_{i,1}$ unipotent and nontrivial and $a \in L_{i,2}$. We let $t_1 \in L_{i,1}$ be a toral element inverting u and $t_2 \in L_{i,2}$ a toral element inverting a. Then we may suppose $t=(t_1,t_2)$ is an involution inverting au (the only special case being the one where $a=1$ and t_2 should be chosen to have order 4).

The inverse image of au under μ consists of pairs $(tx, txua)$ with t inverting x and x centralizing ua. If ua is unipotent than $x \in C_G(i)$ (Proposition 5.14). If a is toral and non-trivial then x centralizes $(ua)^3 = a^3$, and from Lemma 5.10 it follows that x centralizes a, and hence u, taking us back to the first case. So $x \in C_G(i)$. This then gives $2f$ as the rank of the inverse image of the element ua. Accordingly we compute the following ranks for elements ua with a either toral and nontrivial or unipotent (note that an element of this form lies in a unique centralizer of an involution).

Type of a	rk elements	rk inverse image
Toral	$\mathrm{rk}(G)-f$	$\mathrm{rk}(G)+f$
Unipotent	$\mathrm{rk}(G)-2f$	$\mathrm{rk}(G)$

□

So taking these elements into account will not change the lower bound of Lemma 5.19. The question therefore is what occurs outside of centralizers of involutions, and this is where matters become fairly obscure.

5.4. **Invisible elements and dark matter.**

Definition 5.21. Let G be a group of finite Morley rank satisfying Hypothesis 5.1.

An element $x \in G$ is *visible* if x centralizes some involution, and *invisible* otherwise. A subgroup of G will be called visible if all of its elements are visible.

A strongly real invisible element is called *dark*. The set of dark elements is called the *dark matter.* We write $\mathcal{D}$ for the dark matter together with the identity.

Remark 5.22. In the previous section we considered the map

$$\mu I(G) \times I(G) \to \Sigma$$

and showed that if the inverse image of the visible elements is generic then the rank of G is $14f$. This holds in particular if there is no dark matter, a case which seems worth separate examination on its own as a test case.

Now we make an exploration of the dark matter. The results are inconclusive.

Lemma 5.23. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

The centralizer of a nontrivial visible element is visible.

Proof. We suppose that a centralizes the nontrivial element $h \in C_G(i)$. Writing $h = h_1 h_2$ with $h_\ell \in L_{i,\ell}$, we may suppose that h_1 is nontrivial and h_2 is of the same type (toral or unipotent) as h_1, or trivial, replacing h by h^3 otherwise.

Then Lemma 5.10 or 5.15. applies. □

We put this in a more directly useful form as follows.

Lemma 5.24. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

Then the following hold.

(1) *The centralizer of a dark element is an abelian group.*

(2) *The set $\mathcal{D}$ is the union of the centralizers of dark elements.*

(3) *Distinct centralizers of dark elements intersect trivially.*

Proof.
Ad 1,2. Let a be a non-trivial dark element, inverted by the involution i. Then i acts on $C(a)$. If i centralizes a non-trivial element b in $C(a)$ then b is visible, so a is visible, a contradiction.

Thus $C(a)$ is an abelian group contained in $\mathcal{D}$ and both (1) and (2) follow.

Ad 3. This follows from (1): if a, b are nontrivial dark elements which commute then $C(a) = C(b)$. □

As we shall see, the general tendency of the dark elements is to behave like unipotent elements, lying in some approximation to a Borel subgroup, but with no ambient semisimple algebraic group involved.

Lemma 5.25. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

If a is a dark element then the following hold.

(1) $N_G^\circ(C(a)) > C(a)$.

(2) *There is no prime p for which $C(a)$ and $N_G(C(a))/C(a)$ both contain p-torsion.*

(3) *$C(a)$ is connected.*

Proof. We set $A = C(a)$.

Ad 1. If $N_G(A)^\circ = A$ then the dark matter is generic. But the visible elements are generic, so this is a contradiction.

Ad 2. Suppose that A contains a nontrivial p-element. Then any p-element of $N_G(A)$ centralizes a nontrivial p-element of A, and hence lies in A. From this (2) follows.

Ad 3. If $A > A^\circ$ take an element $b \in A$ of prime order p modulo A°. We may take b itself to be a p-element.

$N_G^\circ(A)/(A \cap N_G^\circ(A))$ centralizes the finite group A/A°. So b centralizes N_G°/A, and as $N_G(A)/A$ contains no p-torsion, the group $C(b)$

covers $N_G^\circ(A)/(A \cap N_G^\circ(A))$. But $C(b) = A$, so this is a contradiction. □

Lemma 5.26. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

If a is a dark element and i is an involution inverting a, then the following hold.

(1) For any prime p, $U_p(C(a))$ is p-unipotent.

(2) $N(C(a))/C(a)$ has p-rank at most one for each prime p.

(3) $N(C(a))/C(a)$ has a unique involution.

(4) $N(C(a))/C(a) = C(a) \rtimes C_{N(C(a))}(i)$.

(5) $C^\circ_{N(C(a))}(i)$ is a good torus.

Proof. We set $A = C(a)$

Ad 1. A is connected abelian. Any Π-torus T_A contained in A will be central in $N_G^\circ(A)$, giving a contradiction. So (1) follows.

Ad 2. Suppose that E is an elementary abelian p-subgroup of $N_G(A)/A$ of rank two. Then A is generated by the subgroups $C_A(V_0)$ for $[V : V_0] = p$ and these are trivial, giving a contradiction.

Ad 3. On one hand, i represents an involution of $N_G(A)/A$.

On the other hand, if t_1, t_2 are two involutions of $N_G(A)/A$ then they both invert A and thus t_1t_2 centralizes A, and hence lies in A.

Ad 4. By (3), i centralizes $N_G(A)/A$. So (4) follows from Fact 2.18.

Ad 5. By (2) the 3-unipotent part of $C_{N_G(A)}(i)$ is trivial, so it is inside an algebraic torus in non-zero characteristic, and (5) follows. □

All of this suggests that the dark elements tend to look a good deal like the generic unipotent elements in centralizers of involutions.

Lemma 5.27. *Let G be a group of finite Morley rank satisfying Hypothesis 5.1.*

If a is a dark element and $A = C(a)$ then

$$\mathrm{rk}(N_G(A)/A) \le f.$$

Proof. Let T_A be the good torus $C_{N_G(A)^\circ}(i)$, with p-rank at most one for each prime p. We claim that

$$\mathrm{rk}(T_A) \le f.$$

We let T_1 be the projection of T_A to $L_{i,1}$ and T_2 the kernel and $\mathbb{T}_1$ an algebraic torus of $L_{i,1}$ containing T_1. By the proof of Lemma 5.6 there is a definable isomorphism of $k_2^\times$ with $k_1^\times$ and so T_2 is definably isomorphic to some $T_2^* \le \mathbb{T}_1$.

Then $T_1 \cap T_2^* = 1$ as otherwise there is some element in the intersection of prime order p and then T has p-rank at least two, a contradiction. Thus

$$\mathrm{rk}(T) = \mathrm{rk}(T_1) + \mathrm{rk}(T_2^*) \le \mathrm{rk}\,\mathbb{T}_1 = f. \qquad \square$$

We will now leave this configuration and turn to the less pathological branch of the analysis.

6. Identification of G_2 when $U \not\leq C_G(i)$.

We return to the "main branch."

Hypothesis 6.1. Let G be a group of finite Morley rank satisfying Hypothesis 2.12. Suppose that $N_G^\circ(\mathbb{X})$ is not contained in $C_G(i)$.

We will keep the torus $\mathbb{T}$ fixed, and when we speak of *root subgroups*, we mean root subgroups normalized by $\mathbb{T}$. The Weyl group $\langle \bar{w}_1, \bar{w}_2, \bar{\sigma} \rangle$ is described in Fact 3.2.

6.1. **Unipotent subgroups.**

Recall from Fact 2.10 that the base fields k_1, k_2 have the same characteristic, which we can call the *base field characteristic.*

Lemma 6.2. *Suppose that the base field characteristic is non-zero. Then the following hold.*

(1) The torus $\mathbb{T}$ is a good torus.

(2) $\mathrm{rk}(k_1) = \mathrm{rk}(k_2)$

Proof.

Ad 1.

We have $\mathbb{T} = T_1 *_2 T_2$ with $T_\ell = \mathbb{T} \cap L_{i,\ell}$ an algebraic torus of $L_{i,\ell}$. As the factors are good tori by Fact 2.16, $\mathbb{T}$ is a good torus.

Ad 2.

Suppose toward a contradiction that $\mathrm{rk}(k_2) < \mathrm{rk}(k_1)$. From the actions of $\mathbb{T}$ on $L_{i,2}$ and $L_{j,2}$ we get an action on the product of rank at most $2\,\mathrm{rk}(k_2) < \mathrm{rk}(\mathbb{T})$. Hence the centralizer $\mathbb{T}_0 = C_{\mathbb{T}}(L_{i,2}, L_{j,2})$ is nontrivial.

Let $H = C_G(\mathbb{T}_0)$. By Fact 2.33 we must $E_{alg}(H)$ is of type $\mathrm{SL}_3(k_2)$. As this has Lie rank 2 this forces $\mathbb{T}_0$ to be torsion free. But $\mathbb{T}_0$ is a good torus. □

Lemma 6.3. *Let G be a group of finite Morley rank satisfying Hypothesis 6.1.*

Then for some index $\ell = 1$ or 2 and involution $t = j$ or ij there is a root subgroup of $L_{t,\ell}$ centralizing $\mathbb{X}$.

Proof.

Claim 1. *There is a root subgroup $\mathbb{Y}$ of some $L_{t,\ell}$, with $t = j$ or ij and $\ell = 1$ or 2, for which $\mathbb{Y}$ normalizes $\mathbb{X}$.*

Let $Q = N_G^\circ(\mathbb{X})$. Then

$$Q = \left\langle C_Q^\circ(t) \,:\, t \in I(V) \right\rangle .$$

To fix notation, we may suppose that $B = C_Q^\circ(ij)$ is not contained in $C_G(i)$.

As B is $\mathbb{T}$-invariant we have

$$B = B_1 \times B_2$$

with $B_\ell = B \cap L_{ij,\ell}$. We may suppose B_1 is not contained in $C_G(i)$, hence contains a root subgroup $\mathbb{Y}$ of $L_{ij,1}$.

This proves the claim.

Now we prove that $\mathbb{X}\mathbb{Y}$ is abelian.

The group $\mathbb{X}\mathbb{Y}$ is nilpotent, with $\mathbb{X}$ as a normal subgroup.

Let $Z = (Z(\mathbb{X}\mathbb{Y}) \cap \mathbb{X})^\circ$. Then Z contains $\mathbb{X}_1$ or $\mathbb{X}_2$. The claim is that $Z = \mathbb{X}$.

Otherwise, we have $Z < \mathbb{X}$ and $Z = \mathbb{X}_1$ or $\mathbb{X}_2$. Then $\mathbb{X} \le Z_2^\circ(\mathbb{X}\mathbb{Y})$. So for $x \in \mathbb{X}$, $y \in \mathbb{Y}$ we have $[x, y] \in Z$ and

$$[x, y] = [x, y]^i = [x, y^{-1}] = [x, y]^{-1}$$

and $[x, y] = 1$, and $\mathbb{X}$ centralizes $\mathbb{Y}$, after all. □

Notation 6.4. We have chosen notation so that $\sigma \in N$ acts on $I(V)$ as the 3-cycle (i, j, ij). We let $w_{t,\ell}$ denote a Weyl group element in $L_{t,\ell}$ ($w_\ell = w_{i,\ell}$).

We have fixed $\mathbb{X} = \mathbb{X}_1\mathbb{X}_2$ and we will suppose going forward that notation is chosen as in the proof of the previous lemma so that some root group $\mathbb{Y}$ of $L_{ij,1}$ commutes with $\mathbb{X}$ (since we may switch the indices 1 and 2, and replace σ by σ^{-1}).

We set

$$\mathbb{X}_{t,\ell} = \mathbb{X}_\ell^{\sigma_t}$$

where σ_t is the power of σ taking i to t.

Let $\mathbb{X}^-_{t,\ell}$ be the opposite root group. The action of the Weyl group D_{12} is the natural action on each of the orbits $\mathcal{O}_\ell = (\mathbb{X}^\pm_{t,\ell} : t = i, j, ij)$ for $\ell = 1$ or 2. Here σ acts on the index t and $z = w_1 w_2$ is the central element swapping the pair $\mathbb{X}^\pm_{t,\ell}$ for each choice of t, ℓ. Furthermore in view of the structure of D_{12}, we have $\sigma^{w_1} = \sigma^{-1}$. Hence w_1 swaps the pairs

$$(\mathbb{X}_1, \mathbb{X}^-_1),\ (\mathbb{X}_{1,j}, \mathbb{X}^-_{1,ij}),\ \text{and}\ (\mathbb{X}^-_{1,j}, \mathbb{X}_{\ell,ij}).$$

For example

$$\mathbb{X}_{1,j}^{w_1} = \mathbb{X}_1^{\sigma w_1} = \mathbb{X}_1^{w_1 \sigma^{-1}} = (\mathbb{X}^-_1)^{\sigma^{-1}} = \mathbb{X}^-_{1,ij}.$$

In diagrammatic terms we have the actions shown below.

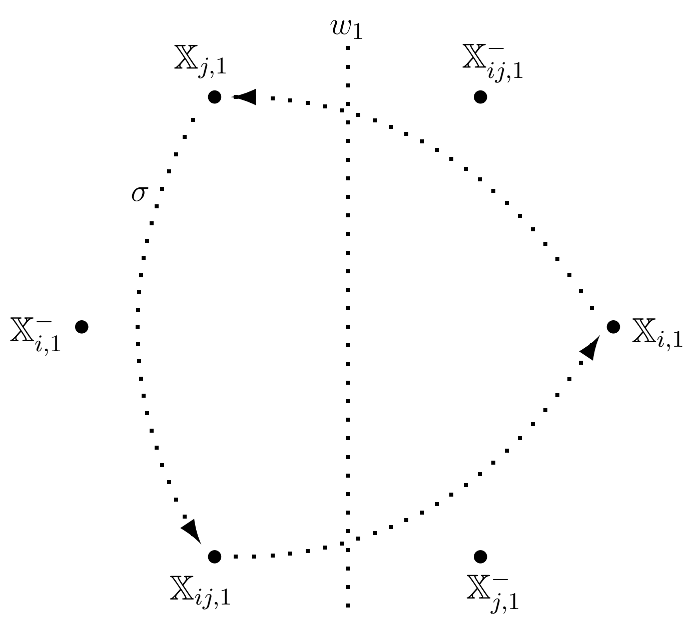

FIGURE 1. Action of the Weyl group, w_1 and σ on $\mathbb{X}_{t,1}$

We have a similar action of σ and w_2 on the root groups $\mathbb{X}_{t,2}$ and this together with the action of $w_1 w_2$ gives a complete description of the action on these root groups, which the usual sort of diagram for G_2 (as shown) summarizes at least as far as the notation is concerned.

We now show that the behavior of these groups is at least roughly in keeping with the Chevalley commutator formula for G_2.

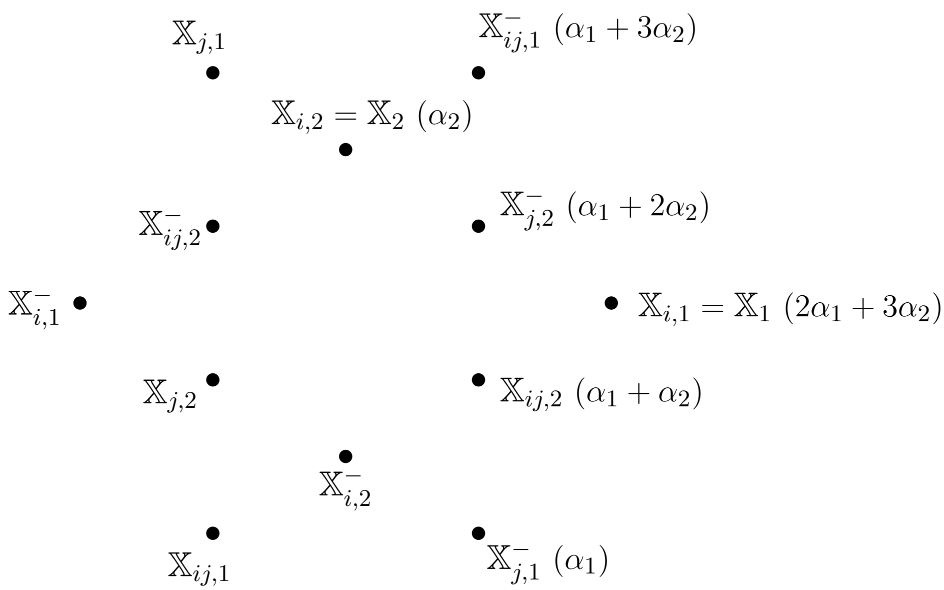

FIGURE 2. All root groups, with G_2-type labels

Lemma 6.5. *Let G be a group of finite Morley rank satisfying Hypothesis 6.1.*

Then

(1) *$\mathbb{X}$ commutes with $\mathbb{X}^-_{ij,1}$.*

(2) *$\mathbb{X}_2$ commutes with $\mathbb{X}^-_{ij,1}$ and $\mathbb{X}_{j,1}$.*

(3) *$\mathbb{X}_1$ commutes with $\mathbb{X}^-_{ij,1}$, $\mathbb{X}^-_{j,2}$, $\mathbb{X}_{ij,2}$ and $\mathbb{X}^-_{j,1}$.*

(4) *The group*
$$Q = \left\langle \mathbb{X}^-_{ij,1}, \mathbb{X}^-_{j,2}, \mathbb{X}_1, \mathbb{X}_{ij,2}, \mathbb{X}^-_{j,1} \right\rangle$$
is nilpotent of class 2 with
$$Q = \mathbb{X}^-_{ij,1}\mathbb{X}^-_{j,2}\mathbb{X}_1\mathbb{X}_{ij,2}\mathbb{X}^-_{j,1} \text{ (product—in any order)};$$
$$Q' = [\mathbb{X}^-_{ij,1}, \mathbb{X}_{j,1}] = \mathbb{X}_1.$$

(5) *$Q = OF(C_G(\mathbb{X}_1))$.*

Proof.

Ad 1. By hypothesis (as of Notation 6.4) $\mathbb{X}$ commutes with at least one of the root subgroups $\mathbb{X}_{ij,1}$, $\mathbb{X}^-_{ij,1}$.

Suppose toward a contradiction that $\mathbb{X}$ commutes with $\mathbb{X}_{ij,1} = \mathbb{X}_1^{\sigma^{-1}}$. Applying σ, we find that $\mathbb{X}_1$ also commutes with $\mathbb{X}_{j,1}$.

We let $H = C^\circ_G(\mathbb{X}_1)$. Then $\langle \mathbb{X}_1 \rangle_\sigma \le H$.

Let $\bar{H} = H/OF(H) = \bar{E} * \bar{K}$ with $\bar{E} = E_{alg}(\bar{H})$.

Now $\bar{E}$ contains $\bar{L}_{i,2}$ and $\bar{E}$ is generated by conjugates of $\bar{L}_{i,2}$, so if $\bar{E} > \bar{L}_{i,2}$ then H contains $\mathbb{T}$ and so $\mathbb{X}_1$ commutes with j, a contradiction. It follows that $\langle \mathbb{X}_1 \rangle_\sigma \le OF(H)$. This contradicts Lemma 4.8.

Ad 2. By (1) $\mathbb{X}_2$ commutes with $\mathbb{X}^-_{ij,1}$. Applying w_1, $\mathbb{X}_2$ commutes with $\mathbb{X}_{j,1}$.

Ad 3. By (1) $\mathbb{X}_1$ commutes with $\mathbb{X}^-_{ij,1}$. Applying w_2, it also commutes with $\mathbb{X}^-_{j,1}$.

Applying σ^{-1} to (2) we find that $\mathbb{X}_1$ commutes with $\mathbb{X}_{ij,2}$, and applying w_2 we find $\mathbb{X}_1$ commutes with $\mathbb{X}^-_{j,2}$.

Ad 4. Again, let $H = C^\circ_G(\mathbb{X}_1)$, $\bar{H} = H/OF(H) = \bar{E} * \bar{K}$. Then $\bar{E} = \bar{L}_2$. The involution i inverts the groups other than $\mathbb{X}_1$ taken as generators of Q, and modulo $OF(H)$ it commutes with them, so we find that $Q \le OF(H)$. Thus Q is nilpotent.

In particular $Q \cap L_{i,2} = 1$. It is then easy to see that $C_Q(i) = \mathbb{X}_1$. But $C_Q(i)$ covers $C_{Q/\mathbb{X}_1}(i)$, so it follows that i inverts $Q/\mathbb{X}_1$ and $Q/\mathbb{X}_1$ is abelian. So Q is nilpotent of class at most 2. In particular we can write an element of Q as a product of elements of root groups in any desired order, as the commutators lie in $\mathbb{X}_1$.

For the final point, as $\mathbb{X}^-_{ij,1} = (\mathbb{X}^-_{j,1})^\sigma$, the groups $\mathbb{X}^-_{ij,1}$ and $\mathbb{X}^-_{j,1}$ do not commute. But their commutator is a $\mathbb{T}$-invariant subgroup of $\mathbb{X}_1$, so

$$[\mathbb{X}^-_{j,1}, \mathbb{X}^-_{ij,1}] = \mathbb{X}_1.$$

Thus $Q' = \mathbb{X}_1$.

This completes the proof of (4).

Ad 5. Let $R = OF(C_G(\mathbb{X}_1))$. Then $Q \le R$ and it suffices to check that $C_R(t) = C_Q(t)$ for $t = i, j, ij$.

For $t = i$ this is clear as $R \cap L_2 = 1$.

For $t = j$ or ij the centralizers in Q are maximal unipotent subgroups of the corresponding centralizers in G.

So (5) holds. □

Lemma 6.6. *Let G be a group of finite Morley rank satisfying Hypothesis 6.1.*

Then the following hold.

(1) *The ranks of the base fields are equal.*

(2) *The group $Q \cdot \mathbb{X}_2$ is a maximal connected definable nilpotent subgroup of G.*

Proof.

Ad 1. We know this already when the base field characteristic is non-zero.

In the case of non-zero characteristic if the ranks of the base fields are different then consider the decomposition

$$Q = \mathbb{U}_{\pi_1}(Q) * U_{\pi_2}(Q)$$

where $\pi_\ell = (0, \mathrm{rk}(k_\ell))$.

Then L_2 acts on Q_1 and as the ranks of the base fields are different the action is trivial. This then puts Q_1 into $C_G(i)$ to get a contradiction.

Ad 2. That $Q \cdot \mathbb{X}_2$ is nilpotent follows from (1).

The maximality follows much as in the case of point (5) of the last lemma by considering the centralizers of i, j, and ij; notably, $C_{Q \cdot \mathbb{X}_2}(i) = \mathbb{X}$. □

At this point we can take a less abstract approach to the treatment of the group U and define it explicitly as follows, with a slight change of notation.

Definition 6.7.

$$\mathbb{U} = Q \cdot \mathbb{X}_2$$

with $Q = OF(C_G(\mathbb{X}_1)) = \left\langle \mathbb{X}_{ij,1}^-, \mathbb{X}_{j,2}^-, \mathbb{X}_1, \mathbb{X}_{ij,2}, \mathbb{X}_{j,1}^- \right\rangle$.

Correspondingly we now take $\mathbb{B} = \mathbb{T} \cdot \mathbb{U}$.

6.2. **The BN-pair.** We now move toward the construction of a spherical split irreducible BN-pair.

Definition 6.8 (BN-pair)**.** Let G be a group. A *BN-pair* for G consists of two subgroups B and N satisfying the following conditions, where $T = B \cap N$.

(BN1) $G = \langle B, N\rangle$ and $T \lhd N$.

(BN2) The group $W_{BN} := N/T$ is generated by a (specified) nonempty set I of involutions.

(BN3) For $v, w \in N$ and $wT \in I$ we have

$$vBw \subseteq BvB \cup BvwB.$$

(BN4) $wBw \neq B$ for all $w \in N$ with $wT \in I$.

The BN-pair is irreducible if W_{BN} is not a direct product of proper subgroups *spherical* if W_{BN} is finite, and *split* if B splits as $\mathbb{U} \cdot T$ with $\mathbb{U}$ a nilpotent normal subgroup of B.

We will use $\mathbb{B} = \mathbb{U}\mathbb{T}$ as in Definition 6.7 and $N = N(\mathbb{T}) = N(T)$. W_{BT} can be identified with $W_T = N(T)/T$, the dihedral group. That is, the notations B, N, T from the definition of BN-pair will refer to $\mathbb{B}$, N and $\mathbb{T}$. If we define I and verify (B2–B4) then the identification of $G2$ will follow since we will then have an irreducible spherical split BN-pair of finite Morley rank, and either of [Ten04], [TVM03] applies.

Now we need to pick a suitable set of generators for the Weyl group.

Notation 6.9. We set

$$\begin{aligned} &s = w_2,\ t = w_1^\sigma = w_{j,1}; \qquad && I = \{\bar{s}, \bar{t}\} \subseteq W_{BN}, \\ &z = w_1 w_2. \end{aligned}$$

So we take $\bar{s}, \bar{t}$ as the fixed generators of our Weyl group, and we also fix notation for a central element $\bar{z}$ of the Weyl group.

Recall that $\sigma^{w_1} = \sigma^{-1}$ and thus $st = z\sigma^{-1}$ is an element of order 6. inverted by s. The action of W on root subgroups was given above and can be read off of Figure 2.

Now we treat condition (B4).

Lemma 6.10. *Let G be a group of finite Morley rank satisfying Hypothesis 6.1.*

Then $\mathbb{B}^s, \mathbb{B}^t \neq \mathbb{B}$.

Proof. We have

$$\mathbb{X}_2^s = \mathbb{X}_2^-; \qquad\qquad (\mathbb{X}_{1,j}^-)^t = \mathbb{X}_{1,j}$$

and as $\mathbb{U}$ is nilpotent it follows that neither of these groups lies in $\mathbb{U}$. Thus $\mathbb{U}^s, \mathbb{U}^t \neq \mathbb{U}$ and $\mathbb{B}^s, \mathbb{B}^t \neq \mathbb{B}$. □

So everything comes down to the condition (B3), which we may restate as follows.

$$v\mathbb{U}w \subseteq \mathbb{B}v\mathbb{B} \cup \mathbb{B}vw\mathbb{B} \text{ for } w = s \text{ or } t.$$

We examine the structure of $\mathbb{U}$ more closely.

Lemma 6.11. *Let G be a group of finite Morley rank satisfying Hypothesis 6.1.*

Then any $\mathbb{T}$-invariant definable subgroup of $\mathbb{U}$ is a product of some of the root groups given as generators of $\mathbb{U}$.

Proof. Let U_0 be such a subgroup.

The corresponding statement is clear for the centralizers $C_{U_0}(t)$ with $t = i$, j, or ij. So U_0 is generated by suitable root subgroups.

It suffices to show that $U_0 \cap Q$ has the stated form. So we may suppose $U_0 \leq Q$.

If U_0 contains $\mathbb{X}_1$ then U_0 is the product of the corresponding root subgroups taken in any order. If U_0 does not contain $\mathbb{X}_1$ then it is abelian and the same conclusion follows. □

Definition 6.12.

$$Q_1 = \mathbb{U} \cap \mathbb{U}^t.$$

Lemma 6.13. *Let G be a group of finite Morley rank satisfying Hypothesis 6.1.*

Then the following hold.

(1) $Q_1 = \left\langle \mathbb{X}_{i,2}, \mathbb{X}^-_{ij,1}, \mathbb{X}^-_{j,2}, \mathbb{X}_{i,1}, \mathbb{X}_{ij,2} \right\rangle$.

(2) $\mathbb{U} = Q_1 \rtimes \mathbb{X}^-_{j,1}$.

Proof.
Ad 1. Recall that $t = w_{j,1}$. It is clear that the root subgroups listed lie in Q_1; these are all of the root subgroups of $\mathbb{U}$ other than $\mathbb{X}^-_{j,1}$.

On the other hand $\mathbb{X}^-_{j,1} = \mathbb{X}^t_{j,1}$ so the root subgroup $\mathbb{X}^-_{j,1}$ does not lie in Q_1.

Q_1 is the product of the root groups it contains.

Ad 2.The group $N_{\mathbb{U}}(Q_1)$ properly contains Q_1 and is a product of its root subgroups, so it is $\mathbb{U}$. Point (2) follows. □

Thus we have the following two decompositions of $\mathbb{U}$.

$$\mathbb{U} = Q \cdot \mathbb{X}_2; \qquad\qquad \mathbb{U} = Q_1 \cdot \mathbb{X}^-_{j,1}.$$

A more sugesstive notation would be

$$\mathbb{U} = \mathbb{X}_w \cdot Q_w,$$

where w is s or t and we set, correspondingly, $Q_w = \mathbb{U} \cap \mathbb{U}^w$ and

$$\mathbb{X}_s = \mathbb{X}_2,\ \mathbb{X}_t = \mathbb{X}^-_{j,1}.$$

In this notation, we have

$$v\mathbb{U}w = v\mathbb{X}_w w Q_w$$

and hence the corresponding instance of (B3) reduces to

$$(\mathrm{B3}_w) \qquad v\mathbb{X}_w w \subseteq \mathbb{B}v\mathbb{B} \cup \mathbb{B}vw\mathbb{B}.$$

Returning to our more explicit notation the condition becomes

$$\text{(B3}_s\text{)} \qquad v\mathbb{X}_2 w_2 \subseteq \mathbb{B}v\mathbb{B} \cup \mathbb{B}vw_2\mathbb{B};$$

$$\text{(B3}_t\text{)} \qquad v\mathbb{X}^-_{j,1} w_{j,1} \subseteq \mathbb{B}v\mathbb{B} \cup \mathbb{B}vw_{j,1}\mathbb{B}.$$

If (B3$_w$) holds for a particular pair (v, w) then it holds also for the pair (vw, w). Indeed, we have

$$w\mathbb{X}_w w \subseteq B_w w B_w \cup B_w$$

where B_w is the corresponding Borel subgroup in $L_w = L_{i,2}$ or $L_{j,1}$ respectively, and then

$$vw\mathbb{X}_w w \subseteq v\mathbb{B}w\mathbb{B} \cup v\mathbb{B}$$

and we can apply the hypothesis to $(v\mathbb{B}w)\mathbb{B}$.

Lemma 6.14. *Let G be a group of finite Morley rank satisfying Hypothesis 6.1.*

Then condition (B3) holds.

Proof. We deal first with (B3$_s$), i.e., $w = w_2$.

If $\mathbb{X}_2^{v^{-1}} \subseteq \mathbb{U}$ then $v\mathbb{X}_2 w_2 = \mathbb{X}_2^{v^{-1}} vw_2 \subseteq \mathbb{B}vw_2$ and the claim follows.

If $(\mathbb{X}_2^-)^{v^{-1}} \subseteq \mathbb{U}$ then $\mathbb{X}_2^{(vw_2)^{-1}} \subseteq \mathbb{U}$, so the claim follows for (vw_2, w_2), and hence for (v, w_2) as well. One of these two cases will apply for any choice of v.

The argument for (B3$_t$) is similar (symmetrical). □

Thus we conclude.

Theorem 6.15. *Let G be a simple group of finite Morley rank, an L^*-group of odd type satisfying* NTA_2*, with one conjugacy class of involutions. With i an involution of G, suppose that $N^\circ_G(\mathbb{X})$ is not contained in $C_G(i)$.*

Then G is $\mathrm{G}_2(k)$ *for some algebraically closed field k.*

Proof. By Fact 2.10 we arrive at Hypothesis 2.12 and hence at hypothesis 6.1. Hence our analysis above gives a BN-pair $(\mathbb{B}, N)$ with dihedral Weyl group. The condition $G = \langle \mathbb{B}, N\rangle$ holds by Fact 2.32.

Then [Ten04] or [TVM03] gives the identification, and as the base field has finite Morley rank, it is algebraically closed. □

7. Appendix

It may be helpful to record the commutatiion relations in G_2, particularly in connection with the exceptional case of characteristic 3. We take this from [DS16]. We give only $[e_\alpha, e_\beta]$ for positive roots, assuming an appropriate normalization.

Roots: α_1, α_2, $\alpha_1+\alpha_2$, $\alpha_1+2\alpha_2$, $\alpha_1+3\alpha_2$, $2\alpha_1+3\alpha_2$ with α_1 long and α_2 short.

$[e_\alpha, e_\beta]$ *(α, β positive; $\alpha+\beta$ a root; $\alpha \leq \beta$)*

α \ β	α_1	$\alpha_1+\alpha_2$	$\alpha_1+2\alpha_2$	$\alpha_1+3\alpha_2$
α_1				$e_{2\alpha_1+3\alpha_2}$
α_2	$e_{\alpha_1+\alpha_2}$	$2e_{\alpha_1+2\alpha_2}$	$3e_{\alpha_1+3\alpha_2}$	
$\alpha_1+\alpha_2$			$-3e_{2\alpha_1+3\alpha_2}$	

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